

Color Sets of Odd Cycles and Products of Odd Cycles

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Abstract

A graph G is tripartite if its chromatic number $\chi(G) = 3$. Let $P(a, b, c)$ denote the color partition of an odd cycle whose color sets have cardinalities a, b , and c . We examine the color partitions of odd cycles and products of odd cycles.

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I. Preliminary remarks

A graph G is *tripartite* if its chromatic number $\chi(G) = 3$. This means the vertex set of a connected graph G , $V(G)$, can be partitioned into three sets each of which is an *independent set*, that is, no two vertices in a partition set are adjacent. This also means that each of the three partition sets can be assigned a distinct color resulting in the graph G being *properly vertex colored* (no pair of adjacent vertices will have the same color) [1]. With this point of view, the partition sets are called *color sets*.

Definition: Let G be an odd cycle whose color sets have cardinalities $a \geq b \geq c$. Then $P(a, b, c)$ will denote this *color partition*.

There may be several ways to partition $V(G)$ into *color sets*. The tripartite cycle C_7 , for example, can have color sets of orders 3, 3, and 1, or of orders 3, 2, and 2. The color of the vertices of a chosen minimum color set shall, without loss of generality, be grey, while the vertices of the two other color sets shall be black and white. We denote the cardinality of a minimum color set of graph G by $f(G)$.

It is easily seen that the Cartesian product, G , of two cycle such that at least one of them is odd is tripartite. The following theorem determines $f(G)$.

Theorem 1:

(a) Given positive integers $m, n \geq 3$, such that m is odd and n is even, then $f(C_m \times C_n) = n$.

(b) Given positive odd integers $m, n \geq 3$, then $f(C_m \times C_n) = \max\{m, n\}$.

Proof: (a) Consider the 2-dimensional mesh $P_m \times P_n$, whose rows and columns are copies of P_n and P_m , respectively. Label the vertices with ordered pairs of positive integers as one would label the entries of an $m \times n$ matrix. (Upper leftmost vertex is $(1, 1)$, upper rightmost is $(1, n)$, lower leftmost vertex is $(m, 1)$, lower rightmost is (m, n) , etc.). After rendering the vertices of the first and last column adjacent in corresponding pairs, and doing the same to the vertices of the first row and the last row, we obtain the toroidal graph $C_m \times C_n$. Each column becomes the odd cycle C_m and therefore requires at least one grey vertex. The proof of the theorem is over when we show that there exists a tripartite labeling of $C_m \times C_n$ that utilizes exactly one grey vertex per column. This is done as follows: Let $(1, 1), (2, 2), (1, 3), (2, 4), (1, 5), (2, 6), \dots, (1, n-1), (2, n)$ be colored grey. In each column, let the vertex directly under the grey one be black while the vertex directly above the grey vertex (if one exists) is to be colored white. The rest of the color assignment using black and white vertices is now

valid and fully determined. It is easily seen that such a coloring enables the required added adjacencies to produce $C_m \times C_n$.

(b) We proceed as in the proof of (a), the only difference being the labeling of the grey vertices. We lay out the mesh so that the width is greater than or equal to the height. That is, we assume without loss of generality, that $n \geq m$. We select exactly one grey vertex per column as follows: Let $(1,1), (2,2), (3,3), \dots (m, m)$ be colored grey. If $m = n$, we complete the proof as in (a). Otherwise, let $(m-1, m+1), (m, m+2), (m-1, m+3), (m, m+4), \dots, (m-1, n-1), (m, n)$ be colored grey. We complete the proof as in (a). ■

Corollary 1: Given positive integers $m, n \geq 3$, such that m is odd, then $f(C_m \times P_n) = n$.

Proof: This follows from part (a) of the theorem. ■

References

- [1]. F.Buckley and M.Lewinter, *A Friendly Introduction to Graph Theory*. Prentice-Hall, 2003.
- [2]. F.Buckley, M.Lewinter, *Introductory Graph Theory with Applications*. Waveland Press. 2014.