

# Weighted SNA Integral Transform and Its Properties

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**Abstract:** In this paper a new Weighted SNA integral transform is introduced which is defined as

$$\mathcal{A}_\sigma[f](s) = \int_0^\infty e^{-st} (1 + \sigma t) f(t) dt; \quad \sigma \geq 0$$

where the transformation depends on the weighted parameter  $\sigma$ . Some theorems and properties of the proposed transformation is proved.

**Key Word:** Weighted Laplace type transform, Integral transform, Laplace transform, Convolution

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## I. Introduction

Integral transform forms the foundation of operational analysis, a branch of mathematics that has powerful applications in applied mathematics as well as in other branch of science like physics, engineering, etc. There are many important integral transformations owing their properties and applications i.e. (Laplace transform, Fourier transform, Mellin transform, Stieltjes transform, Hankel transform Radon transform, Wavelet transform) [2, 4, 5, 7, 10, 14]

In recent years many new integral transforms are established and developed. In 1993 Sumudu transform was introduced by Watugala<sup>[12]</sup>. In 2008 Natural transform was introduced by Khan & Khan<sup>[8]</sup>. In 2013 Elzaki transform was introduced by Atangana & Kilicman<sup>[1]</sup>. In 2019 Shehu transform was introduced by Shehu & Weidon<sup>[11]</sup>.

In this paper we introduced a new integral transform Weighted Laplace type transform. This transform is a powerful generalization that unifies Laplace transform. The aim of this paper is to introduce a simple but effective weighted transform. The kernel consists of the ordinary exponential damping factor  $e^{-st}$  and a linear weight  $1 + \sigma t$ . The weight parameter  $\sigma$  controls the additional influence of the function at positive values of  $t$ .

The proposed transform reduces to the classical Laplace transform when  $\sigma = 0$ . Thus, it is not isolated from the classical theory; rather, it extends it in a natural way.

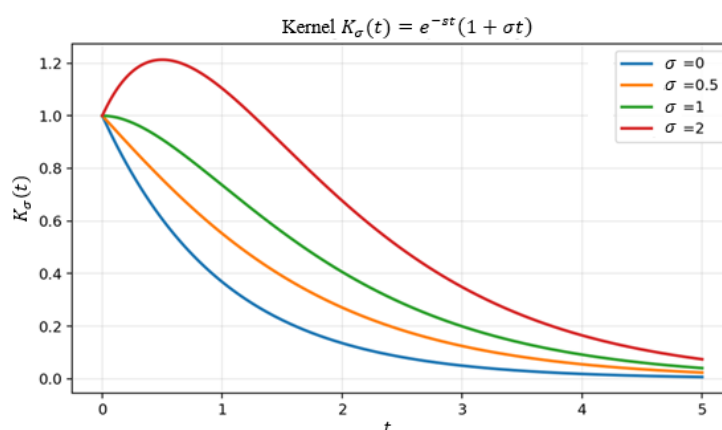


Figure 1: Graph of Kernel function

$$\mathcal{A}_\sigma[f](s) = \int_0^\infty e^{-st} (1 + \sigma t) f(t) dt, \quad \sigma \geq 0 \quad (1.1)$$

Figure 1 show the behavior of the proposed kernel for different values of sigma.

The graph represents the kernel function involved in the weighted integral transform (1.1).

At  $t = 0$ ,  $K_\sigma(s, 0) = e^0(1 + \sigma \cdot 0) = 1$ . Hence, all curves start from the common point  $(0,1)$ . When  $\sigma = 0$ , the kernel becomes  $K_0(s, t) = e^{-st}$ , which is the usual Laplace kernel. Therefore, the curve decreases continuously from 1 to 0.

For  $\sigma > 0$ , the factor  $(1 + \sigma t)$  gives additional weight to the function. This is why the curves for larger  $\sigma$  remain above the ordinary Laplace kernel for a longer interval. In particular, when  $\sigma = 2$ , the graph first rises slightly and then decreases. This happens because the linear factor  $(1 + \sigma t)$  initially dominates, but after some time the exponential factor  $e^{-st}$  becomes stronger.

For large  $t$ ,

$$\lim_{t \rightarrow \infty} e^{-st}(1 + \sigma t) = 0, s > 0.$$

Thus, every curve finally approaches the  $t$ -axis. This shows that the exponential damping controls the convergence of the transform.

## II. Definition of the Transform

The Laplace transform of a function  $f(t)$  defined for all real numbers  $t \geq 0$  is the function  $F(s)$  defined by

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt \tag{2.1}$$

where  $s$  is a complex number.

The linear-weighted Laplace-type transform of  $f(t)$  is denoted by  $\mathcal{A}_\sigma[f](s)$  and defined by (1.1). Also, the kernel of the transform is

$$K_\sigma(s, t) = e^{-st}(1 + \sigma t) \tag{2.2}$$

When  $\sigma = 0$ , the kernel becomes  $e^{-st}$ , and therefore the proposed transform becomes the classical Laplace transform.

$$\mathcal{A}_0[f](s) = L[f](s) \tag{2.3}$$

## III. Relation with the Laplace Transform

Let  $L[f](s)$  denote the classical Laplace transform of  $f(t)$ . From the definition of  $W_\sigma$ , we split the integral into two parts.

$$\mathcal{A}_\sigma[f](s) = \int_0^\infty e^{-st} f(t) dt + \sigma \int_0^\infty t e^{-st} f(t) dt \tag{3.1}$$

$$\mathcal{A}_\sigma[f](s) = L[f](s) + \sigma L[tf(t)](s) \tag{3.2}$$

Since

$$\frac{d}{ds} L[f](s) = -L[tf(t)](s),$$

we also get

$$\mathcal{A}_\sigma[f](s) = L[f](s) - \sigma \frac{d}{ds} L[f](s) \tag{3.3}$$

It shows that the new transform is a first-order differential modification of the Laplace transform.

## IV. Important Theorems

**4.1. Existence Theorem:** We now establish a sufficient condition for the existence of the transform.

**Statement:** Let  $f(t)$  be piecewise continuous on every finite interval of  $[0, \infty)$ . Suppose that  $f(t)$  is of exponential order  $\rho$ ; that is, there exist constants  $M > 0$  and  $T > 0$  such that

$$|f(t)| \leq M e^{\rho t}, \quad t \geq T \tag{4.1}$$

Then  $\mathcal{A}_\sigma[f](s)$  exists for  $s > \rho$ .

**Proof.** By definition,

$$\mathcal{A}_\sigma[f](s) = \int_0^\infty e^{-st}(1 + \sigma t)f(t) dt \tag{4.2}$$

Taking absolute values, we get

$$\begin{aligned} |\mathcal{A}_\sigma[f](s)| &\leq \int_0^\infty e^{-st}(1 + \sigma t)|f(t)| dt \\ \Rightarrow |\mathcal{A}_\sigma[f](s)| &\leq \int_0^T e^{-st}(1 + \sigma t)|f(t)| dt + \int_T^\infty e^{-st}(1 + \sigma t)|f(t)| dt \end{aligned} \tag{4.3}$$

The first integral exists because  $f$  is piecewise continuous on  $[0, T]$ . For  $t \geq T$ , we have  $|f(t)| \leq M e^{\rho t}$ . Then

$$e^{-st}(1 + \sigma t)|f(t)| \leq M(1 + \sigma t)e^{-(s-\rho)t}$$

Therefore, the second integral of (4.3) converges when  $s - \rho > 0$ . Hence the transform exists for  $s > \rho$ . This proves the theorem.

#### 4.2. Theorem (Region of Convergence)

**Statement:** Let  $f: [0, \infty) \rightarrow \mathbb{C}$  be a function of exponential order  $\rho$  i.e there exists constants  $M > 0$  and  $T \geq 0$  such that

$$|f(t)| \leq M e^{\rho t} : t \geq T \tag{4.4}$$

Then the Weighted Laplace type transforms (1.1) is absolutely convergent for  $s \in \mathbb{C}$  satisfying  $\text{Re}(s) > \rho$ .

**Proof:** Let  $s = a + ib$  where  $a = \text{Re}(s)$ . Therefore,

$$\begin{aligned} |e^{-st}| &= e^{-at} \\ \Rightarrow |e^{-st} (1 + \sigma t)f(t)| &\leq e^{-at}(1 + \sigma t)|f(t)| \\ \Rightarrow |e^{-st} (1 + \sigma t)f(t)| &\leq M(1 + \sigma t)e^{-(a-\rho)t} \end{aligned} \tag{by (4.4)}$$

Now suppose that  $a > \rho$  then  $a - \rho > 0$  and the function  $(1 + \sigma t)e^{-(a-\rho)t}$  is integrable on  $[T, \infty)$ . Since an exponential decay dominates the linear factor  $(1 + \sigma t)$ . Therefore,

$$\int_T^\infty |e^{-st} (1 + \sigma t)f(t)| dt < \infty \tag{4.5}$$

On the finite interval  $[0, T]$ . The integrand is integrable and the factor  $e^{-st} (1 + \sigma t)$  is bounded. Consequently, the whole integral converges absolutely. Thus,  $\mathcal{A}_\sigma[f](s)$  exists whenever  $\text{Re}(s) > \rho$ .

#### 4.3. Theorem (Preservation of Exponential Order)

**Statement:** Let  $f(t)$  is of exponential order  $\rho$  then the weighted function  $(1 + \sigma t)f(t)$  is also of exponential order  $\rho$ .

**Proof:** Suppose for some constant  $M > 0$  and  $T \geq 0$

$$|f(t)| \leq M e^{\rho t} : t \geq T.$$

For  $t \geq T$  we have

$$|(1 + \sigma t)f(t)| \leq M(1 + \sigma t)e^{\rho t}.$$

Since the factor grows only linearly whereas  $e^{\rho t}$  is exponential therefore the product function remains of exponential order  $\rho$ . That is, for any  $\varepsilon > 0$ , there exists a constant  $C > 0$  such that

$$(1 + \sigma t) \leq C e^{\varepsilon t}, \quad t \geq 0$$

Therefore,

$$|(1 + \sigma t)f(t)| \leq M C e^{(\rho + \varepsilon)t} \tag{4.4}$$

This shows that weighted function is of exponential order  $\rho$ .

## V. PROPERTIES

**5.1. Linearity:** Let  $f$  and  $g$  be the function for which the weighted transform (1) exists in a common right half plane then for any scalars  $\lambda, \mu \in \mathbb{C}$

$$\mathcal{A}_\sigma[\lambda f + \mu g](s) = \lambda \mathcal{A}_\sigma[f](s) + \mu \mathcal{A}_\sigma[g](s) \tag{5.1}$$

**Proof:** By the definition of  $W_\sigma$  and linearity of the integral we have by (1.1)

$$\begin{aligned} \mathcal{A}_\sigma[\lambda f + \mu g] &= \int_0^\infty e^{-st} (1 + \sigma t)(\lambda f(t) + \mu g(t)) dt \\ &= \lambda \int_0^\infty e^{-st} (1 + \sigma t)f(t) dt + \mu \int_0^\infty e^{-st} (1 + \sigma t)g(t) dt = \lambda \mathcal{A}_\sigma[f](s) + \mu \mathcal{A}_\sigma[g](s) \end{aligned}$$

### 5.2. Change in Scale Property

**5.2.1. Scaling of the Argument:** Let  $a > 0$ , then

$$\mathcal{A}_\sigma[f(at)](s) = \frac{1}{a} \mathcal{A}_{\frac{\sigma}{a}}[f]\left(\frac{s}{a}\right). \tag{5.2}$$

**Proof:** By definition, we have

$$\mathcal{A}_\sigma[f(at)](s) = \int_0^\infty e^{-st} (1 + \sigma t)f(at) dt.$$

Putting

$$z = at, t = \frac{z}{a}, dt = \frac{dz}{a}.$$

Then

$$\mathcal{A}_\sigma[f(at)](s) = \int_0^\infty e^{-s(z/a)} \left(1 + \sigma \frac{z}{a}\right) f(z) \frac{dz}{a}.$$

Therefore,

$$\mathcal{A}_\sigma[f(at)](s) = \frac{1}{a} \int_0^\infty e^{-(s/a)z} \left(1 + \frac{\sigma}{a} z\right) f(z) dz.$$

Hence,

$$\mathcal{A}_\sigma[f(at)](s) = \frac{1}{a} \mathcal{A}_{\sigma/a}[f]\left(\frac{s}{a}\right)$$

### 5.2.2. Time-Dilation Form

Let  $a > 0$ . Then

$$\mathcal{A}_\sigma \left[ f \left( \frac{t}{a} \right) \right] (s) = a \mathcal{A}_{a\sigma}[f](as) \tag{5.3}$$

**Proof:** By definition,

$$\mathcal{A}_\sigma \left[ f \left( \frac{t}{a} \right) \right] (s) = \int_0^\infty e^{-st} (1 + \sigma t) f \left( \frac{t}{a} \right) dt.$$

Putting

$$z = \frac{t}{a}, t = az, dt = a dz.$$

Then

$$\mathcal{A}_\sigma \left[ f \left( \frac{t}{a} \right) \right] (s) = a \int_0^\infty e^{-asz} (1 + a\sigma z) f(z) dz.$$

Thus,

$$\mathcal{A}_\sigma \left[ f \left( \frac{t}{a} \right) \right] (s) = a \mathcal{A}_{a\sigma} [f](as)$$

### 5.2.3. Combined Constant and Argument Scaling

If  $C$  is a constant and  $a > 0$ , then

$$\mathcal{A}_\sigma [Cf(at)](s) = \frac{C}{a} \mathcal{A}_{\frac{\sigma}{a}} [f] \left( \frac{s}{a} \right). \tag{5.4}$$

This follows directly from linearity and the first scaling property.

### 5.2.4. Scaling with Power Factor

For  $a > 0$  and  $r > -1$ ,

$$\mathcal{A}_\sigma [t^r f(at)](s) = \frac{1}{a^{r+1}} \mathcal{A}_{\frac{\sigma}{a}} [t^r f(t)] \left( \frac{s}{a} \right) \tag{5.5}$$

**Proof:**

$$\mathcal{A}_\sigma [t^r f(at)](s) = \int_0^\infty e^{-st} (1 + \sigma t) t^r f(at) dt.$$

Put  $z = at$ . Then

$$t = \frac{z}{a}, dt = \frac{dz}{a}.$$

Therefore,

$$\mathcal{A}_\sigma [t^r f(at)](s) = \int_0^\infty e^{-(s/a)z} \left( 1 + \frac{\sigma}{a} z \right) \left( \frac{z}{a} \right)^r f(z) \frac{dz}{a}.$$

Hence,

$$\mathcal{A}_\sigma [t^r f(at)](s) = \frac{1}{a^{r+1}} \mathcal{A}_{\sigma/a} [t^r f(t)] \left( \frac{s}{a} \right)$$

### 5.2.5. Exponential Shifting Property

For a constant  $a$

$$\mathcal{A}_\sigma [e^{at} f(t)](s) = \mathcal{A}_\sigma [f](s - a). \tag{5.6}$$

#### Important Special Case

When  $\sigma = 0$ ,

$$\mathcal{A}_0 [f](s) = \int_0^\infty e^{-st} f(t) dt,$$

which is the ordinary Laplace transform. Therefore, the scaling property becomes

$$\mathcal{A}_0[f(at)](s) = \frac{1}{a} \mathcal{A}_0[f]\left(\frac{s}{a}\right), \tag{5.7}$$

which is the classical Laplace scaling rule.

Thus, the transform

$$\mathcal{A}_\sigma[f](s) = \int_0^\infty e^{-st} (1 + \sigma t) f(t) dt$$

has the main scaling formula

$$\mathcal{A}_\sigma[f(at)](s) = \frac{1}{a} \mathcal{A}_{\sigma/a}[f]\left(\frac{s}{a}\right), a > 0.$$

### 5.3. Differentiation Property:

**Statement:** If  $f$  be differentiable and suppose that  $f$  and  $f'$  are of exponential order, then

$$\mathcal{A}_\sigma[f'](s) = s\mathcal{A}_\sigma[f](s) - \sigma L\{f(t)\}(s) - f(0) \tag{5.8}$$

where

$$L\{f(t)\}(s) = \int_0^\infty e^{-st} (1 + \sigma t) f'(t) dt$$

**Proof:** On integrating by parts by taking  $u = e^{-st}(1 + \sigma t)$  and  $v = f(t)$ , we get

$$du = [-se^{-st}(1 + \sigma t) + \sigma e^{-st}]dt \text{ and } dv = f'(t).$$

Thus,

$$\mathcal{A}_\sigma[f'](s) = [e^{-st}(1 + \sigma t)f(t)]_0^\infty - \int_0^\infty (e^{-st}(1 + \sigma t) + \sigma e^{-st})f(t)dt \tag{5.9}$$

Now, if  $f(t)$  is of exponential order  $\rho$ , then for  $\text{Re}(s) > \rho$ ,

$$\lim_{t \rightarrow \infty} [e^{-st}(1 + \sigma t)f(t)] = f(0) \tag{5.10}$$

Also

$$\lim_{t \rightarrow 0} [e^{-st}(1 + \sigma t)f(t)] = f(0) \tag{5.11}$$

Therefore, by (5.9), we get

$$\begin{aligned} \mathcal{A}_\sigma[f'](s) &= -f(0) + s \int_0^\infty (e^{-st}(1 + \sigma t)f(t))dt - \sigma \int_0^\infty e^{-st} f(t)dt \\ &= -f(0) + s\mathcal{A}_\sigma[f](s) - \sigma L\{f(t)\}(s) \\ &= s\mathcal{A}_\sigma[f](s) - \sigma L\{f(t)\}(s) - f(0). \end{aligned}$$

**5.4. Second Derivative Property:** Let  $f(t), f'(t)$  and  $f''(t)$  be piecewise continuous on every finite subinterval of  $[0, \infty)$ , and suppose that they are of exponential order  $\rho$ , and that  $f(0)$  and  $f'(0)$  are finite. Then for  $\text{Re}(s) > \rho$  and  $\sigma \geq 0$ ,

$$\mathcal{A}_\sigma[f''](s) = s^2 \mathcal{A}_\sigma[f](s) - 2\sigma s L[f](s) - s f(0) + \sigma f(0) - f'(0) \tag{5.12}$$

**Proof:** Since  $f, f'$  and  $f''$  are of exponential order  $\rho$ , there exist positive constants  $M$  and  $T$  such that, for sufficiently large  $t$ , the functions  $f(t), f'(t)$  and  $f''(t)$  are bounded by a constant multiple of  $e^{\rho t}$ . Therefore, when  $\text{Re}(s) > \rho$ , the factor  $e^{-st}$  dominates the linear factor  $(1 + \sigma t)$ . Consequently,

$$\lim_{t \rightarrow \infty} e^{-st}(1 + \sigma t)f(t) = 0, \quad \lim_{t \rightarrow \infty} e^{-st}(1 + \sigma t)f'(t) = 0. \tag{5.13}$$

At the lower limit, continuity of  $f$  and  $f'$  at  $t = 0$  gives

$$\lim_{t \rightarrow 0^+} e^{-st}(1 + \sigma t)f(t) = f(0), \quad \lim_{t \rightarrow 0^+} e^{-st}(1 + \sigma t)f'(t) = f'(0). \quad (5.14)$$

Putting  $u(t) = e^{-st}(1 + \sigma t)$ , we get  $u'(t) = e^{-st}\{\sigma - s - \sigma st\}$ . Therefore, by the definition of  $\mathcal{A}_\sigma$ , we get

$$\mathcal{A}_\sigma[f''](s) = \int_0^\infty e^{-st}(1 + \sigma t)f''(t) dt. \quad (5.15)$$

Integrating by parts by taking  $u(t) = e^{-st}(1 + \sigma t)$  and  $v(t) = f'(t)dt$ , we have

$$\mathcal{A}_\sigma[f''](s) = [e^{-st}(1 + \sigma t)f'(t)]_0^\infty - \int_0^\infty u'(t)f'(t)dt. \quad (5.16)$$

Using the boundary values in (5.13) and (5.14),

$$[e^{-st}(1 + \sigma t)f'(t)]_0^\infty = 0 - f'(0) = -f'(0). \quad (5.17)$$

Therefore, from (5.15), (5.16) and (5.17), we get

$$\begin{aligned} \mathcal{A}_\sigma[f''](s) &= -f'(0) - \int_0^\infty e^{-st}\{\sigma - s(1 + \sigma t)\}f'(t) dt. \\ \Rightarrow \mathcal{A}_\sigma[f''](s) &= -f'(0) + s \int_0^\infty e^{-st}(1 + \sigma t)f'(t)dt - \sigma \int_0^\infty e^{-st}f'(t)dt. \\ \Rightarrow \mathcal{A}_\sigma[f''](s) &= -f'(0) + s\mathcal{A}_\sigma[f'](s) - \sigma L[f'](s). \end{aligned} \quad (5.18)$$

By using property (5.3) we get

$$\begin{aligned} \mathcal{A}_\sigma[f''](s) &= -f'(0) + s[s\mathcal{A}_\sigma[f](s) - \sigma L\{f(t)\}(s) - f(0)] - \sigma L[f'](s). \\ \Rightarrow \mathcal{A}_\sigma[f''](s) &= -f'(0) + s\{s\mathcal{A}_\sigma[f](s) - \sigma L[f](s) - f(0)\} - \sigma\{sL[f](s) - f(0)\}. \\ \Rightarrow \mathcal{A}_\sigma[f''](s) &= -f'(0) + s^2\mathcal{A}_\sigma[f](s) - \sigma sL[f](s) - sf(0) - \sigma sL[f](s) + \sigma f(0). \\ \Rightarrow \mathcal{A}_\sigma[f''](s) &= s^2\mathcal{A}_\sigma[f](s) - 2\sigma sL[f](s) - sf(0) + \sigma f(0) - f'(0). \end{aligned}$$

This proves the required second derivative property.

**Remark 5.1**

When  $\sigma = 0$ , the transform  $\mathcal{A}_\sigma$  reduces to the usual Laplace transform. In this case, (5.12) becomes

$$\mathcal{A}_0[f''](s) = s^2\mathcal{A}_0[f](s) - sf(0) - f'(0).$$

This is the classical Laplace-transform formula for the second derivative. Hence, equation (5.12) is a weighted extension of the standard Laplace result.

**5.5. Transform of the  $n$ th Derivative:**

Let  $n \in \mathbb{N}$  and  $\sigma \geq 0$ . Suppose that  $f, f', \dots, f^{(n-1)}$  are continuous on  $[0, \infty)$ ,  $f^{(n)}$  is piecewise continuous on every finite subinterval of  $[0, \infty)$ , and all these functions are of exponential order  $\rho$ . Then, for  $\text{Re}(s) > \rho$ , the following formula holds:

$$\mathcal{A}_\sigma[f^{(n)}](s) = s^n\mathcal{A}_\sigma[f](s) - n\sigma s^{n-1}\mathcal{A}_0[f](s) - f^{(n-1)}(0) - \sum_{k=1}^{n-1} (s^k - k\sigma s^{k-1}) f^{(n-1-k)}(0). \quad (5.19)$$

Equivalently, using the convention that  $k\sigma s^{k-1} = 0$  when  $k = 0$ , (5.19) can be written in compact form as

$$\mathcal{A}_\sigma[f^{(n)}](s) = s^n\mathcal{A}_\sigma[f](s) - n\sigma s^{n-1}\mathcal{A}_0[f](s) - \sum_{k=0}^{n-1} (s^k - k\sigma s^{k-1}) f^{(n-1-k)}(0) \quad (5.20)$$

**Proof:** Putting  $K_\sigma(s, t) = e^{-st}(1 + \sigma t)$ , as in (2.2), we get

$$\mathcal{A}_\sigma[f^{(n)}](s) = \int_0^\infty K_\sigma(s, t) f^{(n)}(t) dt.$$

The repeated integration by parts formula gives

$$\int_0^\infty K_\sigma(s, t) f^{(n)}(t) dt = \sum_{k=0}^{n-1} (-1)^k [K_\sigma^{(k)}(s, t) f^{(n-1-k)}(t)]_0^\infty + (-1)^n \int_0^\infty K_\sigma^{(n)}(s, t) f(t) dt. \quad (5.21)$$

Since the functions are of exponential order and  $\text{Re}(s) > \rho$ , the exponential factor  $e^{-st}$  dominates the polynomial factor  $(1 + \sigma t)$ . Hence

$$\lim_{t \rightarrow \infty} K_\sigma^{(k)}(s, t) f^{(n-1-k)}(t) = 0, \quad k = 0, 1, \dots, n-1. \quad (5.22)$$

Therefore, the boundary contribution at infinity vanishes.

Again  $t = 0$ , we have

$$K_\sigma(s, 0) = 1. \quad (5.23)$$

For  $k \geq 1$ , differentiating (2.2)  $k$  times, we get

$$K_\sigma^{(k)}(s, t) = e^{-st} [(-s)^k (1 + \sigma t) + k\sigma(-s)^{k-1}]. \quad (5.24)$$

Putting  $t = 0$ , we obtain

$$K_\sigma^{(k)}(s, 0) = (-s)^k + k\sigma(-s)^{k-1}, \quad k \geq 1. \quad (5.25)$$

Using (5.22)–(5.25), the boundary part of (5.21) becomes

$$\sum_{k=0}^{n-1} (-1)^k [K_\sigma^{(k)}(s, t) f^{(n-1-k)}(t)]_0^\infty = -f^{(n-1)}(0) - \sum_{k=1}^{n-1} (s^k - k\sigma s^{k-1}) f^{(n-1-k)}(0). \quad (5.26)$$

Now we evaluate the integral part of (5.21). From (5.24), for  $k = n$ ,

$$K_\sigma^{(n)}(s, t) = e^{-st} [(-s)^n (1 + \sigma t) + n\sigma(-s)^{n-1}].$$

Hence

$$(-1)^n \int_0^\infty K_\sigma^{(n)}(s, t) f(t) dt = s^n \int_0^\infty e^{-st} (1 + \sigma t) f(t) dt - n\sigma s^{n-1} \int_0^\infty e^{-st} f(t) dt. \quad (5.27)$$

By (1.1) and (2.3), (5.27) becomes

$$(-1)^n \int_0^\infty K_\sigma^{(n)}(s, t) f(t) dt = s^n \mathcal{A}_\sigma[f](s) - n\sigma s^{n-1} \mathcal{A}_0[f](s). \quad (5.28)$$

Combining (5.21), (5.26), and (5.28), we get

$$\mathcal{A}_\sigma[f^{(n)}](s) = s^n \mathcal{A}_\sigma[f](s) - n\sigma s^{n-1} \mathcal{A}_0[f](s) - f^{(n-1)}(0) - \sum_{k=1}^{n-1} (s^k - k\sigma s^{k-1}) f^{(n-1-k)}(0).$$

This proves the required  $n$ th derivative property.

### 5.3. Particular Cases

For  $n = 1$ , (5.19) gives

$$\mathcal{A}_\sigma[f'](s) = s \mathcal{A}_\sigma[f](s) - \sigma \mathcal{A}_0[f](s) - f(0)$$

For  $n = 2$ , we obtain

$$\mathcal{A}_\sigma[f''](s) = s^2 \mathcal{A}_\sigma[f](s) - 2\sigma s \mathcal{A}_0[f](s) - (s - \sigma)f(0) - f'(0)$$

For  $n = 3$ , we get

$$\mathcal{A}_\sigma[f^{(3)}](s) = s^3 \mathcal{A}_\sigma[f](s) - 3\sigma s^2 \mathcal{A}_0[f](s) - (s^2 - 2\sigma s)f(0) - (s - \sigma)f'(0) - f''(0).$$

### 5.4. Reduction to the Classical Laplace Derivative Formula

When  $\sigma = 0$ , the transform  $\mathcal{A}_\sigma$  becomes the ordinary Laplace transform. Substituting  $\sigma = 0$  in (5.19), we obtain

$$\mathcal{A}_0[f^{(n)}](s) = s^n \mathcal{A}_0[f](s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0),$$

which is the usual derivative formula of the Laplace transform. Therefore, (5.19) is a proper extension of the classical result.

### 5.5. Example: Let

$$f(t) = e^{at}, \quad a \in \mathbb{R}.$$

Then

$$f^{(n)}(t) = a^n e^{at}.$$

Directly from the definition,

$$\mathcal{A}_\sigma[e^{at}](s) = \int_0^\infty e^{-(s-a)t} (1 + \sigma t) dt = \frac{1}{s-a} + \frac{\sigma}{(s-a)^2}, \quad \text{Re}(s) > a.$$

Hence

$$\mathcal{A}_\sigma[f^{(n)}](s) = a^n \left[ \frac{1}{s-a} + \frac{\sigma}{(s-a)^2} \right].$$

This agrees with the result obtained from (5.19), because  $f^{(r)}(0) = a^r$  for  $r = 0, 1, \dots, n-1$ . Thus, the formula is consistent for exponential test functions.

### Remark

The existence of  $\mathcal{A}_\sigma[f](s)$  alone does not imply that  $f$  is differentiable. The derivative property requires extra smoothness assumptions on  $f$ . In particular, to apply the  $n$ th derivative formula, the functions  $f, f', \dots, f^{(n-1)}$  should have finite initial values at  $t = 0$ , and the boundary terms at infinity should vanish under the exponential-order condition.

## VI. Transform of Elementary Functions

### 6.1. Transform of 1

$$\begin{aligned} \mathcal{A}_\sigma[1](s) &= \int_0^\infty e^{-st} (1 + \sigma t) dt \\ &= \int_0^\infty e^{-st} dt + \sigma \int_0^\infty t e^{-st} dt = \frac{1}{s} + \frac{\sigma}{s^2}. \end{aligned}$$

### 6.2. Transform of $t^n$

$$\begin{aligned} \mathcal{A}_\sigma[t^n](s) &= \int_0^\infty e^{-st} (1 + \sigma t) t^n dt \\ &= \int_0^\infty t^n e^{-st} dt + \sigma \int_0^\infty t^{n+1} e^{-st} dt \\ &= \frac{n!}{s^{n+1}} + \frac{(n+1)!}{s^{n+2}}. \end{aligned}$$

**6.3. Transform of  $e^{at}$**

$$\begin{aligned} \mathcal{A}_\sigma[e^{at}](s) &= \int_0^\infty e^{-(s-a)t}(1 + \sigma t) dt \\ &= \int_0^\infty e^{-(s-a)t} dt + \sigma \int_0^\infty t e^{-(s-a)t} dt \\ &= \frac{1}{s-a} + \frac{\sigma}{(s-a)^2}, \quad s > a. \end{aligned}$$

**6.4. Transform of  $\cos at$  and  $\sin at$**

$$\begin{aligned} \mathcal{A}_\sigma[e^{iat}](s) &= \frac{1}{s-ia} + \frac{\sigma}{(s-ia)^2} \tag{by 6.3.} \\ \Rightarrow \mathcal{A}_\sigma[e^{iat}](s) &= \frac{s+ia}{s^2+a^2} + \sigma \frac{(s+ia)^2}{(s^2+a^2)^2} = \left[ \frac{s}{s^2+a^2} + \sigma \frac{s^2-a^2}{(s^2+a^2)^2} \right] + i \left[ \frac{a}{s^2+a^2} + \sigma \frac{2as}{(s^2+a^2)^2} \right]. \\ \Rightarrow \mathcal{A}_\sigma[\cos at + i \sin at](s) &= \left[ \frac{s}{s^2+a^2} + \sigma \frac{s^2-a^2}{(s^2+a^2)^2} \right] + i \left[ \frac{a}{s^2+a^2} + \sigma \frac{2as}{(s^2+a^2)^2} \right]. \end{aligned}$$

Therefore, by linearity property, we get

$$\mathcal{A}_\sigma[\cos at](s) = \left[ \frac{s}{s^2+a^2} + \sigma \frac{s^2-a^2}{(s^2+a^2)^2} \right]. \tag{6.1}$$

and

$$\mathcal{A}_\sigma[\sin at](s) = \left[ \frac{a}{s^2+a^2} + \sigma \frac{2as}{(s^2+a^2)^2} \right]. \tag{6.2}$$

**6.5. Transform of  $\cosh at$  and  $\sinh at$**

Similarly, we can find  $\mathcal{A}_\sigma$ -transformation of  $\sinh at$  and  $\cosh at$  as:

$$\mathcal{A}_\sigma[\sinh at](s) = \frac{a}{s^2-a^2} + \sigma \frac{as}{(s^2-a^2)^2} \tag{6.3}$$

and

$$\mathcal{A}_\sigma[\cosh at](s) = \frac{s}{s^2-a^2} + \sigma \frac{s^2+a^2}{(s^2-a^2)^2} \tag{6.4}$$

**6.6. Basic transform pairs for  $\mathcal{A}_\sigma$ .**

Function $f(t)$	$W_\sigma[f](s)$
1	$\frac{1}{s} + \frac{\sigma}{s^2}$
$t$	$\frac{1}{s^2} + 2\frac{\sigma}{s^3}$
$t^n$	$\frac{n!}{s^{n+1}} + \sigma \frac{(n+1)!}{s^{n+2}}$
$e^{at}$	$\frac{1}{s-a} + \frac{\sigma}{(s-a)^2}$

$\sin at$	$\frac{a}{s^2 + a^2} + 2\sigma \frac{as}{(s^2 + a^2)^2}$
$\cos at$	$\frac{s}{s^2 + a^2} + \sigma \frac{(s^2 - a^2)}{(s^2 + a^2)^2}$
$\sinh at$	$\frac{a}{s^2 - a^2} + 2\sigma \frac{as}{(s^2 - a^2)^2}$
$\cosh at$	$\frac{s}{s^2 - a^2} + \sigma \frac{s^2 + a^2}{(s^2 - a^2)^2}$

### VII. Convolution

If  $F(s) = L\{f(t)\}(s)$  and  $G(s) = L\{g(t)\}(s)$ , then

$$L\{f(t) * g(t)\} = L\{f(t)\}L\{g(t)\} = F(s)G(s)$$

where  $f(t) * g(t)$  is called the convolution of  $f(t)$  and  $g(t)$  and is defined by the integral

$$f(t) * g(t) = \int_0^t f(t - \tau)g(\tau)d\tau \tag{7.1}$$

This integral is often referred as the convolution integral and is denoted by  $(f * g)(t)$ .

#### 7.1. Convolution Property

Let  $f$  and  $g$  be a function for which the weighted transform (1.1) exists in a common right half plane.

Let

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)d\tau \tag{7.2}$$

be the ordinary convolution of  $f$  and  $g$ , Then,

$$\mathcal{A}_\sigma[f * g](s) = \mathcal{A}_\sigma[f](s)L\{g(t)\}(s) + L\{f(t)\}(s)\mathcal{A}_\sigma[g](s) \tag{7.3}$$

**Proof:** Suppose  $F(s) = L\{f(t)\}(s)$  and  $G(s) = L\{g(t)\}(s)$ . Therefore, by the classical convolution theorem for the Laplace transform

$$L\{f * g\}(s) = F(s)G(s) \tag{7.4}$$

Now by definition of  $W_\sigma$

$$\mathcal{A}_\sigma[f * g](s) = L\{(1 + \sigma t)f * g\}(s) \tag{7.5}$$

Using linearity by splitting weight we get

$$\mathcal{A}_\sigma[f * g](s) = L[f * g](s) + \sigma L\{t(f * g)(t)\}(s) \tag{7.6}$$

Using (7.4) in (7.6) and using derivative property of Laplace transform we get

$$L\{t(f * g)(t)\}(s) = -\frac{d(F(s)G(s))}{ds} = -F'(s)G(s) - F(s)G'(s) \tag{7.5}$$

Substituting (7.7) in (7.6) we get

$$\begin{aligned} \mathcal{A}_\sigma[f * g](s) &= F(s)G(s) - \sigma(F'(s)G(s) + F(s)G'(s)) \\ \Rightarrow \mathcal{A}_\sigma[f * g](s) &= [F(s) - \sigma F'(s)]G(s) + F(s)[G(s) - \sigma G'(s)] \end{aligned} \tag{7.6}$$

Since

$$\mathcal{A}_\sigma[f](s) = F(s) - \sigma F'(s), \text{ and } \mathcal{A}_\sigma[g](s) = G(s) - \sigma G'(s)$$

Equation (7.7) becomes

$$\mathcal{A}_\sigma[f * g](s) = \mathcal{A}_\sigma[f](s)L\{g(t)\}(s) + L\{f(t)\}(s)\mathcal{A}_\sigma[g](s)$$

This proves the theorem.

**Example 7.2.** Let  $f(t) = 1, g(t) = 1$ , then,

$$(f * g)(t) = \int_0^\infty 1.1. d\tau = t$$

For the weighted transform  $f(t) = 1$

$$\begin{aligned} \mathcal{A}_\sigma[1](s) &= \int_0^\infty e^{-st} (1 + \sigma t) dt \\ &= \frac{1}{s} + \frac{\sigma}{s^2} \end{aligned}$$

Also,

$$F(s) = G(s) = \frac{1}{s}$$

Therefore,

$$\begin{aligned} \mathcal{A}_\sigma[1 * 1](s) &= \mathcal{A}_\sigma[1](s) \frac{1}{s} + \frac{1}{s} \mathcal{A}_\sigma[1](s) - \frac{1}{s^2} \\ &= \left(\frac{1}{s} + \frac{\sigma}{s^2}\right) \frac{1}{s} + \frac{1}{s} \left(\frac{1}{s} + \frac{\sigma}{s^2}\right) - \frac{1}{s^2} \left(\frac{1}{s} + \frac{\sigma}{s^2}\right) \\ &= 2 \left(\frac{1}{s} + \frac{\sigma}{s^2}\right) \frac{1}{s} - \frac{1}{s^2} \\ \Rightarrow \mathcal{A}_\sigma[1 * 1](s) &= \frac{2}{s^2} + \frac{2\sigma}{s^3} - \frac{1}{s^2} = \frac{1}{s^2} + \frac{2\sigma}{s^3} \end{aligned}$$

**Verification:**

Since,  $(1 * 1)(t) = t$ , therefore,

$$\mathcal{A}_\sigma[t](s) = \int_0^\infty e^{-st} (1 + \sigma t) f(t) dt$$

Now,

$$\int_0^\infty t e^{-st} dt = \frac{1}{s^2}, \quad \int_0^\infty t^2 e^{-st} dt = \frac{2\sigma}{s^3}$$

Therefore,

$$\mathcal{A}_\sigma[t](s) = \frac{1}{s^2} + \frac{2\sigma}{s^3}$$

### VIII. Conclusion

In this paper we introduced a new integral transform  $\mathcal{A}_\sigma$  –Transform i.e Weighted Laplace type transform. Some definitions and theorem are presented in which the existence theorem, region of convergence and preservation of exponential order is shown. By giving some examples some of the properties (linearity, change in scale, derivative, convolution) of the new transform is proved.

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