

The Singleton-Triple Swap Graph: Symmetry and Spectrum

Luis Alfredo Dupont García^{1,*}; Evodio Muñoz Aguirre¹

¹Facultad de Matemáticas, Universidad Veracruzana, Xalapa, Veracruz, Mexico

Luis Alfredo Dupont García: ldupont@uv.mx | ORCID: 0000-0002-9044-3547 • Evodio Muñoz Aguirre: evmunoz@uv.mx | ORCID: 0000-0002-7887-5734

* Corresponding author: Luis Alfredo Dupont García (ldupont@uv.mx)

Abstract: We study the singleton-triple swap graph ST_{10} , whose vertices are the partitions of a ten-element set of type $(3,3,3,1)$, with two vertices adjacent when the singleton is exchanged with one element of a triple block. The graph has 2800 vertices, is 9-regular, and is a 280-fold covering of K_{10} . Its natural transitive S_{10} -action has rank 16. We determine its clique and metric structure: $\omega(ST_{10})=4$, $\text{diam}(ST_{10})=7$, and the distance distribution is $(1,9,54,297,1107,972,324,36)$. The maximal-distance relation has a natural connection with a known graph on uniform set partitions: $D_7(ST_{10}) \cong 10X_{3,3}$. This decomposition intrinsically recovers the ten singleton fibers and, together with the geometry of maximal 4-cliques, yields $\text{Aut}(ST_{10}) \cong S_{10}$. We also determine the complete adjacency spectrum $\{9^{(1)}, 7^{(35)}, 6^{(75)}, 5^{(225)}, 4^{(288)}, 2^{(90)}, 1^{(525)}, (-1)^{(261)}, (-2)^{(475)}, (-3)^{(825)}\}$, so ST_{10} is integral. The spectral calculation is obtained from the S_{10} -permutation module and spherical character sums for the adjacency orbital. Finally, we introduce the general singleton-block swap family $ST(m,r)$ and record its basic covering structure.

Keywords: set partitions; orbital graph; graph covering; integral graph; distance graph; automorphism group; uniform partitions; symmetric group.

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I. INTRODUCTION

Graphs defined on set partitions arise naturally in algebraic combinatorics, permutation group theory, extremal set theory, and association-scheme methods; algebraic and spectral viewpoints on such graphs are standard tools in the area [2,3]. Uniform set-partition systems have also been studied extensively from Erdos-Ko-Rado and related extremal viewpoints [1,5]. Their adjacency relations are often prescribed by intersection patterns or refinement conditions. Here we consider a complementary construction generated by a local exchange between a distinguished singleton and a point lying in one of the remaining blocks.

Let $X=\{0,1,\dots,9\}$. A vertex of ST_{10} is a partition of X into one singleton and three unordered triples. We write it as $v=(i;P)$, $P=\{A_1,A_2,A_3\}$, where $\{i\}$ is the singleton and P partitions $X\setminus\{i\}$ into three triples. Two vertices are adjacent when the singleton is exchanged with one element of one triple. The resulting graph is called the singleton-triple swap graph.

The construction belongs to the standard setting of symmetric-group actions on set partitions and orbital graphs. Our focus is the specific orbital selected by the singleton-triple exchange and the structural consequences of this move. To the best of our knowledge, this particular swap graph and the structural results developed below have not previously been studied.

The singleton gives ten canonical fibers $V_0,\dots,V_9$, each of size 280. Between every two distinct fibers the swap operation induces a perfect matching, so $ST_{10}\rightarrow K_{10}$ is a 280-fold covering. The natural S_{10} -action is transitive, with point stabilizer isomorphic to $S_3\wr S_3$, and has rank 16.

The local clique geometry is controlled by the three triple blocks. Each block determines a maximal K_4 , and these are exactly the maximal cliques through a vertex. The metric structure is richer: the graph has diameter 7 and distance distribution $(1,9,54,297,1107,972,324,36)$.

At maximal distance, two vertices necessarily lie in the same singleton fiber. Within a fiber, distance seven is equivalent to the two underlying $3+3+3$ partitions being skew, meaning that every block of one meets every block of the other in exactly one point.

This identifies the distance-seven graph with ten copies of the uniform-partition graph $X_{3,3}$ studied by Meagher, Shirazi and Stevens [1]. Thus $X_{3,3}$ itself is not new; our contribution is its intrinsic occurrence as the maximal-distance structure of ST_{10} . The connected components of $D_7(ST_{10})$ recover the canonical fibers from

the abstract graph. The fiber supports of the three maximal K4's through a vertex then recover its three blocks, yielding $\text{Aut}(\text{ST10}) \cong S_{10}$.

A second algebraic feature is integrality. An exact orbital/character calculation for the permutation module on $S_{10}/(S_3 \wr S_3)$ yields the complete spectrum displayed in the abstract. The graph is highly symmetric without being distance-regular: the stabilizer suborbit counts by distance are (1,1,1,3,5,3,1,1). Appendix A now supplies the complete 16×16 orbital quotient matrix and a reproducible construction certificate.

II. DEFINITION AND BASIC PARAMETERS

Definition 2.1 (Singleton-triple swap). For $i \in X$, let V_i be the set of all vertices $(i;P)$ for which P is a partition of $X \setminus \{i\}$ into three unordered triples. If $j \neq i$ lies in the block $A \in P$, define $\Phi_{ij}(i;P)$ by replacing the singleton i with j and replacing A by $(A \setminus \{j\}) \cup \{i\}$. The vertices $(i;P)$ and $\Phi_{ij}(i;P)$ are adjacent.

The number of partitions of a nine-element set into three unordered triples is $|V_i| = 9!/((3!)^3 3!) = 280$. Hence $|V(\text{ST10})| = 10 \cdot 280 = 2800$. For each $j \neq i$ there is exactly one swap taking $(i;P)$ into V_j , so every vertex has degree 9. Consequently $|E(\text{ST10})| = 2800 \cdot 9/2 = 12600$.

Proposition 2.2 (Covering structure). For every $i \neq j$, $\Phi_{ij}: V_i \rightarrow V_j$ is a bijection and $\Phi_{ji}\Phi_{ij} = \text{id}_{V_i}$. Thus the edges between V_i and V_j form a perfect matching. The projection $\pi: V(\text{ST10}) \rightarrow V(K_{10})$, $\pi(i;P) = i$, is a 280-fold graph covering.

Proof. The swap $i \leftrightarrow j$ is reversed by the swap $j \leftrightarrow i$. Hence Φ_{ij} and Φ_{ji} are mutual inverses. Each vertex in V_i has exactly one neighbor in each V_j , which is precisely the local bijection required for a graph covering.

III. SYMMETRY AND ORBITAL DESCRIPTION

The symmetric group S_{10} acts on vertices by relabelling the ground set. This action preserves adjacency and is transitive. For $v_0 = (0; \{\{1,2,3\}, \{4,5,6\}, \{7,8,9\}\})$, the stabilizer fixes 0 and permutes the three triples and the three points inside each triple. Therefore $H = \text{Stab}_{S_{10}}(v_0) \cong S_3 \wr S_3$, $|H| = (3!)^3 3! = 1296$, and $|S_{10}:H| = 10!/1296 = 2800$. Thus $V(\text{ST10}) \cong S_{10}/H$.

The stabilizer H has sixteen suborbits. Grouped by graph distance from v_0 , their sizes are 1; 9; 54; 27,108,162; 54,81,324,324,324; 162,162,648; 324; 36. Their sum is 2800, and the number of suborbits in the eight distance layers is (1,1,1,3,5,3,1,1). In particular, the natural transitive action has rank 16.

IV. CLIQUE GEOMETRY

Fix $v = (i; \{A_1, A_2, A_3\})$. For $j \neq i$, denote by v_j the unique neighbor of v in V_j .

Lemma 4.1 (Local adjacency among neighbors). For distinct $j, k \neq i$, $v_j \sim v_k$ if and only if j, k lie in the same block of P .

Proof. If j, k lie in the same triple A , swapping i with j replaces A by a triple containing i and k ; the subsequent swap of the new singleton j with k gives exactly the vertex obtained from v by swapping i with k . Conversely, if j and k lie in distinct blocks, the two-step move changes two original blocks, whereas the direct swap $i \leftrightarrow k$ changes only the block containing k . Hence the resulting vertices differ.

Theorem 4.2 (Maximal cliques). The three maximal cliques through v are $K(v, A_t) = \{v\} \cup \{v_j; j \in A_t\}$, $t=1,2,3$. Consequently $\omega(\text{ST10})=4$, $N(K_4)=2100$, $N(K_3)=8400$. Every triangle lies in a unique maximal K4.

Proof. The lemma shows that the neighborhood of v is the disjoint union of three triangles, one for each triple block. Thus the displayed K4's are exactly the maximal cliques through v . Counting vertex-K4 incidences gives $2800 \cdot 3/4 = 2100$. Each K4 has four triangles, and the local description gives unique completion, so there are $4 \cdot 2100 = 8400$ triangles.

V. METRIC STRUCTURE

Let $L_0, \dots, L_{15}$ denote the sixteen H -suborbits, ordered as in Appendix A, and let $qrs = |N(x) \cap L_s|$ for $x \in L_r$. This number is well defined by transitivity of H on each suborbit. Let $Q = (qrs)$ be the orbital quotient matrix and let Q_{supp} be its support graph, with $L_r L_s$ an edge whenever $qrs > 0$.

Lemma 5.1 (Quotient-support distance certificate). For every $x \in L_r$, $d_{\text{ST10}}(v_0, x) = d_{Q_{\text{supp}}}(L_0, L_r)$.

Proof. Every path in ST10 projects to a walk in Q_{supp} , giving one inequality. Conversely, a path in Q_{supp} can be lifted step by step because a positive quotient entry means every vertex in the source suborbit has a neighbor in the target suborbit. The lifted path may end at some $y \in L_r$; an element of H fixes v_0 and maps y to any prescribed $x \in L_r$, yielding a path of the same length.

Theorem 5.2 (Distance distribution). The graph ST10 has diameter 7 and, about every vertex, $(|\Gamma_0|, \dots, |\Gamma_7|) = (1, 9, 54, 297, 1107, 972, 324, 36)$.

Proof. The support graph of the exact sixteen-state quotient matrix Q in Appendix A has distance layers containing 1,1,1,3,5,3,1,1 suborbits. With the suborbit sizes ordered by distance, these layers are 1; 9; 54; (27,108,162); (54,81,324,324,324); (162,162,648); 324; 36. Summing the sizes within each layer gives (1,9,54,297,1107,972,324,36), and the last nonempty layer is 7. Vertex transitivity makes the distribution independent of the base vertex.

The fiber containing a base vertex has the internal distance distribution 1,27,54,162,36 at distances 0,3,4,5,7, respectively. Each of the other nine fibers contributes 1,6,30,117,90,36 at distances 1,2,3,4,5,6. These numbers sum to 280 in each fiber and recover the global distance distribution.

Corollary 5.3 (Maximal distance lies within a fiber). If $d(u,v)=7$, then u and v belong to the same canonical fiber.

VI. THE DISTANCE-SEVEN GRAPH AND UNIFORM PARTITIONS

Let $P=\{A_1,A_2,A_3\}$ and $Q=\{B_1,B_2,B_3\}$ be partitions of the same nine-element set into triples. Define their meet matrix by $M(P,Q)=(|A_r \cap B_s|)_{r,s=1}^3$. We call P and Q skew if $M(P,Q)=J_3$, equivalently if every block of P meets every block of Q in exactly one point.

Meagher, Shirazi and Stevens [1] define $X_{k,l}$ on uniform (k,l) -partitions by joining two partitions when every pair of blocks, one from each partition, intersects in at most one point. For $k=l=3$, the nine meet-matrix entries sum to nine, so this condition is equivalent to all entries being one. Thus adjacency in $X_{3,3}$ is precisely skewness. Neither $X_{3,3}$ nor its spectrum is new; the new phenomenon established here is its intrinsic occurrence, in ten copies, as the maximal-distance graph of ST10 and the resulting reconstruction of the singleton fibers.

Lemma 6.1 (The distance-seven common-fiber orbital). Among pairs of vertices in a common singleton fiber, the unique H -suborbit at distance seven is the one whose meet matrix is J_3 .

Proof. The common-fiber suborbits are determined by the intersection pattern of the two unordered triples systems. In the orbital certificate of Appendix A, exactly one common-fiber suborbit occurs in the last distance layer. A representative of that suborbit has all nine block intersections equal to one, hence meet matrix J_3 . Conversely, the H -action permutes rows, columns, and labels within the fixed base partition, so all pairs with meet matrix J_3 lie in this same suborbit.

Theorem 6.1 (Maximal-distance structure). For vertices in the same fiber, $d((i;P),(i;Q))=7$ if and only if P,Q are skew. Consequently $D_7(ST10) \cong 10X_{3,3}$.

Proof. By Lemma 6.1, a common-fiber pair is at distance seven exactly when its meet matrix is J_3 , equivalently when the two partitions are skew. Corollary 5.3 shows that no distance-seven edge joins distinct fibers. Hence $D_7(ST10)$ is the disjoint union, over the ten fibers, of the skewness graph on the 280 uniform $3+3+3$ partitions, namely $X_{3,3}$.

Lemma 6.2 (Connectivity of $X_{3,3}$). The graph $X_{3,3}$ is connected; in fact, it has diameter 2.

Proof. Given two uniform partitions P,Q , form the bipartite multigraph whose left and right vertices are their three blocks and whose edge multiplicities are the intersection sizes. This is a 3-regular bipartite multigraph. A regular bipartite multigraph decomposes into three perfect matchings. The matchings define a third partition R whose blocks each meet every block of both P and Q in one point. Thus R is skew to both, giving distance at most two. The graph is not complete, so its diameter is two.

The eigenvalues of $X_{3,3}$ are 36,8,2,-4,-12, with multiplicities 1,48,120,84,27, respectively. Therefore $\text{Spec}(D_7(ST10)) = \{36^{(10)}, 8^{(480)}, 2^{(1200)}, (-4)^{(840)}, (-12)^{(270)}\}$.

VII. INTRINSIC RECONSTRUCTION AND AUTOMORPHISMS

Because $X_{3,3}$ is connected, the ten connected components of $D_7(ST10)$ are exactly the ten canonical fibers. Hence the fiber decomposition is intrinsic to the abstract graph ST10. For a clique K , define its fiber support by $\text{fsupp}(K) = \{j: K \cap V_j \neq \emptyset\}$.

Lemma 7.1 (Reconstruction of a vertex). If $v=(i; \{A_1,A_2,A_3\})$, the three maximal K_4 's through v have fiber supports $\{i\} \cup A_1$, $\{i\} \cup A_2$, $\{i\} \cup A_3$. Thus the fiber containing v and these three supports determine v uniquely.

Proof. The maximal clique associated with A_t consists of v and the three neighbors obtained by moving the singleton to the three points of A_t . Removing i from the three supports recovers the three blocks.

Theorem 7.2 (Automorphism group). $\text{Aut}(ST10) \cong S_{10}$.

Proof. Relabelling gives $S_{10} \leq \text{Aut}(ST10)$. Conversely, every automorphism preserves distances and therefore permutes the connected components of $D_7(ST10)$, inducing a homomorphism $\rho: \text{Aut}(ST10) \rightarrow S_{10}$ on the ten fibers. If $f \in \ker \rho$, every fiber is fixed setwise. Automorphisms preserve maximal cliques and their fiber supports.

The reconstruction lemma therefore forces $f(v)=v$ for every vertex v . Thus $\ker p=1$, so $\text{Aut}(\text{ST}10)$ embeds in S_{10} . Together with the natural S_{10} subgroup this proves equality.

VIII. ADJACENCY SPECTRUM

Let $G=S_{10}$ and $H=\text{Stab}_G(v_0)\cong S_3\wr S_3$. Then $C[V(\text{ST}10)]\cong \text{Ind}_H^G 1$. We first isolate the permutation module W on the 280 partitions of a nine-element set into three unordered triples.

Proposition 8.1 (The 3+3+3 Foulkes module).

$$W=\text{Ind}_{\{S_3\wr S_3\}^{\{S_9\}}} 1\cong S^\lambda(9)\oplus S^\lambda(7,2)\oplus S^\lambda(6,3)\oplus S^\lambda(5,2,2)\oplus S^\lambda(4,4,1).$$

Proof. Under the Frobenius characteristic map, $\text{ch}(W)=h_3[h_3]$. Writing $h_3=(p_1^3+3p_1p_2+2p_3)/6$ and expanding the plethysm, the Hall inner products $\langle h_3[h_3], s_\lambda \rangle$ are 1 for $\lambda=(9),(7,2),(6,3),(5,2,2),(4,4,1)$, and 0 for all other $\lambda\vdash 9$. The dimensions 1,27,48,120,84 sum to 280. This is the standard Foulkes-module setting [6,7].

As a finite certificate, the nonzero Hall multiplicities are $\langle h_3[h_3], s_\lambda(9) \rangle = \langle h_3[h_3], s_\lambda(7,2) \rangle = \langle h_3[h_3], s_\lambda(6,3) \rangle = \langle h_3[h_3], s_\lambda(5,2,2) \rangle = \langle h_3[h_3], s_\lambda(4,4,1) \rangle = 1$; all other partitions of 9 have multiplicity zero. Appendix D reproduces these multiplicities independently from fixed-point characters of the 280 uniform 3+3+3 partitions.

By transitivity of induction and Young branching,

$$C[V]\cong S^\lambda(10)\oplus S^\lambda(9,1)\oplus S^\lambda(8,2)\oplus 2S^\lambda(7,3)\oplus S^\lambda(7,2,1)\oplus S^\lambda(6,4)\oplus S^\lambda(6,3,1)\oplus S^\lambda(6,2,2)\oplus S^\lambda(5,4,1)\oplus S^\lambda(5,3,2)\oplus S^\lambda(5,2,2,1)\oplus S^\lambda(4,4,2)\oplus S^\lambda(4,4,1,1).$$

Hence twelve irreducible types occur with multiplicity one and $S^\lambda(7,3)$ occurs with multiplicity two. Thus $\dim \text{End}_G(C[V])=12+2^2=16$, independently recovering the permutation rank.

Lemma 8.2 (Orbital adjacency and spherical normalization). Let $t=(0\ 1)$, where 1 lies in the first triple of v_0 . The H -orbit of tv_0 is the neighborhood of v_0 and has size $k=9$. Put $e_H=|H|^{-1}\sum_{h\in H}h$. For an irreducible constituent S^λ with $\dim(S^\lambda)^H=1$, the adjacency eigenvalue is

$$\theta_\lambda = k \omega_\lambda(t) = (9/1296) \sum_{h\in H} \chi^\lambda(ht).$$

Proof. The orbital of (v_0, tv_0) has valency $[H:H\cap \text{Stab}_H(t)]=9$. On the one-dimensional H -fixed line in S^λ , the matrix coefficient of t is the zonal spherical value $\omega_\lambda(t)$. Since e_H is the orthogonal projection onto the H -fixed space, $\omega_\lambda(t)=\text{tr}(\rho_\lambda(e_H t))=(1/|H|)\sum_{h\in H} \chi^\lambda(ht)$. Multiplication by the orbital valency gives the adjacency eigenvalue.

Exact enumeration of H ($|H|=1296$), followed by exact character evaluation on the cycle types of ht , gives the following spherical sums and eigenvalues; the character computations use the Murnaghan-Nakayama rule in the standard form described in [4]. Appendix D gives a self-contained standard-library Python certificate for both this computation and Proposition 8.1.

S^λ	$\dim S^\lambda$	$\sum_h \chi^\lambda(ht)$	θ_λ
(10)	1	1296	9
(9,1)	9	-144	-1
(8,2)	35	1008	7
(7,2,1)	160	-288	-2
(6,4)	90	288	2
(6,3,1)	315	-288	-2
(6,2,2)	225	720	5
(5,4,1)	288	576	4
(5,3,2)	450	144	1
(5,2,2,1)	525	-432	-3
(4,4,2)	252	-144	-1
(4,4,1,1)	300	-432	-3

Lemma 8.3 (The multiplicity-two block). Let B be the 2×2 matrix induced by adjacency on the multiplicity space of $2S^\lambda(7,3)$. Then $\text{tr}(B)=7$ and $\text{tr}(B^2)=37$. Hence $\text{Spec}(B)=\{1,6\}$.

Proof. The same projected character calculation gives $\sum_{h\in H} \chi^\lambda(7,3)(ht)=1008$, hence $\text{tr}(B)=7$. Because $\text{ST}10$ is 9-regular on 2800 vertices, $\text{tr}(A^2)=2800\cdot 9=25200$. The twelve multiplicity-one constituents in the preceding table contribute 22425. Since $\dim S^\lambda(7,3)=75$, $75 \text{tr}(B^2)=25200-22425=2775$, so $\text{tr}(B^2)=37$. If x,y are the eigenvalues of B , then $x+y=7$ and $x^2+y^2=37$, hence $xy=6$ and $\{x,y\}=\{1,6\}$.

Theorem 8.4 (Integral spectrum).

$$\text{Spec}(\text{ST}10)=\{9^{\wedge}(1),7^{\wedge}(35),6^{\wedge}(75),5^{\wedge}(225),4^{\wedge}(288),2^{\wedge}(90),1^{\wedge}(525),(-1)^{\wedge}(261),(-2)^{\wedge}(475),(-3)^{\wedge}(825)\}.$$

Proof. Combine Lemma 8.2 and the table with Lemma 8.3. Equal eigenvalues combine according to the dimensions of their irreducible constituents: 1 has multiplicity $450+75=525$; -1 has $9+252=261$; -2 has $160+315=475$; and -3 has $525+300=825$. All eigenvalues are integers.

The spectral moments provide independent checks: the multiplicities sum to 2800, $\sum \text{mult}_A(\lambda)\lambda=0$, $\sum \text{mult}_A(\lambda)\lambda^2=25200=2|E|$, and $\sum \text{mult}_A(\lambda)\lambda^3=50400=6\cdot 8400$.

IX. METRIC REGULARITY

Since ST10 is vertex-transitive, the cardinality of every distance sphere $\Gamma_i(v)$ is independent of the choice of v . Nevertheless, ST10 is not distance-regular. At distance three, the three stabilizer suborbits have aggregated intersection data $(c_3, a_3, b_3) = (2, 4, 3), (1, 5, 3), (1, 2, 6)$, so the intersection numbers are not determined solely by distance. In particular, c_3 takes both values 1 and 2. The graph is also not distance-transitive, since $\Gamma_3(v)$ already splits into three stabilizer orbits. The numbers of suborbits by distance are $(1, 1, 1, 3, 5, 3, 1, 1)$.

At the opposite end, the last two layers have well-defined aggregated parameters $(c_6, a_6, b_6) = (4, 4, 1)$ and $(c_7, a_7, b_7) = (9, 0, 0)$.

X. THE GENERAL SINGLETON-BLOCK SWAP FAMILY

The construction extends naturally. Let $ST(m, r)$ have as vertices the partitions of an $(mr+1)$ -element set into one singleton and m unordered blocks of size r . Adjacency is again defined by exchanging the singleton with one point in one of the r -blocks.

Proposition 10. 1 (Basic parameters of $ST(m, r)$). The graph $ST(m, r)$ has $(mr+1)!/((r!)^m m!)$ vertices, is mr -regular, and admits a natural covering $ST(m, r) \rightarrow K_{mr+1}$ of degree $(mr)!/((r!)^m m!)$.

Proof. Choose the singleton and partition the remaining mr points into m unordered r -blocks. From each vertex there is one swap for each of the mr nonsingleton points. Fixing the singleton gives the fibers, and between every two singleton fibers the corresponding swap is a perfect matching.

A systematic investigation of the diameter, maximal-distance relation, spectrum and automorphism group of $ST(m, r)$ is left for future work.

XI. INDEPENDENCE BOUNDS AND FURTHER PROBLEMS

A canonical fiber is an independent set, so $\alpha(ST10) \geq 280$. The least adjacency eigenvalue is -3, and Hoffman's ratio bound, in its standard spectral form [3], gives $\alpha(ST10) \leq 2800 \cdot 3/(9+3) = 700$.

The same upper bound follows from the maximal- K_4 incidence structure: an independent set meets each of the 2100 maximal K_4 's in at most one vertex, while every vertex lies in three such cliques, giving $3\alpha \leq 2100$. Thus $280 \leq \alpha(ST10) \leq 700$, and determining the exact value remains open.

Problem 11.1. Determine the exact independence number $\alpha(ST10)$.

Other natural problems are to determine for which pairs (m, r) the graphs $ST(m, r)$ are integral, to determine their full automorphism groups, and to identify the uniform-partition graphs that arise from their maximal-distance relations.

XII. CONCLUSIONS

The singleton-triple swap graph turns a simple local exchange into a rigid global object. Its ten singleton classes form a 280-fold covering of K_{10} ; its local block geometry determines all maximal K_4 's; its rank-16 orbital structure yields diameter seven and the exact distance distribution; and its maximal-distance graph is the disjoint union of ten copies of $X_{3,3}$. This last decomposition recovers the canonical fibers intrinsically and, together with the clique geometry, determines the full automorphism group S_{10} . The graph is also integral, with a spectrum compatible with the independent combinatorial counts of edges and triangles.

The example illustrates how covering theory, permutation-group orbitals, set-partition geometry, metric structure and spectral methods can reinforce one another in the analysis of a graph generated by a local move.

APPENDIX A. COMPLETE ORBITAL CERTIFICATE

The sixteen H-suborbits are ordered first by distance from v_0 and, within a distance layer, by size (ties fixed by the canonical enumeration used in the reproducible script below). Their distance/size data are:

L0:(0,1); L1:(1,9); L2:(2,54); L3:(3,27); L4:(3,108); L5:(3,162); L6:(4,54); L7:(4,81); L8:(4,324); L9:(4,324); L10:(4,324); L11:(5,162); L12:(5,162); L13:(5,648); L14:(6,324); L15:(7,36).

For $qrs = |N(x) \cap Ls|$ ($x \in Lr$), the complete quotient matrix $Q = (qrs)$ is:

	L0	L1	L2	L3	L4	L5	L6	L7	L8	L9	L10	L11	L12	L13	L14	L15
L0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L1	1	2	6	0	0	0	0	0	0	0	0	0	0	0	0	0
L2	0	1	2	1	2	3	0	0	0	0	0	0	0	0	0	0
L3	0	0	2	0	4	0	0	3	0	0	0	0	0	0	0	0
L4	0	0	1	1	4	0	0	0	0	0	3	0	0	0	0	0
L5	0	0	1	0	0	2	1	1	2	2	0	0	0	0	0	0
L6	0	0	0	0	0	3	0	0	6	0	0	0	0	0	0	0
L7	0	0	0	0	1	0	2	0	2	0	4	0	0	0	0	0
L8	0	0	0	0	0	1	1	0	2	0	1	0	2	2	0	0
L9	0	0	0	0	0	1	0	1	0	4	0	1	0	2	0	0
L10	0	0	0	0	1	0	0	0	1	0	3	1	1	2	0	0
L11	0	0	0	0	0	0	0	0	0	2	2	0	1	4	0	0
L12	0	0	0	0	0	0	0	0	4	0	2	1	2	0	0	0
L13	0	0	0	0	0	0	0	0	1	1	1	1	0	3	2	0
L14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	4	1
L15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	9	0

Every row sums to 9, and the balance identities $|Lr|qrs = |Ls|qsr$ hold. The support graph of Q has 1,1,1,3,5,3,1,1 suborbits in distance layers 0,...,7. In particular, the three distance-three rows aggregate to $(c,a,b) = (2,4,3), (1,5,3), (1,2,6)$, while L14 and L15 give $(4,4,1)$ and $(9,0,0)$, respectively.

APPENDIX B. REPRODUCIBLE CONSTRUCTION OF Q

The following self-contained Python code enumerates the 2800 vertices, constructs adjacency directly from Definition 2.1, constructs $H \cong S_3 \wr S_3$, computes its 16 suborbits, orders them as in Appendix A, and prints Q . It uses only the Python standard library.

```
import itertools
from collections import deque

X = tuple(range(10))
B = ((1,2,3),(4,5,6),(7,8,9))

def parts(items):
    items = tuple(sorted(items))
    if not items:
        yield (); return
    a, rest = items[0], items[1:]
    for pair in itertools.combinations(rest,2):
        block = tuple(sorted((a,)+pair))
        rem = tuple(x for x in rest if x not in pair)
        for tail in parts(rem):
            yield tuple(sorted((block,)+tail))

V = [(i,P) for i in X
      for P in parts([x for x in X if x != i])]
pos = {v:k for k,v in enumerate(V)}

A = [[] for _ in V]
for u,(i,P) in enumerate(V):
    for j in X:
        if j == i: continue
        for t,C in enumerate(P):
            if j in C:
                D = tuple(sorted((set(C)-{j})){i}))
                R = list(P); R[t] = D
                A[u].append(pos[(j,tuple(sorted(R)))])
```

```

        break

v0 = (0,((1,2,3),(4,5,6),(7,8,9)))
root = pos[v0]

H = []
for bp in itertools.permutations(range(3)):
    targets = [B[bp[k]] for k in range(3)]
    for p0 in itertools.permutations(targets[0]):
        for p1 in itertools.permutations(targets[1]):
            for p2 in itertools.permutations(targets[2]):
                f = {0:0}
                for C,T in zip(B,(p0,p1,p2)):
                    for x,y in zip(C,T): f[x]=y
                H.append(tuple(f[x] for x in X))

def act(v,g):
    i,P=v
    return (g[i],tuple(sorted(
        tuple(sorted(g[x] for x in C)) for C in P)))

unseen=set(range(len(V))); O=[]
while unseen:
    u=min(unseen)
    orb={pos[act(V[u],g)] for g in H}
    O.append(sorted(orb)); unseen-=orb

d=[-1]*len(V); d[root]=0; q=deque([root])
while q:
    u=q.popleft()
    for w in A[u]:
        if d[w] < 0:
            d[w]=d[u]+1; q.append(w)

O.sort(key=lambda C:(d[C[0]],len(C),C[0]))
oid={u:r for r,C in enumerate(O) for u in C}
Q=[[0]*len(O) for _ in O]
for r,C in enumerate(O):
    u=C[0]
    for w in A[u]: Q[r][oid[w]] += 1

print([(d[C[0]],len(C)) for C in O])
for row in Q: print(row)

```

APPENDIX C. SPECTRAL CONSISTENCY CERTIFICATE

For the displayed adjacency spectrum, $\sum \text{mult}_A(\lambda)=2800$, $\sum \text{mult}_A(\lambda)\lambda=0$, $\sum \text{mult}_A(\lambda)\lambda^2=25200$, and $\sum \text{mult}_A(\lambda)\lambda^3=50400$. Thus the spectrum independently recovers the order, regular degree, edge count and triangle count. The representation-theoretic derivation in Section 8 is independent of these moment checks: it obtains the eigenvalues from the spherical character sums for the adjacency orbital.

APPENDIX D. REPRODUCIBLE SPECTRAL CERTIFICATE

The following certificate uses only the Python standard library. It implements the Murnaghan-Nakayama rule [4], checks the Foulkes decomposition from fixed-point characters of the 280 partitions of a nine-set into three triples, constructs $H \cong S_3 \wr S_3$, enumerates the cycle types of ht for $t=(0\ 1)$, and reproduces every spherical sum in Section 8. The identical script is supplied with the source files as ST10_spectral_certificate.py.

```
from itertools import permutations, combinations
```

```
from math import factorial
```

```
from collections import Counter
```

```
# ----- partitions and characters (Murnaghan-Nakayama) -----
```

```
def integer_partitions(n, maxpart=None):
```

```
    if n == 0:
```

```
        yield ()
```

```
        return
```

```
    if maxpart is None or maxpart > n:
```

```
        maxpart = n
```

```
    for a in range(maxpart, 0, -1):
```

```
        for tail in integer_partitions(n-a, a):
```

```
            yield (a,) + tail
```

```
def cells(shape):
```

```
    return {(r,c) for r,row in enumerate(shape) for c in range(row)}
```

```
def subpartitions(shape, target):
```

```
    # all partitions nu contained in shape, of size target
```

```
    def rec(r, prev, rem, cur):
```

```
        if r == len(shape):
```

```
            if rem == 0:
```

```
                yield tuple(x for x in cur if x)
```

```
            return
```

```
            hi=min(shape[r], prev, rem)
```

```
            for x in range(hi,-1,-1):
```

```
                yield from rec(r+1,x,rem-x,cur+[x])
```

```
    yield from rec(0,10**9,target,[])
```

```
def is_border_strip(shape, nu):
```

```
    lam=cells(shape); nc=cells(nu)
```

```
    skew=lam-nc
```

```
    if not skew: return False,0
```

```
    # connected by edge adjacency
```

```
    seen={next(iter(skew))}; stack=list(seen)
```

```
    while stack:
```

```
        r,c=stack.pop()
```

```
        for q in ((r+1,c),(r-1,c),(r,c+1),(r,c-1)):
```

```
            if q in skew and q not in seen:
```

```
                seen.add(q); stack.append(q)
```

```
    if len(seen)!=len(skew): return False,0
```

```
    # no 2x2 square
```

```
    for r,c in skew:
```

```
        if {(r,c),(r+1,c),(r,c+1),(r+1,c+1)} <= skew:
```

```
            return False,0
```

```

rows={r for r,c in skew}
return True, len(rows)-1

_char_cache={}
def chi(shape, cycle_type):
    shape=tuple(shape); cycle_type=tuple(cycle_type)
    key=(shape,cycle_type)
    if key in _char_cache: return _char_cache[key]
    if not cycle_type:
        return 1 if sum(shape)==0 else 0
    r=cycle_type[0]
    total=0
    target=sum(shape)-r
    for nu in subpartitions(shape,target):
        ok,height=is_border_strip(shape,nu)
        if ok:
            total += (-1)**height * chi(nu,cycle_type[1:])
    _char_cache[key]=total
    return total

def dim(shape):
    n=sum(shape)
    hooks=1
    for r,row in enumerate(shape):
        for c in range(row):
            below=sum(1 for rr in range(r+1,len(shape)) if shape[rr]>c)
            hooks *= (row-c)+below
    return factorial(n)//hooks

# ----- permutations -----
def compose(p,q):
    return tuple(p[q[i]] for i in range(len(p)))

def cycle_type(p):
    n=len(p); seen=[False]*n; out=[]
    for i in range(n):
        if not seen[i]:
            j=i; k=0
            while not seen[j]:
                seen[j]=True; k+=1; j=p[j]
            out.append(k)
    return tuple(sorted(out, reverse=True))

def representative(mu):
    n=sum(mu); p=list(range(n)); s=0
    for r in mu:
        cyc=list(range(s,s+r))
        for a,b in zip(cyc,cyc[1:]+cyc[:1]): p[a]=b
        s+=r
    return tuple(p)

```

```

def z(mu):
    C=Counter(mu); ans=1
    for r,m in C.items(): ans*= (r**m)*factorial(m)
    return ans

# ----- partitions into triples -----
def triple_partitions(items):
    items=tuple(sorted(items))
    if not items:
        yield (); return
    a=items[0]
    for pair in combinations(items[1:],2):
        block=tuple(sorted((a,)+pair))
        rem=tuple(x for x in items[1:] if x not in pair)
        for tail in triple_partitions(rem):
            yield tuple(sorted((block,)+tail))

def act_partition(P,g):
    return tuple(sorted(tuple(sorted(g[x] for x in B)) for B in P))

# ----- Foulkes certificate  $H^{(3^3)}$  -----
P9=list(triple_partitions(range(9)))
mult={lam:0 for lam in integer_partitions(9)}
for mu in integer_partitions(9):
    g=representative(mu)
    fixed=sum(act_partition(P,g)==P for P in P9)
    for lam in mult:
        mult[lam] += fixed*chi(lam,mu)/z(mu)
print('Foulkes  $H^{(3^3)}$  nonzero multiplicities:')
for lam,m in mult.items():
    if round(m): print(lam, int(round(m)), 'dim', dim(lam))

# -----  $H=S_3$  wr  $S_3$  in  $S_{10}$  and spherical sums -----
B=((1,2,3),(4,5,6),(7,8,9))
H=[]
for bp in permutations(range(3)):
    targets=[B[bp[k]] for k in range(3)]
    for p0 in permutations(targets[0]):
        for p1 in permutations(targets[1]):
            for p2 in permutations(targets[2]):
                f={0:0}
                for C,T in zip(B,(p0,p1,p2)):
                    for x,y in zip(C,T): f[x]=y
                H.append(tuple(f[x] for x in range(10)))
assert len(H)==1296
# t=(0 1)
t=list(range(10)); t[0],t[1]=1,0; t=tuple(t)
cts=Counter(cycle_type(compose(h,t)) for h in H)
print("\nCycle types of ht (type: count):")
for mu,c in sorted(cts.items(), reverse=True): print(mu,c)

```

```
lams=[(10),(9,1),(8,2),(7,2,1),(6,4),(6,3,1),(6,2,2),
      (5,4,1),(5,3,2),(5,2,2,1),(4,4,2),(4,4,1,1),(7,3)]
```

```
print("\nSpherical sums and adjacency data:")
```

```
for lam in lams:
```

```
    s=sum(c*chi(lam,mu) for mu,c in cts.items())
```

```
    theta=9*s/1296
```

```
    print(lam, 'dim',dim(lam),'sum',s,'9*sum/1296',theta)
```

Expected certificate output: the only nonzero multiplicities in $H^*(3^3)$ are $(9),(7,2),(6,3),(5,2,2),(4,4,1)$, each once; $|H|=1296$; and the spherical sums for $(10),(9,1),(8,2),(7,2,1),(6,4),(6,3,1),(6,2,2),(5,4,1),(5,3,2),(5,2,2,1),(4,4,2),(4,4,1,1),(7,3)$ are respectively 1296, -144, 1008, -288, 288, -288, 720, 576, 144, -432, -144, -432, 1008.

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