

On the Norm of Multiplication of Right and Left Multiplication Operators Induced by A and B

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Abstract:

Elementary operators have been widely studied based on properties such as numerical range, spectrum, compactness, norm, and rank. Previous studies have considered elementary operators on bounded operators on Hilbert spaces, matrix algebras, C^* -algebras, von Neumann algebras, and JB^* -algebras. This study focused on the multiplication elementary operator $M_{A,B} = L_A R_B$, defined by $M_{A,B}(X) = AXXB$, acting on a C^* -algebra. The norm of $M_{A,B}$ was determined using the Cauchy-Schwarz inequality together with properties of bounded operators, finite rank operators, inner products, definition of elementary operator, and functionals. The study established the norm of $M_{A,B}$ on a C^* -algebra.

Keywords: Elementary operators; C^* -algebras; Operator norm; Multiplication operators.

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I. Introduction

Consider a C^* -algebra $\mathfrak{A}$ and fixed elements $A_i, B_i \in \mathfrak{A}$. An elementary operator on $\mathfrak{A}$ is a mapping $T: \mathfrak{A} \rightarrow \mathfrak{A}$ of the form

$$T(X) = \sum_{i=1}^n A_i X B_i, \quad X \in \mathfrak{A}$$

The integer n represents the number of terms occurring in the elementary operator. Hence,

$$T(X) = A_1 X B_1 + A_2 X B_2 + \dots + A_n X B_n.$$

When $n = 1$, and $A_1 = A$ and $B_1 = B$, the above expression reduces to

$$T(X) = AXB, \quad X \in \mathfrak{A}$$

We denote Basic elementary operator by $M_{A,B}$ and define

$$M_{A,B}: \mathfrak{A} \rightarrow \mathfrak{A}$$

$$M_{A,B}(X) = AXB, \quad X \in \mathfrak{A}.$$

For $A, B \in \mathfrak{A}$, the left multiplication operator L_A and the right multiplication operator R_B are defined by

$$L_A: \mathfrak{A} \rightarrow \mathfrak{A}$$

$$L_A(X) = AX, \quad X \in \mathfrak{A},$$

and

$$R_B: \mathfrak{A} \rightarrow \mathfrak{A}$$

$$R_B(X) = XB, \quad X \in \mathfrak{A}$$

Thus, L_A and R_B describe multiplication by A from the left and by B from the right, respectively. For every $X \in \mathfrak{A}$,

$$\begin{aligned} L_A R_B(X) &= L_A(R_B(X)) \\ &= L_A(XB) \\ &= AXXB. \end{aligned}$$

Therefore,

$$M_{A,B}(X) = L_A R_B(X), \quad X \in \mathfrak{A}, \text{ and}$$

hence,

$$M_{A,B} = L_A R_B.$$

Thus, the multiplication operator $M_{A,B}$ is represented as the composition of the right multiplication operator R_B followed by the left multiplication operator L_A .

II. Literature

Mathieu (2001) examined the norm of elementary operators on C^* -algebras and obtained theorem 2.1

Theorem 2.1: (Mathieu, 2001)

Let $\mathfrak{A}$ be C^* -algebra, then

$$\|M_{A,B}\| = \sup\{\|M_{A,B}(U)\| : U \in U(\mathfrak{A})\} = \sup\{\|AUB\| : U \in U(\mathfrak{A})\}$$

where $U(\mathfrak{A})$ denotes the set of unitaries in $\mathfrak{A}$.

Timoney (2003) determined the norm of Jordan elementary operators on C^* -algebra that contributed to theorem 2.2.

Theorem 2.2: (Timoney, 2003)

Assume that H is two dimensional Hilbert space and let $A, B \in B(H)$, if $M_{A,B} \geq AXB + BXA$ then

$$\|M_{A,B}\|_{cb} \geq \|A\|_2 \|B\|_2$$

On the same, it showed the inequality holds:

$$\|M\|_K \leq \|M\|_{CB} \leq \frac{1}{2} (\sum_{\ell_j=1} a_j a_{a_j} + \sum_{\ell_j=1} b_j b_{a_j})$$

$$\text{iff } W_{m,e}(a^*_{1}, a^*_{2}, \dots, a^*_{n}) \cap W_{m,e}(b^*_{1}, b^*_{2}, \dots, b^*_{n}) = \phi$$

Magajna (2004) investigated the norm of symmetric (symmetrized) elementary operators acting on prime C^* -algebra to achieve theorem 2.3

Theorem 2.3: (Magajna, 2004)

For all $A, B \in B(H)$ the norm of the operator

$$S_{A,B} : B(H) \rightarrow B(H), S_{A,B}(X) = A^*XB + B^*XA$$

is equal to the norm of its restriction to the space $B(H)_{sa}$ of self adjoint operators and is given by

$$\|S_{A,B}\| = \inf \|pA^*A + rB^*B - q(A^*B - B^*A)\| : p, q, r \in \mathbb{R}, p, r > 0, pr - q^2 = 1 \text{ Furthermore,}$$

$$\|S_{A,B}\| = \|S_{A,B}\|_{cb}$$

Seddik (2007) studied the norm of elementary operators on standard operator algebras using rank-one operators and obtained theorem 2.4

Theorem 2.4:(Seddik,2007)

Let $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ be two n-tuples of non-zero elements in

A. The following properties are equivalent:

- i. $d(R_{A,B}) = D(R_{A,B})$,
- ii. there exist two unit elements f, g in A' and n unit scalars $\lambda_1, \dots, \lambda_n$ such that $f(A_i) = \lambda_i \|A_i\|$ and $g(B_i) = \lambda_i \|B_i\|$ for $i = 1, \dots, n$, iii. $\|\sum_{i=1}^n \lambda_i A_i\| = \|\sum_{i=1}^n A_i\|$ and $\|\sum_{i=1}^n \bar{\lambda}_i B_i\| = \|\sum_{i=1}^n B_i\|$ for some unit scalars $\lambda_1, \dots, \lambda_n$

Seddik (2007) Studied the norm of elementary operators using rank-one operators, providing a foundation for determining the norm of multiplication operators such as $M_{A,B} = L_A R_B$

Okelo and Agure (2011) determined the norm Basic elementary operator to attain theorem 2.5

Theorem 2.5: (Okelo and Agure, 2011)

Let H be a complex Hilbert space, $B(H)$ the algebra of all bounded linear operators on

H . Let $M_{A,B} : B(H) \rightarrow B(H)$ be denoted by $M_{A,B}(X) = AXB, \forall X \in B(H)$ where $A,$

B are fixed in $B(H)$, Then

$$\|M_{A,B}(X)\| = \|A\| \|B\|.$$

Wickstead (2015) examined norms of basic elementary operators on algebras as illustrated in theorem 2.6

Theorem 2.6: (Wickstead, 2015)

If E is an atomic Banach lattice with an order continuous norm and $A, B \in L_r(E)$ then

$$|M_{A,B}| = M_{|A|,|B|}. \|M_{A,B}\|_r = \|A\|_r \|B\|_r$$

Kingangi (2017) investigated the lower bound of the norm of an elementary operator of length two and obtained theorem 2.7. Furthermore, it resulted to corollary 2.7.1 and 2.7.2.

Theorem 2.7: (King'angi, 2017)

Let E_2 be an elementary operator of length two on $B(H)$ Then:

$$\|E_2\| \geq \sup_{\lambda \in WB(A^*A)} \|B_1\| \|A_1 + \frac{\bar{\lambda}}{\|B\|} A_2\|$$

Corollary 2.7.1

Let H be a complex Hilbert space and T_i, S_i be bounded linear operators on H for

$i = 1, 2$. Let $0 \in W_{s1}(S_2^* S_1) \cup W_{s2}(S_1^* S_2)$. Then,

$$\|E_2\| \geq \|T_1\| \|S_1\|$$

Corollary 2.7.2

Let H be a complex Hilbert space and T_i, S_i be bounded linear operators on H for $i = 1, 2$. Let E_2 be an elementary operator of length two. If $\|T_1\| \|T_2\| \in W_{T_1}^*(T_2 T_1^*)$ and $\|S_1\| \|S_2\| \in W_{S_2}(S_1^* S_2)$, then,

$$\|E_2\| = \|T_1\| \|S_1\|$$

Kawira et al (2018) determined the norm of an elementary operator of length n in a C^* -algebra and proved theorem 2.8 **Theorem**

2.8: (Kawira et al., 2017)

let H be a complex Hilbert space and $\zeta(H)$ be the algebra of all bounded Hilbert operators on H . Let E_n be the elementary operator on $\zeta(H)$. If for an operator $X \in \zeta(H)$ with $\|X\| = 1$, we have $X(f) = f$ for all unit vectors $f \in H$ then;

$$\|E_2\| = \sum_{i=1}^2 \|A_i\| \|B_i\|, X \in H$$

Muiruri et al (2018) determined bounds of the norm of a basic elementary operator in a tensor product to obtain theorem 21 which also resulted to corollary 2.9

Theorem 2.9: (Muiruri et al., 2018)

Let H and K be complex Hilbert spaces and $B(H \otimes K)$ be the set of bounded linear operators on $H \otimes K$. Then for all $X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$, we have

$$\|M_{A \otimes B, C \otimes D}\| = \|A\| \|B\| \|C\| \|D\|$$

Muiruri (2024) determined bounds of the norm of finite length elementary operator in a tensor of C^* -algebras to arrive theorem 2.10.

Theorem 2.10: (Muiruri et al., 2024)

If H and K are complex Hilbert spaces and $B(H \otimes K)$, the set of all bounded linear operators on $H \otimes K$. if $X \otimes Y \in B(H \otimes K)$ and $\|X \otimes Y\| = 1$ then;

$$\|T_n\| = \sum_{i=1}^n \|A_i\| \|B_i\| \|C_i\| \|D_i\|$$

where T_n is the Elementary operator of finite length as defined earlier and A_i

$$, C_i \in B(H) \text{ and } B_i, D_i \in B(K).$$

Now, as our main result, we determine the norm of the multiplication elementary operator $M_{A,B}$ on a C^* -algebra.

III. Main Results

3.1 Norm of Multiplication of Right and Left Multiplication Operators Induced By A and B

3.1.1 Proposition:

Let H be a complex Hilbert space and $B(H)$ be set of bounded linear operators on H .

Then $\forall X \in B(H)$, with $\|X\| = 1$ then $\|M_{A,B}\| \leq \|A\| \|B\| \forall A, B \in B(H)$

Proof

Since $M_{A,B} = L_A R_B = AXXB$ Thus by definition:

$$\begin{aligned} \|M_{A,B} \setminus B(H)\| &= \sup\{\|M_{A,B}(X)\| : X \in B(H), \|X\| = 1\} \\ &= \sup\{\|AXXB\| : X \in B(H), \|X\| = 1\} \end{aligned}$$

Therefore $\|M_{A,B} \setminus B(H)\| \geq \|M_{A,B}(X)\| \forall X \in B(H)$ and $\|X\| = 1$

Taking an arbitrary $\varepsilon > 0$ we have

$$\|M_{A,B} \setminus B(H)\| - \varepsilon < \|M_{A,B}(X)\| \quad \forall X \in B(H) \text{ and } \|X\| = 1$$

$$\text{so } \|M_{A,B} \setminus B(H)\| - \varepsilon < \|AXXB\|$$

$$\text{since } \|AXXB\| \leq \|AX\| \|XB\| \leq \|A\| \|X\| \|X\| \|B\| \quad \forall X \in B(H) \text{ and } \|X\| = 1$$

$$\|AXXB\| \leq \|A\| \|B\|$$

and letting $\varepsilon \rightarrow 0$ then we have that

$$\|M_{A,B} \setminus B(H)\| \leq \|A\| \|B\|.$$

3.1.2 proposition

Let H be a complex Hilbert space and let $B(H)$ denote the algebra of bounded linear operators on H .

For arbitrary operators $A, B \in B(H)$, the multiplication operator $M_{A,B} = L_A R_B$, satisfies $\|M_{A,B}\| \geq \|A\| \|B\|$. **Proof**

let $s, y, z \in H$ be unit vectors. Let u, v be functionals so that: $u \otimes y : H \rightarrow \mathbb{C}$ and $v \otimes z : H \rightarrow \mathbb{C}$ are finite rank operators defined by $(u \otimes y)s = u(s)y$ and $(v \otimes z)s = v(s)z$ respectively $\forall s \in H$ and $\|s\| = 1$ $\|u \otimes y\| = \sup\{|(u \otimes y)s| : s \in H, \|s\| = 1\}$

$$= \sup\{|u(s)y| : s \in H, \|s\| = 1\}$$

$$= \sup\{|u(s)| \|y\| : s \in H, \|s\| = 1\}$$

$$= |u(s)| = \|u\| \quad \text{Similarly } \|v \otimes z\| = \|v\|$$

so if we let $A = u \otimes y$

and $B = v \otimes z$, then

$$\|A\| = |u(s)| = \|u\| \text{ and } \|B\| = |v(s)| = \|v\| \quad \text{If}$$

$$\|M_{A,B} \setminus B(H)\| \geq \|M_{A,B}(X)\| \geq \|M_{A,B}(X)s\|$$

where $X \in B(H)$, with $\|X\| = 1$

$$M_{A,B}(X)s = (u \otimes y)XX(v \otimes z)s$$

$$= (u \otimes y)XXv(s)z =$$

$$(u \otimes y)v(s)z$$

$$= u(v(s)z)$$

$$\|M_{A,B} \setminus B(H)\| \geq \|u(v(s)z)\|$$

$$= \|u\| \|v(s)z\| = \|u\| \|v(s)\| \|z\|$$

$$= \|u\| \|v(s)\|$$

But $|v(s)| = \|B\|$ and $\|u\| = \|A\|$ then

$$\|M_{A,B} \setminus B(H)\| \geq \|A\| \|B\|.$$

By proposition 3.1.1 and 3.1.2

$$\|A\| \|B\| \leq \|M_{A,B} \setminus B(H)\| \leq \|A\| \|B\| \quad \|M_{A,B}$$

$$\setminus B(H)\| = \|A\| \|B\|$$

3.3 Corollary:

Let $A, B \in B(H)$, with H a Hilbert space. For each operator, consider its closest approximation by a scalar multiple of the identity. The corresponding distances are given by

$$d(A) = \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|, \quad d(B) = \inf_{\mu \in \mathbb{C}} \|B - \mu I\|$$

Hence, $d(A)$ and $d(B)$ quantify how far A and B, respectively, are from the collection

$\lambda I : \lambda \in \mathbb{C}$ of scalar multiples of the identity operator.

Proof

For arbitrary $\lambda, \mu \in B(H)$, we have

$$M_{A,B}(X) = L_A R_B = AXXB = (AX - \lambda)X X(B - \mu) + (\mu + \lambda)X$$

$$\|M_{A,B}(X)\| = \|(A - \lambda)X X(B - \mu) + (\mu + \lambda)X\| \leq (\|A - \lambda\| \|B - \mu\| + \|\mu + \lambda\|) \|X\|$$

For $\|X\| = 1$, Letting $\mu = -\lambda$

$$\|M_{A,B}(X)\| \leq \|A - \lambda\| \|B + \lambda\| = \|A\| \|B\|$$

Previously we know that,

$$\|M_{A,B} \setminus B(H)\| \geq \|M_{A,B}(X)\| = \|AXXB\| = \|A\| \|B\| \quad \forall \|X\| = 1$$

$$\text{Therefore } \|A\| \|B\| \leq \|M_{A,B} \setminus B(H)\| \leq \|A\| \|B\|$$

Hence,

$$\|M_{A,B} \setminus B(H)\| = \|A\| \|B\|$$

IV. Conclusion

The study determined the norm of the multiplication elementary operator $M_{A,B} = L_A R_B$ acting on a C^* -algebra. The result provides a clear description of the norm of $M_{A,B}$ and adds to the existing results on elementary operators. The findings further support the study of norm properties of multiplication elementary operators in C^* -algebras.

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