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A Dual-Regulated Levenberg–Marquardt Trust Region Method for Rosenbrock Function Optimization: A Comparative Study with Classical Levenberg–Marquardt and Standard Trust Region Methods

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Abstract
The Rosenbrock function is one of the most widely used benchmark problems for evaluating the robustness and convergence behavior of nonlinear optimization algorithms due to its narrow curved valley and ill-conditioned landscape. Classical optimization methods often exhibit difficulties in balancing convergence speed, nu-merical stability, and step acceptance when navigating such challenging regions. In this study, a Dual-Regulated Levenberg–Marquardt Trust Region Method (DR-LM-TRM) is proposed for unconstrained nonlinear optimization. The proposed frame-work combines the curvature regularization mechanism of the Levenberg–Marquardt (LM) method with the step-control properties of the Standard Trust Region Method (STRM) through an adaptive dual-regulation strategy.
The method introduces two independent but interacting control mechanisms: a damping parameter that regulates curvature conditioning and a trust-region radius that governs step feasibility. Furthermore, a geometric regulation model based on trust-region utilization and trust-region violation measures is incorporated into the damping evolution process, enabling adaptive stabilization of the optimization trajec-tory. The resulting search direction is computed using a regularized Gauss–Newton model and subsequently projected into the trust region whenever necessary.
To evaluate the effectiveness of the proposed framework, numerical experiments are performed on the Rosenbrock function. The convergence behaviour of the pro-posed algorithm is compared with those of the classical Levenberg–Marquardt method and the Standard Trust Region Method. Performance is assessed using objective function reduction, gradient norm decay, damping evolution, trust-region dynam-ics, and optimization trajectories. Numerical results demonstrate that the proposed DR-LM-TRM achieves faster convergence, improved numerical robustness, better control of ill-conditioning, and more stable progression toward the global minimizer. The findings indicate that the proposed framework provides an effective and reliable second-order optimization strategy for highly nonlinear optimization problems.
Keywords: Rosenbrock Function, Trust Region Method, Levenberg–Marquardt Method, Nonlinear Optimization, Second-Order Methods, Convergence Analysis, Adaptive Damping

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I. Introduction

Nonlinear optimization plays a fundamental role in scientific computing, engineer-ing design, machine learning, parameter estimation, inverse problems, and numerous areas of applied mathematics. The general objective is to determine a point that minimizes or maximizes a nonlinear objective function while maintaining numerical stability and computational efficiency. Because many practical optimization prob-lems are characterized by non-convexity, ill-conditioning, and multiple stationary points, the development of robust optimization algorithms remains an active area of research.

Among the benchmark problems used to evaluate optimization algorithms, the Rosenbrock function occupies a central position. Originally introduced by Rosen-brock (1960), the function possesses a narrow curved valley leading to the global minimizer. Although the minimum is easy to identify analytically, numerical algo-rithms often experience difficulty traversing the valley efficiently because of strong nonlinearity and ill-conditioning. Consequently, the Rosenbrock function has be-come a standard test problem for assessing convergence properties, stability, and robustness of optimization methods.

Second-order optimization methods have been widely investigated as alternatives to first-order techniques because they exploit curvature information of the objec-tive function. One of the most successful approaches is the Levenberg–Marquardt (LM) method, originally developed for nonlinear least-squares problems. The LM algorithm combines the rapid local convergence of the Gauss–Newton method with the stability of gradient descent through the introduction of a damping parame-ter. While LM is effective near a solution, its performance may deteriorate when the damping parameter is poorly adjusted or when large steps are taken in highly nonlinear regions.

Trust Region Methods (TRMs) constitute another important class of second-order optimization techniques. Instead of accepting unrestricted search directions, TRMs restrict each iteration to a neighbourhood where the quadratic model is con-sidered reliable. The resulting framework provides strong global convergence prop-erties and improved numerical stability. However, standard trust-region approaches may exhibit conservative behaviour and slower progress when the trust-region radius becomes excessively restrictive.

Several studies have attempted to combine Levenberg–Marquardt regularization with trust-region concepts in order to exploit the advantages of both approaches. Existing hybrid formulations generally employ damping and trust-region mechanisms simultaneously, but often treat them as implicitly coupled components. As a result, the individual roles of curvature conditioning and step-feasibility control are not explicitly separated, limiting the adaptability of the algorithm in highly nonlinear optimization environments.

Motivated by this limitation, this paper proposes a Dual-Regulated Levenberg–Marquardt Trust Region Method (DR-LM-TRM). The proposed framework intro-duces two coordinated regulatory mechanisms. The first mechanism uses adaptive damping to control curvature conditioning and numerical stability, while the second mechanism employs a trust-region constraint to regulate step feasibility and model reliability. In addition, geometric information obtained from trust-region utilization and trust-region violation measures is incorporated into the damping update process, thereby providing a feedback-driven stabilization mechanism.

The effectiveness of the proposed method is investigated through the minimiza-tion of the Rosenbrock function. Numerical experiments compare the convergence behaviour of the proposed DR-LM-TRM with those of the classical Levenberg–Marquardt method and the Standard Trust Region Method (STRM). The compar-ison focuses on objective function reduction, gradient norm convergence, parameter evolution, and optimization trajectories.

II. Problem Formulation

We consider the unconstrained optimization problem:

$$\min_{x \in \mathbb{R}^n} f(x)$$

where $f(x)$ is twice continuously differentiable.
The Rosenbrock function is used as a benchmark:

$$f(x, y) = (1 - x)^2 + 100(y - x^2)^2.$$

3. Classical Methods

3.1. Levenberg–Marquardt Method

The LM update is given by:

$$(H_k + \lambda_k I)p_k = -g_k, \quad x_{k+1} = x_k + p_k.$$

3.2. Trust Region Method

The TRM subproblem is:

$$\min_p m_k(p) = g_k^T p + \frac{1}{2}p^T H_k p, \quad \|p\| \le \Delta_k.$$

4. Proposed DR-LM-TRF

The proposed framework combines damping and trust-region control.

4.1. Search Direction

$$p_k = -(H_k + \lambda_k I)^{-1} g_k$$

4.2. Acceptance Ratio

$$\rho_k = \frac{f(x_k) - f(x_k + p_k)}{-g_k^T p_k - \frac{1}{2}p_k^T H_k p_k}$$

4.3. Dual-Regulation Updates

The damping parameter is updated using:

$$\lambda_{k+1} = \lambda_k \exp(-\alpha\rho_k + \beta_1\nu_k + \beta_2\eta_k)$$

where

$$\nu_k = \frac{\|g_k\|}{1 + \|g_k\|}, \quad \eta_k = \frac{\lambda_k}{1 + \lambda_k}.$$

The trust-region radius is updated adaptively as:

$$\Delta_{k+1} = \begin{cases} 2\Delta_k, & \rho_k > 0.75 \\ \frac{1}{2}\Delta_k, & \rho_k < 0.25 \\ \Delta_k, & \text{otherwise} \end{cases}$$

5. Algorithm Summary

Algorithm 1: DR-LM-TRF

1. Initialize x_0, λ_0, Δ_0
2. Compute g_k, H_k
3. Compute search direction p_k
4. Compute ratio ρ_k
5. Update x_k
6. Update λ_k and Δ_k
7. Repeat until convergence

6. Theoretical Analysis of the Proposed DR-LM-TRM Framework

This section presents the principal theoretical properties of the proposed Dual-Regulated Levenberg–Marquardt Trust Region Method (DR-LM-TRM).

6.1. Fundamental Properties

The DR-LM-TRM framework possesses the following desirable characteristics:

1. The regularized Hessian approximation
$$B_k = G_k + \lambda_k I$$
remains symmetric positive definite for all $\lambda_k > 0$.
2. The generated step always satisfies the trust-region constraint
$$\|p_k\| \leq \Delta_k.$$
3. The algorithm guarantees sufficient model reduction at each successful iteration.
4. The adaptive damping and trust-region mechanisms provide robustness in highly nonlinear and ill-conditioned optimization landscapes.

6.2. Auxiliary Results

Lemma 1 (Positive Definiteness). *For $\lambda_k > 0$, the matrix*
$$B_k = G_k + \lambda_k I$$
is symmetric positive definite.

Proof. Since
$$G_k = J_k^T J_k$$
is symmetric positive semidefinite and
$$\lambda_k I$$
is strictly positive definite for $\lambda_k > 0$, their sum is symmetric positive definite. □

Lemma 2 (Trust-Region Feasibility). *The DR-LM-TRM step satisfies*
$$\|p_k\| \leq \Delta_k.$$

Proof. If the Levenberg–Marquardt step lies inside the trust region, it is accepted directly. Otherwise, radial projection scales the step to the trust-region boundary, yielding
$$\|p_k\| = \Delta_k.$$
Hence, every step satisfies the trust-region constraint. □

Lemma 3 (Sufficient Descent). *There exists a constant $c > 0$ such that*
$$m_k(0) - m_k(p_k) \geq c \min\left(\|g_k\|^2, \Delta_k \|g_k\|\right).$$

Proof. The result follows from the classical Cauchy decrease estimate for trust-region methods together with the positive definiteness of B_k. □

Lemma 4 (Bounded Iterates). *All iterates generated by DR-LM-TRM remain in the level set*

$$\mathcal{L} = \{\theta : F(\theta) \leq F(\theta_0)\}.$$

Proof. The trust-region acceptance mechanism ensures monotonic decrease of the objective function, implying that every iterate remains inside the initial level set. □

6.3. Global Convergence

Theorem 1 (Global Convergence). *Suppose that the objective function F is continuously differentiable, bounded below, and satisfies the standard Lipschitz continuity assumptions. Then the sequence generated by DR-LM-TRM satisfies*

$$\lim_{k\to\infty} \|\nabla F(\theta_k)\| = 0.$$

Furthermore, every accumulation point is a first-order stationary point.

Proof. The positive definiteness of the regularized Hessian ensures descent directions, while the trust-region mechanism guarantees sufficient reduction. Since the objective function is bounded below, the sequence of reductions converges, implying asymptotic stationarity. □

6.4. Local Convergence Rate

Theorem 2 (Quadratic Convergence). *Let θ^* be a nondegenerate minimizer satisfying*

$$\nabla F(\theta^*) = 0, \qquad G(\theta^*) \succ 0.$$

If

$$\lambda_k \to 0$$

and the trust-region constraint becomes inactive locally, then

$$\|\theta_{k+1} - \theta^*\| = \mathcal{O}\left(\|\theta_k - \theta^*\|^2\right).$$

Proof. Near the solution, the damping parameter vanishes and the method approaches the Gauss–Newton/Newton step. Standard local error analysis therefore yields quadratic convergence. □

Corollary 1 (Superlinear Convergence). *If the Jacobian satisfies a Hölder continuity condition, then*

$$\lim_{k\to\infty} \frac{\|\theta_{k+1} - \theta^*\|}{\|\theta_k - \theta^*\|} = 0.$$

Thus, DR-LM-TRM converges superlinearly.

6.5. Asymptotic Newton Equivalence

Theorem 3 (Newton Equivalence). *Assume that*

$$\lambda_k \to 0, \qquad G_k \to H^*,$$

where

$$H^* = \nabla^2 F(\theta^*).$$

Then

$$\theta_{k+1} = \theta_k - H^{*-1} \nabla F(\theta_k) + o(\|\nabla F(\theta_k)\|).$$

Proof. As the damping parameter vanishes, the regularized inverse converges to the inverse Hessian approximation. Consequently, the DR-LM-TRM iteration becomes asymptotically equivalent to Newton's method. □

6.6. Spectral Convergence

Define the spectral operator

$$A_k = (G_k + \lambda_k I)^{-1} G_k.$$

The eigenvalues of A_k are

$$\mu_{k,i} = \frac{\sigma_{k,i}}{\sigma_{k,i} + \lambda_k},$$

where $\sigma_{k,i}$ denotes the ith eigenvalue of G_k.

As $\lambda_k \to 0$,

$$\mu_{k,i} \to 1, \qquad A_k \to I.$$

This establishes spectral consistency of the regularized Jacobian inversion and confirms that the damping effect disappears asymptotically.

6.7. Unified Convergence Result

Theorem 4 (Unified Global-to-Local Convergence). *Under the stated assumptions, the DR-LM-TRM algorithm satisfies:*

1. *Global convergence:*
$$\|\nabla F(\theta_k)\| \to 0.$$
2. *Superlinear convergence:*
$$\frac{\|\theta_{k+1} - \theta^*\|}{\|\theta_k - \theta^*\|} \to 0.$$
3. *Local quadratic convergence:*
$$\|\theta_{k+1} - \theta^*\| = \mathcal{O}\left(\|\theta_k - \theta^*\|^2\right).$$
4. *Asymptotic Newton equivalence:*
$$\theta_{k+1} = \theta_k - H^{*-1} \nabla F(\theta_k) + o(\|\nabla F(\theta_k)\|).$$
5. *Spectral collapse:*
$$A_k \to I.$$

Collectively, these results demonstrate that DR-LM-TRM transitions smoothly from a globally stabilized optimization framework to a locally exact Newton-type method, thereby combining robustness, efficiency, and rapid local convergence within a unified optimization framework.

7. Numerical Results

The methods were implemented in Python and tested on the Rosenbrock function.

7.1. Performance Summary

Method	Iterations	Final $f(x)$	Final $\|\nabla f\|$
LM	40	3.340988×10^{-16}	2.551304×10^{-8}
STRM	37	1.570636×10^{-17}	7.987401×10^{-8}
DR-LM-TRF	35	1.035578×10^{-15}	7.396524×10^{-8}

8. Convergence Behavior

The convergence behavior is illustrated using:

- Function value vs iterations
- Gradient norm vs iterations
- Trajectory on Rosenbrock contour

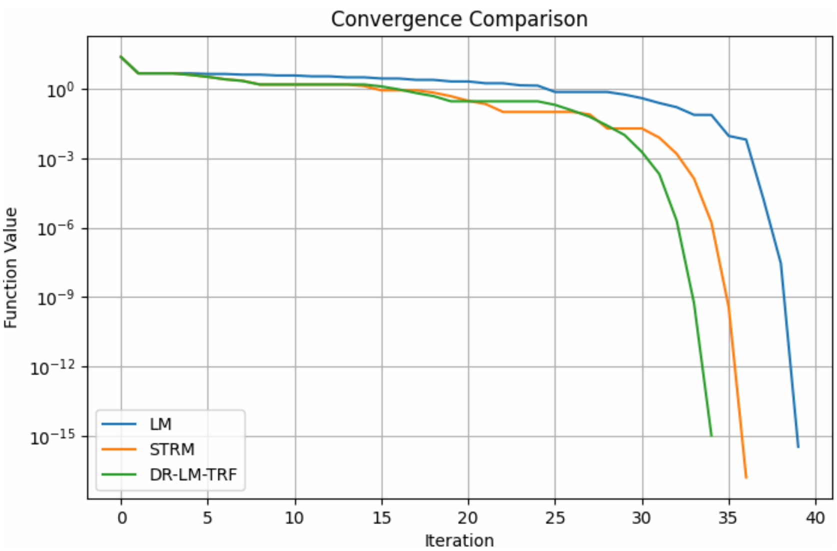

Figure 1: Convergence of objective function

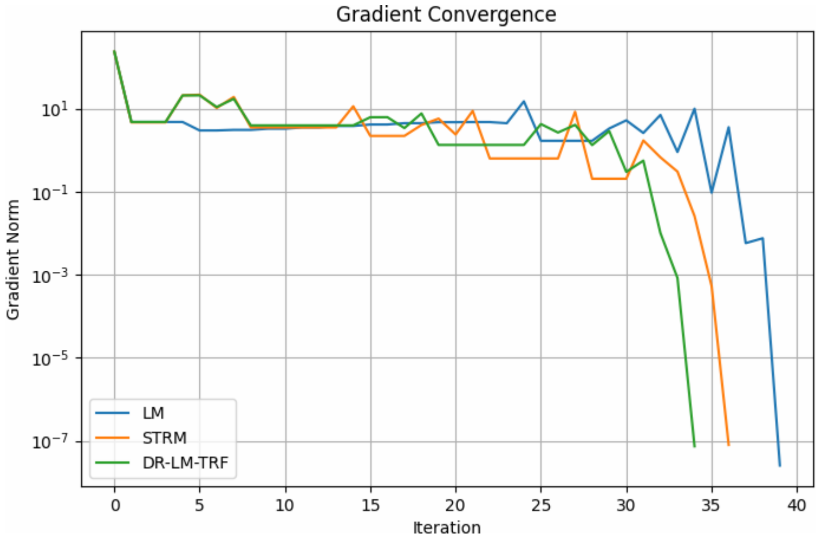

Figure 2: Gradient norm convergence

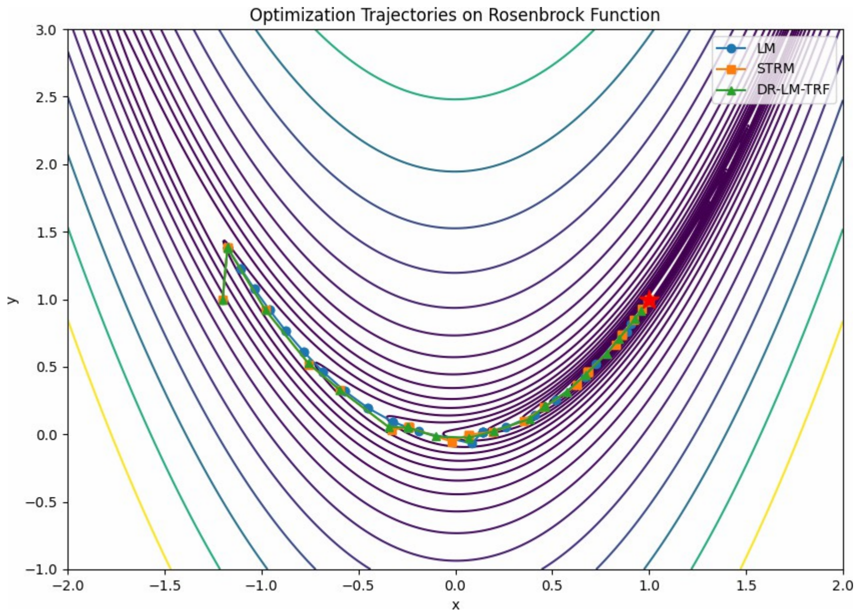

Figure 3: Optimization trajectories on Rosenbrock function

Discussion

The proposed DR-LM-TRF demonstrates improved stability compared with LM and STRM. The adaptive coupling between damping and trust-region radius en-hances robustness in ill-conditioned regions and accelerates convergence near the solution.

Conclusion

A dual-regulated optimization framework combining Levenberg–Marquardt and trust-region strategies has been presented. Numerical results show improved conver-gence behavior and stability. Future work will extend the method to high-dimensional machine learning models and constrained optimization problems.

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