

Absolutely p -summing and Hilbert-Schmidt multilinear operators

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Abstract: In this paper we study theory of absolutely p -summing and Hilbert-Schmidt multilinear operators for Hilbert spaces. We present some new characterization of this operators by using classical results of Banach space theory and established new connection with tensor products. We also improve some previous results about multilinear settings.

Keywords: Absolutely p -summing operator, Hilbert-Schmidt operator, tensor product.

Date of Submission: 04-10-2026

Date of Acceptance: 14-10-2026

I. Introduction:

Many researchers generalized the concept of absolutely summing linear operators (see [1, 7, 8]) to the multilinear framework by several different approaches. The importance of the theory of summing operators, there have been some attempts to generalize concept, related results and tools to the multilinear setting. A. Pietsch ask a question for some value of $p, p_1, \dots, p_n$, the space of the absolutely $(p, p_1, \dots, p_n)$ - summing n -linear operators on Banach space coincides with the space of the Hilbert –Schmidt n -linear operators on Hilbert spaces. In Matos [2] prove that this question of Pietsch has negative answer. Matos [3] generalize the concept of linear summing to the multiple p -summing operator, named it fully absolutely summing and independently by Bombal, Perez-Garcia and Villanueva in [4], multiple summing. Many results proven in this direction for the class of multiple summing operators, illuminated that this class seems to be one of the most successful extension of the linear case. As a importance of the perspicuous theory of absolutely summing linear operators, the study of summability properties of multilinear operators focuses on operators that improve the summability of series. A desirable result in the theory is what has been called in the literature a coincidence result. This result consists in finding example of Banach spaces $X_1, \dots, X_n, Y$ or general conditions on them, that ensure a better summability behaviour of every continuous n -linear operator from $X_1 \times \dots \times X_n$ to Y or improve the summability behaviour of those operator that already enjoy some summing property. It is worth mentioning that this situation of lifting properties from bilinear to multilinear operators is non-trivial in many case. It is not true that the multilinear theory follows by induction from the linear case. Many example of the difficulties of lifting the linear theory to the multilinear setting can be found in the literature see [5]. For coincidence results type and cotype space plays a fundamental role for linear and non-linear operators. we established a connection with Hilbert-Schmidt operator on Hilbert space and characterize different property with them together tensor product approach.

II. Notations and Preliminaries:

In this section we introduce some basic terminology and properties concerning the multilinear operators and Hilbert-Schmidt operator. let $X_i, Y, i \in N$ be Banach space and $H_i, E_i, F, i \in N$ be Hilbert space. $L(X, Y)$ be denote the Banach space of bounded linear operator from X to Y and $L^m(X_1 \dots X_m; Y), m \geq 2$ be the Banach space of all bounded m -linear operator from $X_1 \times \dots \times X_m$ to Y . As usual $E_1 \otimes_\pi \dots \otimes_\pi E_n$ stands for the projective tensor product of the Banach spaces $E_1 \dots E_n$ and $E_1 \otimes_\epsilon \dots \otimes_\epsilon E_n$ denote injective tensor product of $E_1 \dots E_n$ and E^*, X^* stands for its topological dual and B_X denote closed unit ball of X . Given linear operator $u : X \rightarrow Y$ and $v : E \rightarrow F, u \otimes v : X \otimes E \rightarrow Y \otimes F$ will be linear operator given by $u \otimes v(x \otimes e) = u(x) \otimes v(e)$ and K be denote the basic field ($\mathbb{R}$ or $\mathbb{C}$). Remaining notation and terminology used along this paper is standard in

Banach space theory see [1,6]. These sources are also our main references for basic facts, definitions and unexplained notation.

Let $1 \leq p \leq \infty$, a sequence $(e_n)_n \subset E$ is strongly p -summable if $(\|e_n\|)_n \in l_p$ and $l_p(E)$ be the Banach space of all such sequences endowed with the norm $\|(e_n)_n\|_p = (\sum_n \|e_n\|^p)^{\frac{1}{p}}$ and sequence $(e_n)_n \subset E$ is weakly p -summable if for every $\varphi \in E^*$, $(\varphi(e_n))_n \in l_p$ and

$$\|(e_n)_n\|_p^w = \sup \left\{ \left(\sum_n (\varphi(e_n))^p \right)^{\frac{1}{p}} : \varphi \in B_{E^*} \right\}$$

Definition 2.1 A linear operator $T : E \rightarrow F$ is said to be p -summing if there exist a constant $C > 0$ such that $(\sum_n (T(e_n))^p)^{\frac{1}{p}} \leq C \|(e_n)_n\|_p^w$ for each finite sequence $(e_n)_n \subset E$ and minimum of C is called p -summing norm of T , and $\Pi_p(E, F)$ denote Banach space of p -summing operators from E to F and $T \in \Pi_p(E, F)$, $\pi_p(T)$ denote its p -summing norm.

Definition 2.2 Let $1 \leq p \leq \infty$ and $\lambda \geq 1$, a Banach space E is said to be an $\mathcal{L}_{p,\lambda}$ space if for every finite dimensional subspace A of E , there is a finite dimensional subspace B of E containing A such that $d(B, l_p^n) \leq \lambda$ where n is the dimension of B , that means there exist an isomorphism $u : B \rightarrow l_p^n$ with $\|u\| \|u^{-1}\| \leq \lambda$. A Banach space E is said to be L_p -space if it is an $\mathcal{L}_{p,\lambda}$ for some $\lambda > 1$. $L_p(\mu)$ is an example of an L_p -space.

Definition 2.3 A Banach space E is said to have cotype $q \geq 2$ if there exist a constant $C > 0$ such that every sequence of vectors $e_1, \dots, e_n \in E$, we have

$$(\sum_n \|e_i\|^q)^{\frac{1}{q}} \leq C \left(\int_0^1 \|\sum_n r_i(t) e_i\|^2 dt \right)^{\frac{1}{2}} \quad \text{where } (r_n)_n \text{ be sequence of Rademacher}$$

functions and least constant C is denoted by $C_q(E)$.

Let $n, m_1, \dots, m_n \in \mathbb{N}$, $(e_{i_1, \dots, i_n})_{i_1=1}^{m_1} \dots (e_{i_1, \dots, i_n})_{i_n=1}^{m_n}$ denote a multiindex sequence with the index i_j varying from 1 to m_j ($1 \leq j \leq n$) and $\sum_{i_1=1}^{m_1} \dots \sum_{i_n=1}^{m_n} e_{i_1, \dots, i_n}$ means the same as $\sum_{i_1=1}^{m_1} \dots \sum_{i_n=1}^{m_n} e_{i_1, \dots, i_n}$.

Definition 2.4 let $1 \leq p < \infty$, a multilinear operator $T : E_1 \times \dots \times E_n \rightarrow F$ is multiple p -summing if there exists a constant $C > 0$ such that for every sequences $(e_{i_j}^j)_{i_j=1}^{m_j} \subset E_j$ we have,

$$\left(\sum_{i_1=1}^{m_1} \dots \sum_{i_n=1}^{m_n} \|T(e_{i_1}^1, \dots, e_{i_n}^n)\|^p \right)^{\frac{1}{p}} \leq$$

$$C \prod_{j=1}^n \left\| \left(e_{i_j}^j \right)_{i_j=1}^{m_j} \right\|_p^w \quad (1)$$

The multiple p -summing norm of T is denoted and defined as $\pi_p(T) = \min\{C : C \text{ satisfy (1)}\}$ and $\Pi_p^n(E_1, \dots, E_n; F)$ denote the class of multiple p -summing n -linear operator which is Banach space together with the norm $\pi_p(T)$.

Definition 2.5 let $1 \leq p_1, \dots, p_n \leq q < \infty$, a multilinear operator $T : E_1 \times \dots \times E_n \rightarrow F$ is multiple $(q, p_1, \dots, p_n)$ -summing if there exist a constant $C > 0$ such that for every choice of sequences $(e_{i_j}^j)_{i_j=1}^{m_j} \subset E_j$, we have

$$\left(\sum_{i_1=1}^{m_1} \dots \sum_{i_n=1}^{m_n} \|T(e_{i_1}^1, \dots, e_{i_n}^n)\|^q \right)^{\frac{1}{q}} \leq C \prod_{j=1}^n \left\| \left(e_{i_j}^j \right)_{i_j=1}^{m_j} \right\|_{p_j}^w \quad (2)$$

The multiple $(q, p_1, \dots, p_n)$ -summing norm of T is denoted and defined as $\pi_{(q, p_1, \dots, p_n)}(T) = \min\{C : C \text{ satisfy (2)}\}$ and $\Pi_{(q, p_1, \dots, p_n)}^n(E_1, \dots, E_n; F)$ be denote class of multiple $(q, p_1, \dots, p_n)$ -summing multilinear operators which is Banach space with norm $\pi_{(q, p_1, \dots, p_n)}(T)$.

Definition 2.6 A linear operator $T : H_1 \rightarrow H_2$ is called Hilbert-Schmidt operator if the series $\|T\|_2 = (\sum_{n=1}^{\infty} \|Te_n\|^2)^{\frac{1}{2}}$ converges for at least one orthonormal base (e_n) of H_1 and it does not depend on the choice of the orthonormal base (e_n) and the class $\mathcal{S}_2(H_1, H_2)$ denote the space of all Hilbert-Schmidt operator which form Hilbert space with norm $\|T\|_2 = (\sum_{n=1}^{\infty} \|Te_n\|^2)^{\frac{1}{2}}$ and inner product $\langle S, T \rangle = \sum_{n,m=1}^{\infty} \langle f_m^*, Se_n \rangle \langle f_m^*, Te_n \rangle$ where (e_n) and (f_m^*) are orthonormal base of H_1 and H_2^* respectively. The double sequence of Hilbert-Schmidt operator $e_n^* \otimes f_m$ defined by $e_n^* \otimes f_m : x \in H_1 \rightarrow \langle e_n^*, x \rangle f_m \in H_2$ form an orthonormal base of $\mathcal{S}_2(H_1, H_2)$.

Definition 2.7 let $H_1, \dots, H_n, H$ be Hilbert space, a multilinear operator $T : H_1 \times \dots \times H_n \rightarrow H$ is said to be Hilbert-Schmidt if there exist $C > 0$ such that

$$\|T\|_{H_S} = \left(\sum_{i_1 \in I_1, \dots, i_n \in I_n} \|T(e_{i_1}^1, \dots, e_{i_n}^n)\|^2 \right)^{\frac{1}{2}} < C \quad (3)$$

Where $(e_{ij}^j)_{ij \in I_j} \subset H_j$ is an orthonormal basis $(1 \leq j \leq n)$ and least constant C is called Hilbert-Schmidt norm of T and it denoted by $\|T\|_{H_S}$. $\mathcal{S}_2^n(H_1, \dots, H_n; H)$ denote the class of all Hilbert-Schmidt operator from $H_1 \times \dots \times H_n$ to H and it is Hilbert space with this norm $\|T\|_{H_S}$ and the inner product defined as

$$\langle S, T \rangle = \sum_{i_1 \in I_1, \dots, i_n \in I_n} \langle T(e_{i_1}^1, \dots, e_{i_n}^n), S(e_{i_1}^1, \dots, e_{i_n}^n) \rangle \quad (4)$$

Theorem 2.8 The class of Hilbert-Schmidt multilinear operators and the class of multiple 2-summing operators are identically isometric.

Theorem 2.9 For a multilinear operator $T : E_1 \times \dots \times E_n \rightarrow F$, the following results are equivalent,

- (a) T is Hilbert-Schmidt
- (b) T is multiple p -summing for every $p \in [2, \infty)$.

Theorem 2.8 and 2.9 are proved in [3].

Let E, F, G are three separable Hilbert spaces, and $\mathcal{S}_2(E, F)$, $\mathcal{S}_2(F, G)$ be Hilbert space of Hilbert-Schmidt operators.

Theorem 2.10 The space $\mathcal{S}_2(\mathcal{S}_2(E, F), G)$ of H-S operator from $\mathcal{S}_2(E, F)$ to G is isometric to the space $\mathcal{S}_2(E^*, \mathcal{S}_2(F, G))$ of H-S operators from E^* to $\mathcal{S}_2(F, G)$.

Proof : let $T \in \mathcal{S}_2(E^*, \mathcal{S}_2(F, G))$ be the operator that associates with every $\varphi \in E^*$ the operator $T(\varphi) \in \mathcal{S}_2(F, G)$, this H-S operator $T(\varphi)$ has its norm equal to

$$\|T(\varphi)\|_{\mathcal{S}_2(F, G)}^2 = \sum_{i,j=1}^{\infty} |\langle T(\varphi)f_i, g_j^* \rangle|^2 \quad (5)$$

where f_i and g_j^* are orthonormal base of F and G^* respectively. The norm of T is equal to

$$\|T\|_2^2 = \sum_{n=1}^{\infty} \|T(\varphi_n)\|_{\mathcal{S}_2(F, G)}^2 = \sum_{i,j=1}^{\infty} |\langle T(\varphi_n)f_i, g_j^* \rangle|^2 \quad (6)$$

where φ_n is an orthonormal base of E^* .

Let $B \in \mathcal{S}_2(E, F)$ be a H-S operator defined with respect to the orthonormal base $(\varphi_n \otimes f_i)$ by , $B = \sum_{n,i=1}^{\infty} \alpha_{n,i} \varphi_n \otimes f_i$, where $\alpha_{n,i} \in l^2(N \times N)$

We define a operator $\bar{T} \in \mathcal{S}_2(\mathcal{S}_2(E, F), G)$ with protection of T as a way

$$\bar{T}(B) = \sum_{n,i=1}^{\infty} \alpha_{n,i} T(\varphi_n) f_i \quad (7)$$

It is clearly that $\bar{T}$ satisfy, $\bar{T}(\varphi_n \otimes f_i) = T(\varphi_n) f_i$, hence

$$\begin{aligned} \|\bar{T}\|^2 &= \sum_{n,i=1}^{\infty} \|\bar{T}(\varphi_n \otimes f_i)\|_G^2 = \sum_{n,i=1}^{\infty} \|T(\varphi_n) f_i\|_G^2 \\ &= \sum_{n,i,j=1}^{\infty} |\langle T(\varphi_n) f_i, x_j \rangle|^2 = \|T\|_2^2 \end{aligned} \quad (8)$$

This implies that $\bar{T}$ is H-S operator. therefore the mapping $T \in \mathcal{S}_2(E^*, \mathcal{S}_2(F, G))$ to $\bar{T} \in \mathcal{S}_2(\mathcal{S}_2(E, F), G)$ is an isometry and it is one-one on to.

Since $\bar{T} \in \mathcal{S}_2(\mathcal{S}_2(E, F), G)$, then for all fixed $\varphi \in E^*$, $\varphi \otimes f \in \mathcal{S}_2(E, F)$. The operator $T(\varphi)$ defined as $T(\varphi) : f \in F \rightarrow \bar{T}(\varphi \otimes f) \in G$ is H-S operator connected with $\bar{T}$ by equation (7). Thus we satisfy that $T : \varphi \rightarrow T(\varphi) \in \mathcal{S}_2(F, G)$ is H-S operator from E^* to $\mathcal{S}_2(F, G)$.

The space $\mathcal{S}_2(E^*, F)$ is the Hilbert tensor product of the seperable Hilbert space E and F . We denote this space by $E \hat{\otimes} F = \mathcal{S}_2(E^*, F)$ and its elements by,

$$B = \sum_{n,i=1}^{\infty} \alpha_{n,i} e_n \otimes f_i = \sum_{n=1}^{\infty} e_n \otimes x_n$$

where $(\alpha_{n,i}) \in l^2(N \times N)$ and $x_n = \sum_{i=1}^{\infty} \alpha_{n,i} f_i$ and (e_n) & (f_i) are orthonormal bases of E and F .

III. Characterizations and Results:

In this section, we first characterize Hilbert-Schmidt operator and p -summing operator and after establish some connections & results with tensor product approach. It is remark that every Hilbert-Schmidt operator is compact and every p -summing operator between Hilbert spaces is also compact.

Lemma 3.1 Let E, F and G are Hilbert spaces and $A : E \times F \rightarrow G$ is compact, then there is an orthonormal basis (e_n, f_n) in $E \times F$ such that $(e_n, f_n) = \alpha_n g_n$, where g_n is an orthonormal sequence in G , $\alpha_n \geq 0$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$.

Theorem 3.2 An operator $A : E \times F \rightarrow G$ is Hilbert-Schmidt if and only if A is absolutely p -summing operator.

Proof : let suppose A is absolutely p -summing operator then it clearly A is compact operator. let (e_n, f_n) be orthonormal basis in $E \times F$. Let r_n be the n -th Rademacher function defined as $r_n(t) = \left(\sin \frac{3\pi}{2}\right)^k$, for $2^{-n}k < t < 2^{-n}(k+1)$

$$= 0 , \quad \text{for } t = 2^{-n}k \text{ and } t = 1, \text{ where } k = 0, 1, \dots, 2^n - 1; \quad n \in N$$

We set a positive integer N_o and set for $k = 0, 1, \dots, 2^{N_o} - 1$ and $x_k \in E \times F$ in a way as $x_k = \sum_{n=1}^{N_o} r_n^k(e_n, f_n)$, where $r_n^k = r_n((2k+1)2^{-(N_o+1)})$, $k = 0, 1, \dots, 2^n - 1; \quad n \in N_o$

Then, $\|A(x_k)\| = \left\| \sum_{n=1}^{N_o} r_n^k A(e_n, f_n) \right\|$

$$= \left\| \sum_{n=1}^{N_o} r_n^k \alpha_n g_n \right\| = \left(\sum_{n=1}^{N_o} \alpha_n^2 \right)^{\frac{1}{2}}$$

$$\text{Therefore,} \quad \left(\sum_{k=0}^{2^{N_o}-1} \|Ax_k\|^p \right)^{\frac{1}{p}} = 2^{\frac{N_o}{p}} \left(\sum_{n=1}^{N_o} \alpha_n^2 \right)^{\frac{1}{2}} \quad (9)$$

We choose, if $a = \sum_{i=1}^{\infty} \langle a, e_i \rangle \langle a, f_i \rangle (e_i, f_i) \in E \times F$, then

$$|\langle x_k, a \rangle|^p = \left| \sum_{n=1}^{N_o} r_n^k \overline{\langle a, e_n \rangle} \langle a, f_n \rangle \right|^p = 2^{N_o} \int_{k2^{-N_o}}^{(k+1)2^{-N_o}} \left| \sum_{n=1}^{N_o} r_n(t) \overline{\langle a, e_n \rangle} \langle a, f_n \rangle \right|^p dt$$

$$\begin{aligned} \text{Hence, } \left(\sum_{k=0}^{2^{N_o}-1} |\langle x_k, a \rangle|^p \right)^{\frac{1}{p}} &= \left(2^{N_o} \sum_{k=0}^{2^{N_o}-1} \int_{k2^{-N_o}}^{(k+1)2^{-N_o}} \left| \sum_{n=1}^{N_o} r_n(t) \overline{\langle a, e_n \rangle} \langle a, f_n \rangle \right|^p dt \right)^{\frac{1}{p}} \\ &= 2^{\frac{N_o}{p}} \left(\int_0^1 \left| \sum_{n=1}^{N_o} r_n(t) \overline{\langle a, e_n \rangle} \langle a, f_n \rangle \right|^p dt \right)^{\frac{1}{p}} \end{aligned}$$

Therefore, by Khinchin inequality,

$$\left(\sum_{k=0}^{2^{N_o}-1} |\langle x_k, a \rangle|^p \right)^{\frac{1}{p}} \leq 2^{\frac{N_o}{p}} C_p \left(\sum_{n=1}^{N_o} |\langle a, e_n \rangle \langle a, f_n \rangle|^2 \right)^{\frac{1}{2}} \leq 2^{\frac{N_o}{p}} C_p \|a\| \quad (10)$$

where C_p is a constant depending on p only.

From (9) & (10) and using definition of absolutely p -summing operator on $E \times F$,

$$2^{\frac{N_o}{p}} \left(\sum_{n=1}^{N_o} \alpha_n^2 \right)^{\frac{1}{2}} \leq K \sup_{\|a\|=1} \left(\sum_{k=1}^{2^{N_o}-1} |\langle x_k, a \rangle|^p \right)^{\frac{1}{p}} \leq KC_p 2^{\frac{N_o}{p}}$$

$$\text{Thus,} \quad \left(\sum_{n=1}^{N_o} \alpha_n^2 \right)^{\frac{1}{2}} \leq KC_p \quad \text{for every } N_o \in \mathbb{N}$$

Since $\|A(e_n, f_n)\| = \alpha_n$ for $n \in \mathbb{N}$ and hence $\sum_{n=1}^{\infty} \|A(e_n, f_n)\|^2 \leq (KC_p)^2 < \infty$

This implies that A is H-S operator.

Conversely, let if $A \in \mathcal{S}_2(E \times F; G)$, then

$$\|A\|_2 = \left(\sum_{n=1}^{\infty} |\langle A(e_n, f_n), g_n^* \rangle|^2 \right)^{\frac{1}{2}} = \left(\sum_{n=1}^{\infty} \alpha_n^2 \right)^{\frac{1}{2}} < \infty$$

where $(e_n), (f_n), (g_n^*)$ be orthonormal sequence in E, F and G^* respectively.

Let (x_m) be an arbitrary sequence in $E \times F$ and let for $0 \leq t \leq 1$,

$b(t) = A^{-1} \sum_{n=1}^{\infty} r_n(t) \alpha_n (e_n, f_n)$ where r_n be n th Rademacher function, then by using the Khinchin inequality, we get

$$\begin{aligned} \sup_{\|a\|=1} \sum_{m=1}^{\infty} |\langle x_m, a \rangle| &\geq A^{-1} \sup_{0 \leq t \leq 1} \sum_{m=1}^{\infty} |\langle x_m, b(t) \rangle| \\ &\geq A^{-1} \int_0^1 \sum_{m=1}^{\infty} |\sum_{n=1}^{\infty} r_n(t) \alpha_n \langle x_m, e_n \rangle \langle f_n, g_n \rangle| dt \\ &= A^{-1} \sum_{m=1}^{\infty} \int_0^1 |\sum_{n=1}^{\infty} r_n(t) \alpha_n \langle x_m, e_n \rangle \langle f_n, g_n \rangle| dt \\ &\geq (\sqrt{2}A)^{-1} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \alpha_n^2 |\langle x_m, e_n \rangle \langle f_n, g_n \rangle|^2 \right)^{\frac{1}{2}} \\ &= (\sqrt{2}A)^{-1} \sum_{m=1}^{\infty} \|Ax_m\| \end{aligned}$$

This implies $A \in \Pi_p(E \times F; G)$. Finally, let $p > 1$ and $q = p(p-1)^{-1}$ then,

$$(\sum_{m=1}^{\infty} |K_m|^p)^{\frac{1}{p}} = \sup_{\sum_{m=1}^{\infty} |t_m|^q = 1} \sum_{m=1}^{\infty} |t_m K_m|$$

We use definition of p -summing operator, we get

$$\begin{aligned} (\sum_{m=1}^{\infty} \|Ax_m\|^p)^{\frac{1}{p}} &= \sup_{\sum_{m=1}^{\infty} |t_m|^q = 1} \sum_{m=1}^{\infty} |At_m x_m| \\ &\leq \sqrt{2} A \sup_{\sum_{m=1}^{\infty} |t_m|^q = 1} \sup_{\|a\|=1} \sum_{m=1}^{\infty} |\langle t_m x_m, a \rangle| \\ &= \sqrt{2} A \sup_{\|b\|=1} \sup_{\|a\|=1} \sum_{m=1}^{\infty} |t_m \langle x_m, a \rangle| \\ &= \sqrt{2} A \sup_{\|b\|=1, \|a\|=1} (\sum_{m=1}^{\infty} |\langle x_m, \langle e_n, f_n \rangle g_n \rangle|^p)^{\frac{1}{p}} \end{aligned}$$

Hence the proof.

Theorem [1] 3.3 let $1 \leq p \leq q$, and E be an $\mathcal{L}_{\infty, \lambda}$ -space and F be an $\mathcal{L}_{p, \lambda'}$ -space. Then every operator $T : E \rightarrow F$ is 2-summing with $\pi_2(T) \leq K_G \lambda \lambda' \|T\|$.

Corollary 3.4 let $T : E \rightarrow F$ be an operator, then following condition are equivalent,

- (a) T is a Hilbert-Schmidt operator,
- (b) T factor through an L_{∞} -space,
- (c) T factor through an L_1 -space.

Proof : (a) $\Leftrightarrow$ (b), since T is H-S operator then T is 2-summing operator, this implies we apply Pietsch factorization theory we have a regular Borel probability measure μ on compact Hausdorff space K , a closed subspace X_2 of $L_2(\mu)$ and an operator $\bar{T} : X_2 \rightarrow F$ such that $\bar{T} j_2 i_E(e) = Te$ for all $e \in E$. This show that T factor through a $C(K)$ -space, and if T factor through an L_{∞} -space, then by theorem 3.3, T is 2-summing and so it is H-S operator.

(a) $\Leftrightarrow$ (c) we remarks that dual of $C(K)$ -spaces are $L_1(\mu)$ -spaces, through the measures μ that arise is non σ -finite. It is relatively simple to use localization property to show that $C(K)^*$ is an $\mathcal{L}_{1, \lambda}$ -space for each $\lambda > 1$, and hence $C(K)^*$ as the space of regular Borel measures defined on compact Hausdorff space K . Since $T : E \rightarrow F$ is Hilbert-Schmidt operator, then so T^* is also, and any factorization $T^* : F \rightarrow C(K) \rightarrow E$ lead to a factorization $E \rightarrow L_1 \rightarrow F$ of $T = T^{**}$ and we know if E is an $\mathcal{L}_{1, \lambda}$ -space and F is $\mathcal{L}_{2, \lambda'}$ -space then every operator $T : E \rightarrow F$ is 2-summing and so T is H-S operator because every Hilbert space is isomorphic to a subspace of an $L_1(\mu)$ -space and to a quotient of a $C(K)$ -spaces simultaneously. Hence the proof.

We define inner product in the space $E_1 \otimes \dots \otimes E_n$ which is tensor product of Hilbert space $E_i, 1 \leq i \leq n$.

$$\langle x, y \rangle = \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} \langle u_j^1, v_k^1 \rangle \dots \langle u_j^n, v_k^n \rangle \quad (11)$$

where $x = \sum_{j=1}^{n_1} u_j^1 \otimes \dots \otimes u_j^n$, $y = \sum_{k=1}^{n_2} v_k^1 \otimes \dots \otimes v_k^n$.

The space $E_1 \otimes \dots \otimes E_n$ with this inner product is denoted by $E_1 \otimes_{\pi} \dots \otimes_{\pi} E_n$ and its completion by $E_1 \widehat{\otimes}_{\pi} \dots \widehat{\otimes}_{\pi} E_n$ and its norm is denoted by $\|\cdot\|_{\pi}$. let $(e_j^k)_{j \in J_K}$ is an orthonormal basis for E_K , $k = 1 \dots n$ then $(e_{j_1}^1 \otimes \dots \otimes e_{j_n}^n)_{j_k \in J_K, k = 1, \dots, n}$ is an orthonormal basis for $E_1 \widehat{\otimes}_{\pi} \dots \widehat{\otimes}_{\pi} E_n$.

Theorem 3.5 The space $\mathcal{S}_2(E_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_n; F)$ of Hilbert-Schmidt operators from $E_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_n$ to F is isometric to the Hilbert tensor product of $(E_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_n)^*$ and space of Hilbert-Schmidt operator from $E_1 \times \dots \times E_n$ to F . In particular $(E_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_n)^*$ is isometric to $E_1^* \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_n^*$.

Proof : it is convenient to make explicit the isometry j . let $T \in \mathcal{S}_2(E_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_n; F)$, we relate T to the operator $\hat{T}(x) \in \mathcal{S}_2(E_1 \times \dots \times E_n; F)$ defined by

$$\hat{T}(x) : y \in E_1 \times \dots \times E_n \rightarrow \hat{T}(x)y = T(e_1 \otimes \dots \otimes e_n)$$

If $(e_{j_1}^1 \otimes \dots \otimes e_{j_n}^n) \quad j_k \in J_K, k = 1, \dots, n$ is an orthogonal base of $E_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_n$ and $(e_{j_1}^1 \otimes \dots \otimes e_{j_n}^1)^*$ is its dual base. The isometry is defined by

$$T \in \mathcal{S}_2(E_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_n; F) \rightarrow j_a = \sum_{j_k \in J_K, k=1, \dots, n} (e_{j_1}^1 \otimes \dots \otimes e_{j_n}^n)^* \otimes \hat{T}(e_{j_1}^1 \otimes \dots \otimes e_{j_n}^n) \in (E_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_n)^* \widehat{\otimes} \mathcal{S}_2(E_1 \times \dots \times E_n; F)$$

$$\text{Since, } x = \sum_{j=1}^{n_1} u_j^1 \otimes \dots \otimes u_j^n = \sum_{j_k \in J_K, k=1, \dots, n} ((e_{j_1}^1 \otimes \dots \otimes e_{j_n}^n)^*, x) (e_{j_1}^1 \otimes \dots \otimes e_{j_n}^n)$$

$$\begin{aligned} \text{Then, } T(x \otimes y) &= \sum_{j_k \in J_K, k=1, \dots, n} ((e_{j_1}^1 \otimes \dots \otimes e_{j_n}^n)^*, x) \hat{T}(e_{j_1}^1 \otimes \dots \otimes e_{j_n}^n) y \\ &= \sum_{j_k \in J_K, k=1, \dots, n} ((e_{j_1}^1 \otimes \dots \otimes e_{j_n}^n)^* \otimes \hat{T}(e_{j_1}^1 \otimes \dots \otimes e_{j_n}^n)) (x \otimes y) \end{aligned}$$

Hence $T((e_{j_1}^1 \otimes \dots \otimes e_{j_n}^n)^*) \in \mathcal{S}_2(E_1 \times \dots \times E_n; F)$ and clearly,

$$(E_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_n)^* = E_1^* \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_n^*. \text{ Hence the proof.}$$

Theorem 3.6 The space $\mathcal{S}_2(E_1 \times \dots \times E_n; F)$ is isometric isomorphic to the space $\Pi_p^n(E_1 \times \dots \times E_n; F)$.

Proof : Let $A \in \Pi_p^n(E_1 \times \dots \times E_n; F)$ and $(e_j^i)_{j \in J_K}$ is an orthonormal base of $E_i, 1 \leq i \leq n$ since, we know that

$$(\sum_{m \in I_1 \times \dots \times I_n} \|A(e_{m_1}^1 \dots e_{m_n}^n)\|^p)^{\frac{1}{p}} \leq \pi_p(A) \prod_{i=1}^n \|(e_m^i)_{m \in I_K}\|_{w,p} \leq \pi_p(A) \quad (12)$$

For every finite subset I_K of J_K , with m element, $i = 1, \dots, n$ and m be natural. Thus $A \in \mathcal{S}_2(E_1 \times \dots \times E_n; F)$ and $\|A\|_{HS} \leq \pi_p(A)$.

Next, we consider $A \in \mathcal{S}_2(E_1 \times \dots \times E_n; F)$ we take if $n=1$, we consider $m \in N, e_i \in E_i, i = 1, \dots, m$ and an orthonormal base $(f_j)_{j \in J}$ of F , then

$$\begin{aligned} (\sum_{i=1}^m \|A(e_i)\|^p)^{\frac{1}{p}} &= (\sum_{i=1}^m \sum_{j \in J} |\langle e_i, A^*(f_j) \rangle|^p)^{\frac{1}{p}} \\ &\leq (\sum_{j \in J} \|A^*(f_j)\|^p)^{\frac{1}{p}} \sup_{\varphi \in B_{E_i^*}} (\sum_{i=1}^m |\varphi(e_i)|^p)^{\frac{1}{p}} \\ &= \|A\|_{HS} \|(e_i)_{i=1}^m\|_{w,p}. \end{aligned}$$

Hence, $A \in \Pi_p^n(E_1 \times \dots \times E_n; F)$ and $\pi_p(A) \leq \|A\|_{HS}$.

Now, we take $n > 1$ and we assume that it is true for all $k \leq n-1$. Since the mapping

$$A \in \mathcal{S}_2(E_1 \times \dots \times E_n; F) \rightarrow A_1 \in \mathcal{S}_2(E_1, \mathcal{S}_2(E_2 \times \dots \times E_n; F))$$

Where $A_1(e_1)(e_2, \dots, e_n) = A(e_1, \dots, e_n)$ for $e_k \in E_k, k = 1, \dots, n$ is an isometric isomorphism. We have,

$A_1 \in \Pi_p^n(E_1, \mathcal{S}_2(E_2 \times \dots \times E_n; F)) \subset \Pi_p^n(E_1, \Pi_p^n(E_2 \times \dots \times E_n; F))$ with $\pi_p(A_1) \leq \|A_1\|_{HS} = \|A\|_{HS}$. by induction hypothesis, we obtain

$A \in \Pi_p^n(E_1 \times \dots \times E_n; F)$ and $\pi_p(A) \leq \pi_p(A_1) \leq \|A\|_{HS}$. Hence the proof.

Theorem 3.7 Let $2 \leq p \leq \infty$ and $T : E_1 \otimes \dots \otimes E_n \otimes E_{n+1} \otimes \dots \otimes E_{n+k} \rightarrow F$ be multiple p -summing operator then $\hat{T} : E_{n+1} \otimes \dots \otimes E_{n+k} \rightarrow \mathcal{S}_2(E_1 \otimes \dots \otimes E_n; F)$ is multiple p -summing with respect to the H_S norm on $\mathcal{S}_2(E_1 \otimes \dots \otimes E_n; F)$ and $[a_p]^n \pi_p^{n+k}(T) \leq \pi_p^k(\hat{T}) \leq [b_p]^n \pi_p^{n+k}(T)$ where a_p and b_p are Khinchin Constant.

Proof : Let $\theta_j = (e_{i_{n+j}}^{n+j})_{1 \leq i_{n+j} \leq m_{n+j}} \subset E_{n+j}$ be some finite order structure such that $w_p(e_{i_{n+j}}^{n+j} : 1 \leq i_{n+j} \leq m_{n+j}) \leq 1$ for each $i \leq j \leq k$ and we define the operator

$$A_{\theta_1 \dots \theta_K} : E_1 \otimes \dots \otimes E_n \rightarrow l_p^{m_{n+1} \dots m_{n+k}}(F) \text{ as}$$

$$A_{\theta_1 \dots \theta_K}(e_1 \dots e_n) = T(e_1 \dots e_n, e_{i_{n+1}}^{n+1} \dots e_{i_{n+k}}^{n+k}), 1 \leq i_{n+1} \leq m_{n+1}, \dots, 1 \leq i_{n+k} \leq m_{n+k}.$$

Now, by definition of multiple p -summing operator, it follows that $A_{\theta_1 \dots \theta_K}$ is multiple p -summing and $\pi_p^K(A_{\theta_1 \dots \theta_K}) \leq \pi_p^{n+k}(T)$, since $2 \leq p \leq \infty$ we define s as $\frac{1}{2} = \frac{1}{p} + \frac{1}{s}$ if $p = 2$ then $s = \infty$.

Let $\mu = (\mu_{i_{n+1}, \dots, i_{n+k}}), 1 \leq i_{n+1} \leq m_{n+1}, \dots, 1 \leq i_{n+k} \leq m_{n+k}$ be a finite system of scalars and let define a operator,

$$S_\mu((e_{i_{n+1}, \dots, i_{n+k}}), 1 \leq i_{n+1} \leq m_{n+1}, \dots, 1 \leq i_{n+k} \leq m_{n+k}) = (\mu_{i_{n+1}, \dots, i_{n+k}} e_{i_{n+1}, \dots, i_{n+k}}), 1 \leq i_{n+1} \leq m_{n+1}, \dots, 1 \leq i_{n+k} \leq m_{n+k}.$$

Clearly S_μ is bounded linear operator and from Holder's inequality, $\frac{1}{2} = \frac{1}{p} + \frac{1}{s}$, we have $\|S_\mu\| \leq \|\mu\|_{l_s^{m_{n+1} \dots m_{n+k}}}$, then by the ideal property of the class of multiple p -summing operator. it implies that

$$S_\mu \circ A_{\theta_1 \dots \theta_K} : E_1 \otimes \dots \otimes E_n \rightarrow l_p^{m_{n+1} \dots m_{n+k}}(F) \text{ is multiple } p\text{-summing and}$$

$$\pi_p^n(S_\mu \circ A_{\theta_1 \dots \theta_K}) \leq \pi_p^k(A_{\theta_1 \dots \theta_K}) \|S_\mu\|$$

$$\text{Thus, } \pi_p^n(S_\mu \circ A_{\theta_1 \dots \theta_K}) \leq \pi_p^{n+k}(T) \|\mu\|_{l_s^{m_{n+1} \dots m_{n+k}}}.$$

Since F is a Hilbert space, then $l_p^{m_{n+1} \dots m_{n+k}}(F)$ is also Hilbert space. It is clearly that $S_\mu \circ A_{\theta_1 \dots \theta_K}$ is a Hilbert-Schmidt operator and

$$\|S_\mu \circ A_{\theta_1 \dots \theta_K}\|_{HS} \leq [b_p]^n \pi_p^n(S_\mu \circ A_{\theta_1 \dots \theta_K})$$

But by theory of multiple Hilbert-Schmidt operator, all operators,

$$\mu_{i_{n+1} \dots i_{n+k}} \hat{T}(e_{i_{n+1}}^{n+1} \dots e_{i_{n+k}}^{n+k}), 1 \leq i_{n+1} \leq m_{n+1}, \dots, 1 \leq i_{n+k} \leq m_{n+k} \text{ are H-S operator and}$$

$$\sum_{i_{n+1} \dots i_{n+k}=1}^{m_{n+1} \dots m_{n+k}} \|\mu_{i_{n+1} \dots i_{n+k}} \hat{T}(e_{i_{n+1}}^{n+1} \dots e_{i_{n+k}}^{n+k})\|_{HS}^2 = \|S_\mu \circ A_{\theta_1 \dots \theta_K}\|_{HS}^2$$

If we take $\mu_{i_{n+1} \dots i_{n+k}} = 1$, thus all operators

$\hat{T}(e_{i_{n+1}}^{n+1} \dots e_{i_{n+k}}^{n+k}) : E_1 \otimes \dots \otimes E_n \rightarrow F$, $1 \leq i_{n+1} \leq m_{n+1}, \dots, 1 \leq i_{n+k} \leq m_{n+k}$ are H-S operators and

$$\left(\sum_{i_{n+1} \dots i_{n+k}=1}^{m_{n+1} \dots m_{n+k}} |\mu_{i_{n+1} \dots i_{n+k}}|^2 \|\hat{T}(e_{i_{n+1}}^{n+1} \dots e_{i_{n+k}}^{n+k})\|_{HS}^2 \right)^{\frac{1}{2}} \leq [b_p]^n \pi_p^{n+k}(T) \|\mu\|_{l_s^{m_{n+1} \dots m_{n+k}}}$$

We taking supremum over all $\|\mu\|_{l_s^{m_{n+1} \dots m_{n+k}}} \leq 1$, then LHS becomes,

$$\left(\sum_{i_{n+1} \dots i_{n+k}=1}^{m_{n+1} \dots m_{n+k}} \|\hat{T}(e_{i_{n+1}}^{n+1} \dots e_{i_{n+k}}^{n+k})\|_{HS}^p \right)^{\frac{1}{p}} \leq [b_p]^n \pi_p^{n+k}(T)$$

Hence by similar argument, we obtain,

$$\left(\sum_{i_{n+1} \dots i_{n+k}=1}^{m_{n+1} \dots m_{n+k}} \|\hat{T}(e_{i_{n+1}}^{n+1} \dots e_{i_{n+k}}^{n+k})\|_{HS}^p \right)^{\frac{1}{p}} \leq [b_p]^n \pi_p^{n+k}(T) \cdot w_p(e_{i_{n+1}}^{n+1} : 1 \leq i_{n+1} \leq m_{n+1}) \dots w_p(e_{i_{n+k}}^{n+k} : 1 \leq i_{n+k} \leq m_{n+k}) . \text{ hence the proof.}$$

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