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A Shifted Chebyshev Operational Matrix Method for Solving Fractional Abel Integral Equations with Weakly Singular Kernel

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Abstract:
This paper develops a shifted Chebyshev Operational matrix method for the numerical solution of fractional Abel integral equations involving weakly singular kernel. The proposed technique of the paper is based on the fractional order operational matrices of integration and differentiation of shifted Chebyshev polynomials.

A theoretical error analysis of the proposed method is presented. Convergence results are established and an upper bound for an approximate error is derived, demonstrating that the accuracy improves as the degree of approximation increases.

Key Word: *Fractional Abel integral equations, Fractional order operational matrix, Shifted Chebyshev polynomials.*

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I. Introduction

Fractional calculus has the ability to model memory and hereditary effects that cannot be adequately described by the classical integer order differential and integral operators. It is observed that in the past few decades, fractional operators have found applications in different fields of physics [1], bioengineering [2], viscoelasticity [3-4], control theory [5] etc. As a result, considerable attention has been devoted to the development of analytical and numerical techniques for solving fractional differential and integral equations [6].

Among the various class of fractional integral equations, Abel integral equations occupy a prominent position because of their occurrence in many practical applications such as inverse problems [7], plasma diagnostics [8] and tomography [9]. The classical Abel integral equation involves a weakly singular kernel of the form $(x-t)^{-\alpha}$, where $0<\alpha<1$. The presence of singularity often prevents the derivation of exact analytical solutions and creates significant challenges for numerical computations.

To overcome these difficulties, several numerical approaches have been proposed in the literature including collocation method [10], quadrature techniques, wavelet methods [11], spectral methods and operational matrix methods [12-14]. Operational matrix techniques provide an efficient framework for transforming integral and differential equations into system of algebraic equations. The main advantage of this approach lies in replacing integral operators by matrix representations, thereby significantly reducing computational complexity. In recent years, operational matrices associated with Legendre [15], Chebyshev [16], Bernstein [17], Jacobi [18] and other orthogonal polynomial bases [19] have been successfully employed for solving various classes of fractional integral and differential equations.

Motivated by these developments, this paper presents a shifted Chebyshev operational matrix method for the numerical solution of fractional Abel integral equations. The shifted Chebyshev polynomials defined on the interval [0,1] are employed to approximate the unknown function. An operational matrix corresponding to the Abel integral operator with a weakly singular kernel is constructed. Furthermore, an operational matrix for the Riemann-Liouville fractional integral operator is derived within the same polynomial framework. By exploiting the orthogonal properties of the shifted Chebyshev basis, the original fractional Abel integral equation is transformed into a finite-dimensional algebraic system. The resulting coefficients are then used to obtain an accurate approximation of the solution.

In addition, a rigorous error analysis is developed to investigate the convergence behavior of the proposed method. Error estimates are established, demonstrating that the approximation error decreases as the degree of the shifted Chebyshev expansion increases. Numerical examples are provided to validate the theoretical results and to illustrate the accuracy and effectiveness of the proposed technique.

II. Preliminaries and Properties of Shifted Chebyshev Polynomials

In this section, some basic definitions and properties of fractional calculus and shifted Chebyshev polynomials are presented. These results form the mathematical foundation for construction the operational matrices developed in the subsequent sections.

2.1. Fractional Calculus

We begin with the definition of the Gamma function, which plays a fundamental role in the fractional calculus.

Definition 2.1 The Gamma function is defined as

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt \, , \alpha > 0 \tag{2.1}$$

where (2.1) satisfies the recurrence relation $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

Definition 2.2 The Riemann-Liouville fractional order integral operator of order $\beta > 0$ is defined as

$$I^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-x)^{\beta-1} f(x) dx, \ \ t > 0 \tag{2.2}$$

Definition 2.3 The Caputo fractional derivative of order γ is defined as

$$D^\gamma f(t) = I^{m-\gamma} D^m f(t) = \frac{1}{\Gamma(m-\gamma)} \int_0^t (t-x)^{m-\gamma-1} \frac{d^m}{dx^m} f(x) dx \tag{2.3}$$

$$m-1 < \gamma < m, t > 0$$

2.2. Shifted Chebyshev Polynomials

The well-known Chebyshev Polynomials of first kind with degree n are defined on the interval $[-1,1]$ as

$$T_n(t) = \cos(n \cos^{-1} t).$$

We, also define the shifted Chebyshev polynomials on the interval $[0,1]$ by using the change of variable $t = 2x - 1$. So, the shifted Chebyshev polynomials $T_i(2x-1)$ are denoted by $\phi_i(x)$ and satisfy the orthogonality condition

$$\int_0^1 \phi_i(x)\, \phi_j(x) w(x) dx = \delta_{ij} h_j \tag{2.4}$$

where δ_{ij} is Kronecker delta function, $w(x) = 1/\sqrt{x - x^2}$ is weight function, $h_0 = \pi$ and $h_k = \pi/2$ $(k \geq 1)$.

Shifted Chebyshev polynomials can be determined with the aid of the recurrence relation

$$\phi_{i+1}(x) = 2(2x-1)\phi_i(x) - \phi_{i-1}(x), \tag{2.5}$$

where $\phi_0 = 1$ and $\phi_1(x) = 2x - 1$.

The analytic form of the n degree shifted Chebyshev polynomials is given by

$$\phi_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+k-1)!\, 2^{2k}}{(n-k)!\,(2k)!} x^k \qquad (2.6)$$

A function $f(x)$ square integrable in $[0,1]$ may be expressed in terms of shifted Chebyshev polynomials as

$$f(x) = \sum_{i=0}^{\infty} c_i \phi_i(x) \qquad (2.7)$$

where c_i can be determined by

$$c_i = \frac{1}{h_i} \int_0^1 f(x)\, \phi_i(x) w(x) dx$$

We only consider the first $(N+1)$ terms of (2.7) and so

$$f(x) = \sum_{i=0}^{N} c_i \phi_i(x) = C\Phi(x) \qquad (2.8)$$

where, $C = [c_0, c_1, \dots, c_N]^T$ and $\Phi(x) = [\phi_0, \phi_1, \dots, \phi_N]$.

III. Methodology

Consider the fractional Abel integral equation of order $0 < \beta < 1$ as

$$I(y) = \int_y^1 \frac{D^\beta \eta(r)}{(r-y)^b} dr, \qquad (3.1)$$

subject to the initial condition $\eta(0) = a, a$ being a constant.

To solve the above integral equation, we approximate the unknown function $D^\beta \eta(r)$ in terms of shifted Chebyshev polynomials as described earlier.

$$D^\beta \eta(r) = C^T \phi(r). \qquad (3.2)$$

Now, integrating it fractionally using Riemann-Liouville definition, we get,

$$\eta(r) = C^T I^\beta \phi(r) + A^T \phi(r).$$

Since, we have,

$$\eta(0) = a = A^T \phi(r),$$

and

$$I(y) = F^T \phi(y). \qquad (3.3)$$

Substituting from (3.2) and (3.3) in the equation (3.1), we get

$$F^T \phi(y) = \int_y^1 \frac{C^T \phi(r)}{(r-y)^b} dr = C^T \int_y^1 \frac{\phi(r)}{(r-y)^b} dr \qquad (3.4)$$

and as described earlier,

$$\int_y^1 \frac{\phi(r)}{(r-y)^b} dr = Q\phi(y),$$

so that $F^T \phi(y) = C^T Q \phi(y)$.

Comparing coefficients we get, $C^T = F^T Q^{-1}$.

IV. Error Analysis

Lemma 4.1. Let $g(x) = D^{\alpha} f(x)$ and assume that $g \in C^{m}[0,1], m \geq 1$, and that its mth derivative $g^{(m)}$ is of bounded variation on $[0,1]$. Let c_i be the coefficients of the shifted Chebyshev expansion

$$g(x) = \sum_{i=0}^{\infty} c_i \phi_i(x), \quad \phi_i(x) = T_i(2x-1) \tag{4.1}$$

Then there exists a constant $M > 0$, independent of i, such that

$$|c_i| \leq \frac{M}{i^m}, \qquad i \geq 1 \tag{4.2}$$

Proof. Using the change of variable $x = (1 + \cos\, \theta)/2$, the shifted Chebyshev expansion coefficients can be written as Fourier cosine coefficients:

$$c_i = \frac{2}{\pi}\int_0^{\pi} g\left(\frac{1+\cos\theta}{2}\right)\cos(i\theta)\, d\theta = \frac{2}{\pi}\int_0^{\pi} G(\theta)\cos(i\theta)\, d\theta \tag{4.3}$$

where,

$$G(\theta) = g\left(\frac{1+\cos\theta}{2}\right).$$

Since, $g \in C^{m}[0,1]$, the function $G \in C^{m}[0,\pi]$.

We now apply integration by parts m times. Each integration introduces a factor $1/i$, giving

$$c_i = \frac{1}{i^m}\cdot\frac{2}{\pi}\int_0^{\pi} G^{(m)}(\theta)\psi_i(\theta) d\theta \tag{4.4}$$

where, $\psi_i(\theta)$ is a bounded trigonometric function.

Hence,

$$|c_i| \leq \frac{1}{i^m \| G^{(m)} \|_{L^1(0,\pi)}} \tag{4.5}$$

Since, $G^{(m)} \in L^1(0,\pi)$, we conclude that there exists a constant $M > 0$ such that

$$|c_i| \leq \frac{M}{i^m}$$

Theorem 4.1. Let $D^{\alpha} f \in C^{m}[0,1], m \geq 1$, and assume that $(D^{\alpha} f)^{(m)}$ is of bounded variation on $[0,1]$. Let $D^{\alpha} f(x) = \sum_{i=0}^{\infty} c_i \phi_i(x), \quad \phi_i(x) = T_i(2x-1)$, be the shifted Chebyshev expansion of $D^{\alpha} f$, and define $D^{\alpha} f_N(x) = \sum_{i=0}^{N-1} c_i \phi_i(x)$. Then, $\| D^{\alpha} f - D^{\alpha} f_N \|_{L^2_w(0,1)} \to 0,\ N \to \infty$ where $w(x) = 1/\sqrt{x(1-x)}$.

Moreover, the order of the convergence is

$$\| D^{\alpha} f - D^{\alpha} f_N \|_{L^2_w} = O\left((N-1)^{-\left(m-\frac{1}{2}\right)}\right) \tag{4.6}$$

Proof. Let us approximate the function using shifted Chebyshev polynomials as

$$D^{\alpha} f(x) = \sum_{i=0}^{\infty} c_i \phi_i(x), \tag{4.7}$$

and let f_N denotes that the truncation level is $N-1$ so that

$$D^{\alpha} f_N(x) = \sum_{i=0}^{N-1} c_i \phi_i(x) \tag{4.8}$$

Now,

$$D^{\alpha} f(x) - D^{\alpha} f_N(x) = \sum_{i=N}^{\infty} c_i \phi_i(x)\,. \tag{4.9}$$

Define the weighted inner product as

$$\langle u, v \rangle_w = \int_0^1 u(x) v(x) w(x) dx$$

where, for the shifted Chebyshev polynomials, $w(x) = 1/\sqrt{x(1-x)}$.

The corresponding weighted norm is

$$\|u\|_{L_w^2} = \left(\int_0^1 |u(x)|^2 w(x) dx \right)^{1/2}$$

where,

$$L_w^2[0,1] = \left\{ u\colon \int_0^1 |u(x)|^2 w(x) dx < \infty \right\}$$

Then,

$$\|D^{\alpha} f(x) - D^{\alpha} f_N(x)\|^2{}_{L_w^2} = \int_0^1 \left(\sum_{i=N}^{\infty} c_i \phi_i(x) \right)^2 w(x) dx \tag{4.10}$$

Since, the shifted Chebyshev polynomials are orthogonal with respect to $w(x)$,

$$\int_0^1 \phi_i(x) \phi_j(x) w(x) dx = h_i \delta_{ij}$$

and therefore,

$$\|D^{\alpha} f - D^{\alpha} f_N\|_{L_w^2}^2 = \sum_{i=N}^{\infty} c_i^2 h_i\,. \tag{4.11}$$

But, $h_i \leq \pi$

and therefore,

$$\|D^{\alpha} f - D^{\alpha} f_N\|_{L_w^2}^2 \leq \pi \sum_{i=N}^{\infty} c_i^2\,. \tag{4.12}$$

Now, using lemma, we have $|c_i| \leq \frac{M}{i^m}$, so that

$$\|D^{\alpha} f - D^{\alpha} f_N\|_{L_w^2}^2 \leq \pi M^2 \sum_{i=N}^{\infty} \frac{1}{i^{2m}} \tag{4.13}$$

The integral test yields

$$\sum_{i=N}^{\infty} \frac{1}{i^{2m}} \leq \frac{(N-1)^{-(2m-1)}}{2m-1}, \qquad m > \frac{1}{2}.$$

Hence,

$$\|D^{\alpha} f - D^{\alpha} f_N\|_{L_w^2}^2 \leq \frac{\pi M^2}{2m-1} (N-1)^{-(2m-1)}$$

$$\Rightarrow \|D^{\alpha} f - D^{\alpha} f_N\|_{L_w^2} \leq \frac{\sqrt{\pi} M}{\sqrt{2m-1}} (N-1)^{-\left(m-\frac{1}{2}\right)} \tag{4.14}$$

Thus,

$$\|D^{\alpha} f - D^{\alpha} f_N\|_{L_w^2} = O\left((N-1)^{-\left(m-\frac{1}{2}\right)}\right).$$

V. Numerical results

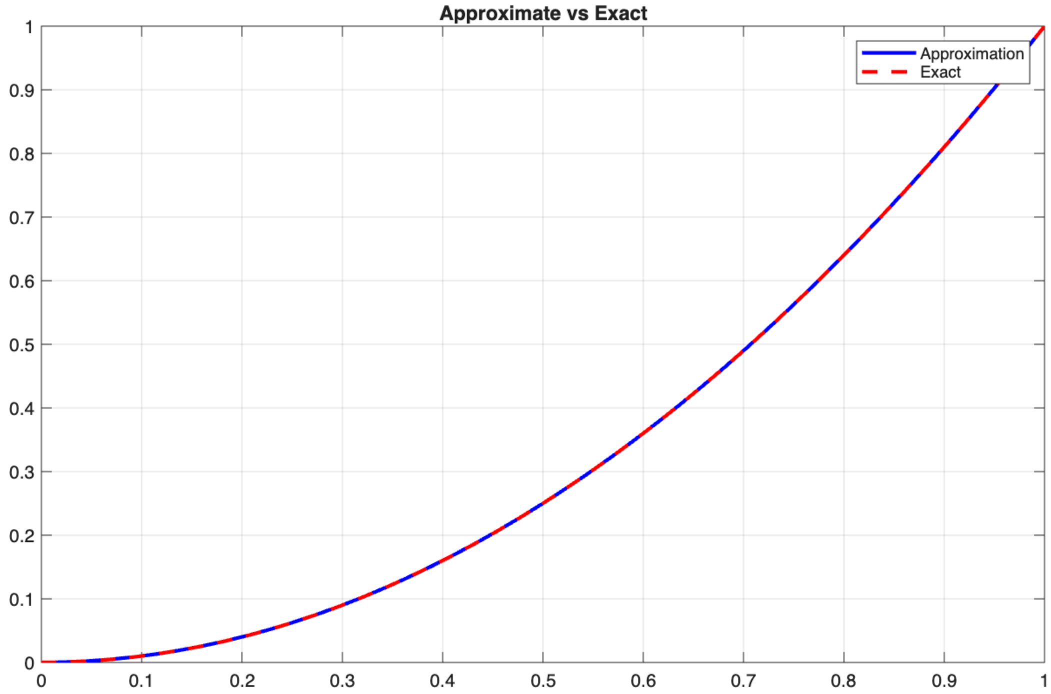
In this section we discuss the implement our proposed numerical scheme based on fractional operational matrix of integration.

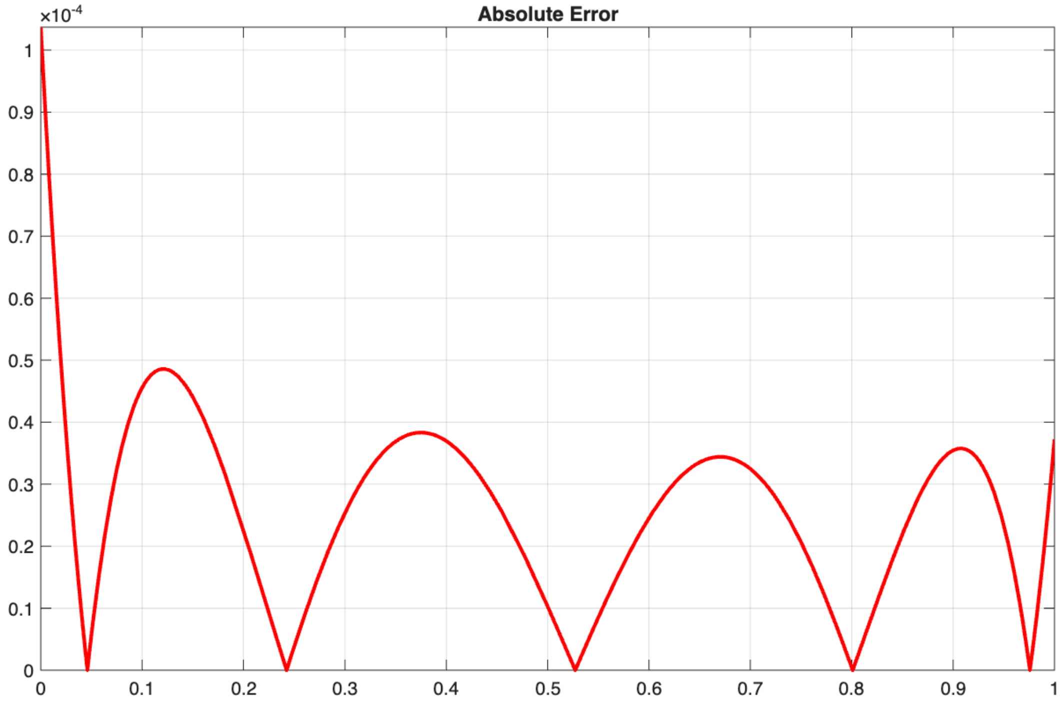
Example 5.1. Consider the fractional Abel integral with $\eta(r) = r^2$ for $0 \leq r \leq 1$,

$$I(y) = \frac{2}{\Gamma\left(\frac{9}{4}\right)} \left[\frac{4}{7}(1-y)^{\frac{1}{2}} + \frac{\sqrt{\pi} y^{\frac{7}{4}} \Gamma\left(-\frac{7}{4}\right)}{\Gamma\left(-\frac{5}{4}\right)} + \frac{20}{21} y \right] \ {}_2F_1\left(-\frac{3}{4}, \frac{1}{2}; \frac{1}{4}; y\right), \text{for } 0 < y < 1$$

with initial conditions $\eta(0) = 0, \beta = 3/4$.

Fig.1 shows the exact and approximate solution computed by the proposed scheme and *Fig.2* shows the absolute error occurred.





6. Table

In this section, we discuss the absolute error and root mean square error for the test function.

y	Approx	Exact	Error	Abs Error
0.1	0.01004556	0.01	4.556765e-05	4.556765e-05
0.2	0.04002254	0.04	2.254760e-05	2.254760e-05
0.3	0.08997465	0.09	-2.534896e-05	2.534896e-05
0.4	0.15996304	0.16	-3.695899e-05	3.695899e-05
0.5	0.24998970	0.25	-1.029205e-05	1.029205e-05

0.6	0.36002455	0.36	2.4552591e-05	2.4552591e-05
0.7	0.49003246	0.49	3.2468861e-05	3.2468861e-05
0.8	0.64000042	0.64	4.2681405e-05	4.2681405e-05
0.9	0.80996455	0.81	-3.544451e-05	3.544451e-05
1.0	1.00003722	1.0	3.7225870e-05	3.7225870e-05

RMS error $= 2.9997645e-05$

Table presents the approximate solution, exact solution and absolute error. It is observed that the proposed shifted Chebyshev Operational matrix method produces highly accurate results throughout the interval [0,1]. Furthermore, the computed root mean square error confirms the excellent agreement between the approximate and exact solution.

VII. Conclusion

In this work, a shifted Chebyshev operational matrix method has been developed for the numerical solution of fractional Abel integral equations with weakly singular kernels. The proposed approach utilizes shifted Chebyshev polynomials to approximate the unknown function and constructs an operational matrix of fractional integration, enabling the original integral equation to be transformed into a system of algebraic equations. This transformation significantly simplifies the computational procedure while preserving the accuracy of the solution.

The numerical experiment indicate that the shifted Chebyshev operational matrix approach provides a computationally efficient and stable framework for solving fractional Abel integral equations. Owing to its simplicity and high accuracy, the proposed method can be extended to a wider class of fractional integral and integro-differential equations arising in applied mathematics, physics, and engineering.

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