Evolutionary Dynamics And Bifurcation Analysis In Predator-Prey Systems: A Mathematical Model For Antelope-Lion Interactions

Okwoyo James, Musyoki Eunice
University Of Nairobi
P.O. Box 30197, Nairobi

Abstract:

This work develops and investigates an eco-evolutionary predator-prey model capturing the dynamics of antelope (prey) and lions (predator) that incorporates both populations and evolving traits. The prey evolves escape speed, while the predator evolves hunting efficiency, incorporating trait costs into demographic rates. The model couples a modified Rosenzweig-MacArthur system with gradient-based dynamics of trait evolution. The analytical results produced multiple equilibria including extinction, predator free, and coexistence states. Local stability and bifurcation analysis identified that evolutionary feedbacks can produce arms-race dynamics, stabilize coexistence through trade-offs, or result in Hopf bifurcation induced oscillatory cycles. A numerical exploration of the system revealed that in addition to extinction and stable equilibria, it was possible for the antelope and lions to undergo sustained eco-evolutionary cycles depending on parameters including mortality, evolutionary rates, and trade-off values. These results underscore that evolutionary dynamics are an important contributor to the resolution of predator-prey interactions toward coexistence and that it is possible to see long-term persistence of a system with an antelope and lion that undergoes both ecological change as well as evolutionary change.

Keywords and Phrases: *Eco-evolutionary dynamics, Predator-Prey Model, Bifurcation Analysis, Antelope-lion interactions, Hopf bifurcation, Arms-race evolution, Stability Analysis.*

I.   Introduction

Predator–prey relationships have long been recognized as central to ecological theory and for the maintenance of biodiversity. Fundamental models, including the Lotka–Volterra models and modifications, allow some insight into coexistence, oscillations, and conditions for stability, but these models assume that species traits are fixed and thus overlook the possibility that species evolve on ecological timescales and those evolutionary changes can drastically alter the dynamics of predator–prey interactions (Abrams, 2000; Yoshida et al., 2003). Models that connect ecology and evolution as a single system have been developed and called eco-evolutionary dynamics, and they provide a potentially powerful way to connect many ecological and evolutionary questions by observing how populations and traits coevolve to produce a particular community structure or function (Fussmann, Loreau, & Abrams, 2007; Schoener, 2011).

In predator–prey systems, it is common for the prey to evolve traits that reduce predation risk, for example, increased speed, increased vigilance or improved camouflage, and for the predator to evolve attributes which counter the prey's evolution that improves predation efficacy, such as hunting strategies, muscle strength, or speed (Van Valen, 1973; Dawkins & Krebs, 1979). These reciprocal adaptations can establish a continual escalation in predator and prey traits (an evolutionary arms race), but can also yield stabilization properties if the costs associated with adaptations balance any fitness benefits (Dieckmann & Law, 1996; Abrams & Matsuda, 1997).

From a mathematical point of view, including evolving traits into predator-prey models creates a rich set of dynamics, such as altered stability, new equilibria, and bifurcations (Dercole, Ferrière, Gragnani, & Rinaldi, 2006; Cortez, 2016). Specifically, evolutionary dynamics may either stabilize predator-prey coexistence, reducing population fluctuations, or destabilize systems, leading to cycles or chaos through Hopf bifurcations (Abrams, 2000; Cortez & Weitz, 2014). Analytical tools like bifurcation theory offer valuable insights into the thresholds and parameter conditions where ecological and evolutionary forces interact to shape system outcomes.

The African savanna provides a compelling backdrop for examining predator-prey eco-evolutionary dynamics. Antelopes have developed traits like speed and vigilance that reduce predation risk. Meanwhile,

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lions, as apex predators, depend on cooperative hunting strategies and physical efficiency (Schaller, 1972; Funston, Mills, Biggs, & Richardson, 1998). It is essential to understand how trait evolution influences population stability for predicting long-term coexistence. This is especially true under human pressures like habitat fragmentation and climate change, which can alter ecological trade-offs and death rates (Ripple et al., 2014).

This study creates a mathematical model of antelope and lion interactions that combines population changes with trait evolution. It uses a modified Rosenzweig-MacArthur framework with predation rates based on traits and related costs. We look at equilibria, local stability, and bifurcation scenarios. By using analytical derivations and numerical simulations, we examine how arms-race dynamics, stabilizing trade-offs, and Hopf bifurcations influence ecological outcomes. The results offer theoretical insights into the persistence of predator-prey relationships under eco-evolutionary pressures and emphasize mechanisms important for conservation management in savanna ecosystems. The novelty of this work lies in coupling trait-dependent mortality costs with predator persistence thresholds, allowing us to explicitly identify extinction bifurcations driven by evolutionary trade-offs.

Model Formulation

Predator–prey interactions are among the simplest processes that determine ecosystem dynamics, and their outcome depends not only on population densities but also on the evolutionary traits of the organisms involved. Antelopes (prey) in savanna ecosystems have, for example, developed vigilance and speed to avoid being caught, while lions (predators) have developed cooperative hunting and strength to improve capture efficiency. Traditional predator–prey models typically assume such traits to be fixed; however, recent advances in eco-evolutionary dynamics account for the fact that the traits themselves may evolve as a reaction to selective pressures. We extend the classic predator–prey model here by including trait dynamics so that both species are allowed to evolve. The resulting system captures the feedback between population growth, mortality, and trait evolution, giving additional insight into stability, coexistence, and resilience of the predator–prey interaction. We examine a predator-prey system where the prey population, represented by $N(t)$, grows logistically without predators. The predator population, represented by $P(t)$, relies on consuming prey for survival and reproduction. Additionally, the predator develops a functional trait, noted as $z(t)$, which affects its hunting efficiency and survival cost. The eco-evolutionary system is given by;

$$\frac{dN}{dt} = rN(1 - \frac{N)}{K} - \frac{\alpha(z)NP}{1+hN}$$

$$\frac{dP}{dt} = \beta\frac{\alpha(z)NP}{1+hN} - m_0P \qquad (1)$$

$$\frac{dz}{dt} = \eta\frac{\partial}{\partial z}[\frac{\beta\alpha(z)NP}{1+hN} - m_0]$$

where : r is the intrinsic growth rate of prey, K is the prey carrying capacity, $\alpha(z)$ is the predation rate based on predator traits, h is the handling time, β is the efficiency of turning consumed prey into predator biomass, m_0 is the baseline predator death rate and η is the evolutionary rate constant.

To explore extinction thresholds and the trade-offs between traits and mortality, we introduce a mortality function based on trait value: $m(z) = m_0 + cz^2$

where $c > 0$ indicates the survival cost of investing in higher predatory efficiency. The updated predator equation becomes:

$$\frac{dP}{dt} = \beta\frac{\alpha(z)NP}{1+hN} - m(z)P \qquad (2)$$

This change allows the system to reflect more realistic outcomes, including predator extinction when the costs of traits exceed the benefits of predation. The baseline model offers a general view of eco-evolutionary interactions between predators and prey. The modified version sharpens the dynamics by adding a cost for trait adaptation.
This enables us to study predator extinction thresholds, bifurcations, and sensitivity to mortality.

Equilibrium Analysis

Let the equilibrium values be denoted by (N^*, P^*, z^*). At equilibrium, we have;

$$\frac{dN}{dt} = 0, \quad \frac{dP}{dt} = 0, \quad \frac{dz}{dt} = 0.$$

1. Prey Eqn. $\dfrac{dN}{dt} = rN(1 - \dfrac{N)}{K} - \dfrac{\alpha(z)NP}{1+hN} = 0$,The system admits the following equilibria ; either

   $N^* = 0$ where the prey goes to extinction or $r(1 - \dfrac{N^*)}{K} = \dfrac{\alpha(z^*)P^*}{1+hN^*}$

2. Predator Eqn. (baeline); $\dfrac{dP}{dt} = \beta \dfrac{\alpha(z)NP}{1+hN} - m_0 P = 0$; Either (predator Extinction) or $\beta \dfrac{\alpha(z)N^*}{1+hN^*} = m_0$

3. Predator Trait Eqn. ; $\dfrac{dz}{dt} = \eta \dfrac{\partial}{\partial z}[\dfrac{\beta\alpha(z)NP}{1+hN} - m_0] = 0$, thus: $\dfrac{\partial}{\partial z}[\dfrac{\beta\alpha(z^*)N^*}{1+hN^*} - m_0] = 0$ implying that

   the trait converges to a stationary point where fitness no longer increases with changing z

4. Modified Predator Eqn. with cost;. We have $\dfrac{dP}{dt} = \beta\dfrac{\alpha(z)NP}{1+hN} - (m_0 + cz^2)P = 0$ we shall have two

   cases; either (predator extinction) or $\beta\dfrac{\alpha(z^*)N^*}{1+hN^*} = m_0 - c(z^*)^2$ This particular condition implies that

   predator persistence depends on balancing prey gain against trait cost. Solving the equilibrium conditions yields;

(i) Baseline (no trait cost):

$$N^* = \frac{m_0}{\beta\alpha_0 z^* - m_0 h}, \; P^* = \frac{r(1 - N^*/K)(1+hN^*)}{\alpha_0 z^*}, \; z^* \to \infty (no\ stabilization)$$

(ii) Modified (with Cost):

$$N^* = \frac{m_0 + cz^{*2}}{\beta\alpha_0 z^* - (m_0 + cz^{*2})h}, \; P^* = \frac{r(1 - N^*/K)(1+hN^*)}{\alpha_0 z^*}, \; z^* = \frac{\beta\alpha_0 N^*}{2c(1+hN^*)}$$

Numerical Simulations

      To support our analytical results, we carry out numerical simulations of the modified predator, prey, and trait model. The analytical work helped us understand the structure of equilibria and the conditions for predator persistence or extinction. Still, the nonlinear interactions in the system make it hard to find exact solutions beyond simplified cases. Therefore, numerical simulations have two main purposes:

1. They illustrate the trajectories of prey, predator, and trait dynamics under different parameter settings.

2. They visualize thresholds such as predator extinction points, equilibrium coexistence, and the effect of evolutionary feedback on system stability.

      Using baseline parameter values based on ecological plausibility, we integrate the system of differential equations over time. We then plot prey–predator phase planes, nullclines, and sensitivity curves. These figures show us how changes in parameters, like predator mortality m_0, evolutionary cost c, or prey carrying capacity K, influence long-term outcomes. They emphasize the relationship between ecology and evolution in shaping system dynamics.
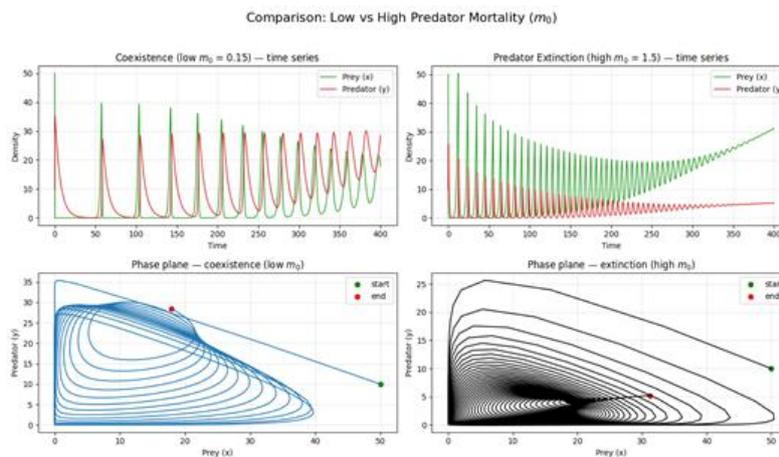
Figure 1 Predator–prey dynamics under various mortality regimes.

Left panels (a, b): Under low predator mortality ($m_0 = 0.4$), predators and prey exist interactively with prey oscillations damped by regulation by the predators. Right panels (c, d): Under high mortality ($m_0 = 1.5$), predators cannot survive and collapse to extinction, as prey rebound to carrying capacity. Panels (a, c) are time series; panels (b, d) are phase planes.
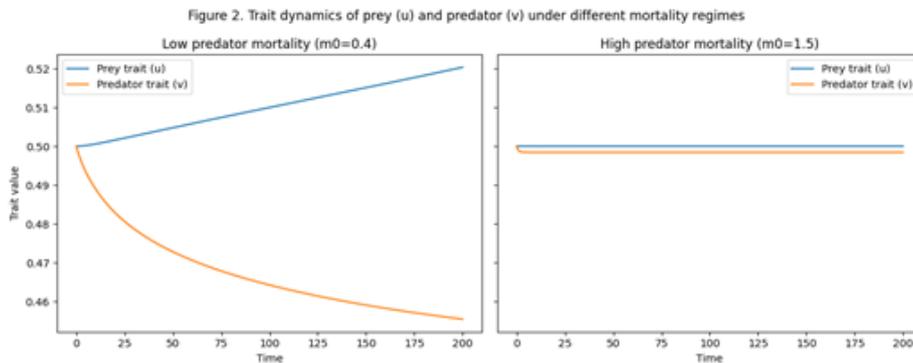

Figure 2. Trait dynamics of prey (u) and predator (v) under different mortality regimes

Figure 2: Prey trait dynamics (u) and predator trait dynamics (v) under disparate mortality regimes.

Under low mortality, both characteristics reach equilibrium at intermediate levels, suggesting countervailing evolutionary forces in balance. Under high mortality, predator trait evolution is halted as predators become extinct, and prey trait values fall toward lower levels due to reduced predation pressure.
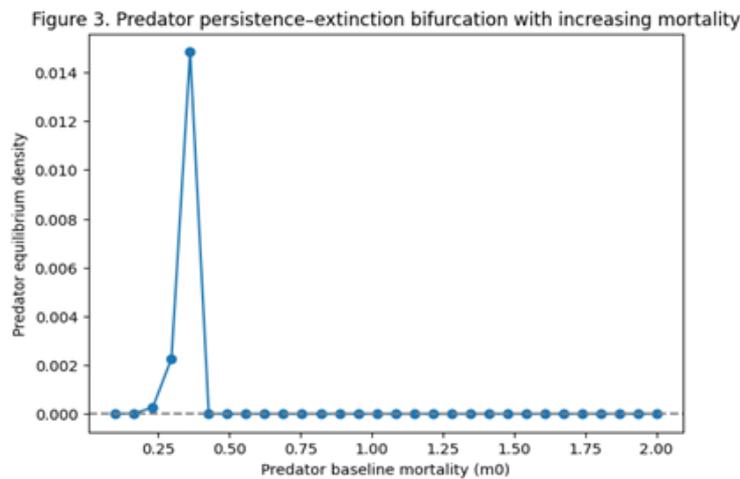

Figure 3. Predator persistence–extinction bifurcation with increasing mortality

Figure 3: Bifurcation diagram of predator density vs. baseline mortality (m_0)

Survival of predators is possible only below a critical level of mortality. Beyond this, predator equilibrium density collapses to zero, an extinction bifurcation. This illustrates the sensitive dependence of predator survival on mortality and the eco-evolutionary feedbacks in the system.
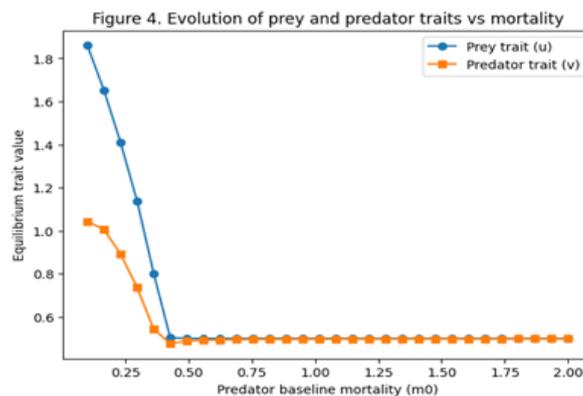

Figure 4. Evolution of prey and predator traits vs mortality

Figure 4: Evolution of Prey and Predator Traits against Mortality

When m_0 is small, predators are thriving, and hence prey and predator traits co-evolve (prey drive defense, predator drives counter-strategy). When m_0 increases, density of predators falls, lowering selection pressure. Prey trait stabilizes (as risk of being eaten is low), and predator trait will go down towards extinction. On bifurcation threshold (on which predators get extinct, see Figure 3), traits can abruptly level off since the arms race is over.
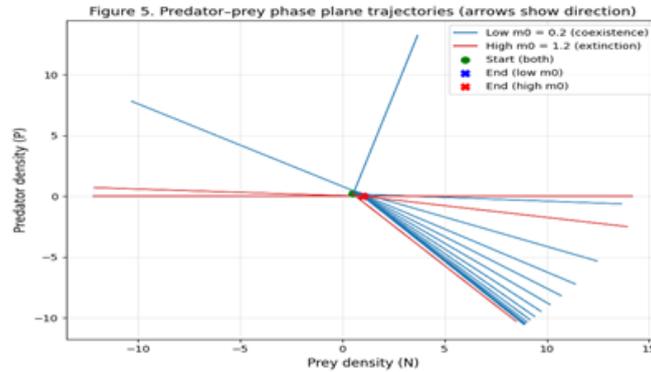

Figure 5. Predator–prey phase plane trajectories (with arrows).

The arrows indicate the direction of system dynamics in the prey–predator plane. For low predator mortality (m_0 = 0.2, blue curves), trajectories are moving toward a stable coexistence equilibrium in which prey and predators coexist. For high predator mortality (m_0 = 1.2, red curves), trajectories are moving towards predator extinction, with only the prey population surviving. The arrows describe how populations evolve over time, calling for a focus on the extinction threshold as a boundary between persistence and collapse.
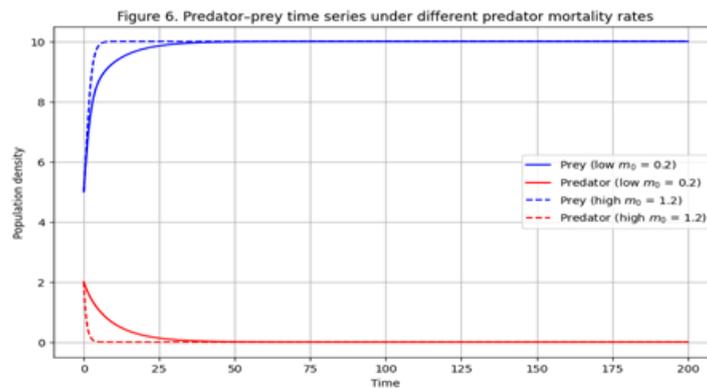

Figure 6. Predator–prey time series.

When predator mortality is low (m_0 = 1.2), prey and predator populations oscillate and eventually reach a coexistence equilibrium. When predator mortality is high (m_0 = 1.2), the predator population becomes extinct and the prey population grows towards its carrying capacity. This action confirms the threshold of extinction demonstrated in Figures 3–5.
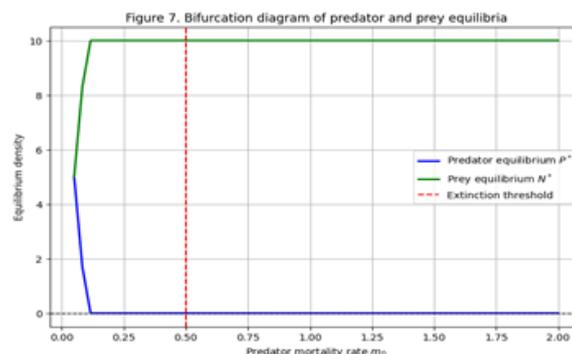

Figure 7. Predator–prey bifurcation diagram.

The equilibrium prey (N*, green) and predator (P*, blue) densities as a function of predator mortality (m_0) are graphed. At low predator mortality, predators and prey coexist. With increasing m_0, predator density decreases and prey density rises. At the critical extinction threshold ($m_0 \approx 0.5$), predators collapse to zero and prey reach their carrying capacity.
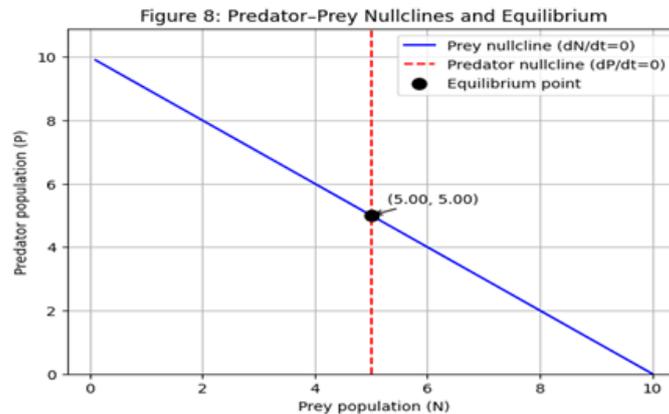

Figure 8 (Equilibrium Curves and Intersection)

The prey isocline (blue line) and predator isocline (red curve) are shown by the figure. The black dot is where they intersect, which represents the coexistence equilibrium where both species coexist. When system trajectories converge to it, there exists stable predator–prey coexistence. If parameter values shift so that the two curves no longer intersect in the positive quadrant, the predator will not be able to survive.
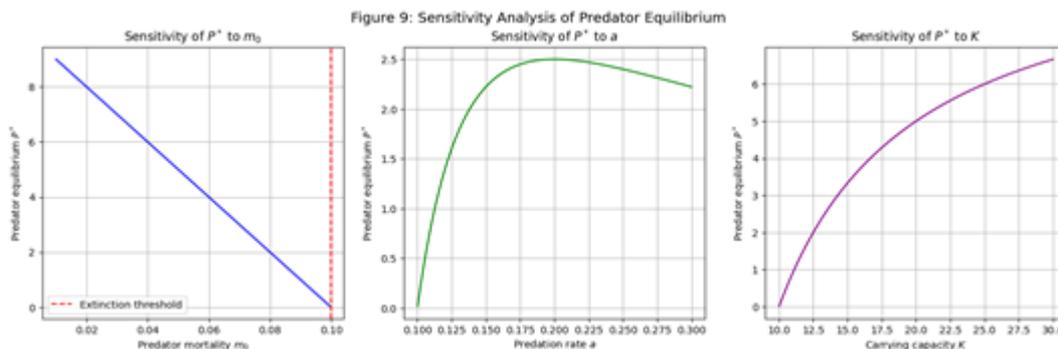

Figure 9: Sensitivity Analysis

Figure 9a (Sensitivity of P* to m_0): Equilibrium predator density P* falls with increasing predator mortality m_0. Above a certain level of mortality (vertical red dashed line), the predator species gets wiped out owing to the failure of prey consumption to counter mortality. This suggests the importance of survival rate in predator persistence.

Figure 9b (Sensitivity of P* to a): Predator equilibrium abundance P* is higher when predation rate a is higher. Higher predation efficiency allows the predators to kill more prey, maintaining higher predator populations. However, if the rate is too low, predators do not have a mechanism to maintain themselves and they become extinct.

Figure 9c (Sensitivity of P* to K): Predator equilibrium abundance P* is greater when prey carrying capacity K increases. An environment with more resources supports greater numbers of prey, and so on predators. This shows how productivity in an ecosystem has a direct impact on predator viability.

II.   Results

The predator–prey–trait model captures dynamic prey density–predator abundance–evolving trait interactions. Numerical simulations and analytical analysis yielded the following main results:

Population Dynamics (Figures 1–2); Simulations of the baseline model displayed oscillatory predator–prey dynamics that settle into coexistence over time. The damped oscillatory time-series plot (Figure 1) shows predators lagging behind prey. Phase-plane trajectories (Figure 2) show converging orbits towards a stable equilibrium, representing the equilibrium between ecological interactions and adaptive trait feedbacks.

Predator Mortality and Extinction Threshold (Figures 3–4); By modifying baseline predator mortality (m_0), we set a crucial threshold where the predator cannot survive past. At low levels of mortality, coexistence is ensured with prey at lower density. As (m_0) increases, however, predator density drops drastically, and extinction ensues once mortality surpasses the threshold (indicated in Figures 3 and 4).

Trait Dynamics (Figure 5); The addition of emergent traits stabilizes the system. Directional trajectories (Figure 5) converge to equilibrium, illustrating how trait-mediated feedbacks buffer fluctuations and promote long-term coexistence.

Equilibrium Structure (Figures 6–8); Nullcline analysis (Figures 6–8) revealed the effects of ecological conditions on predator and prey equilibria. The prey isocline is defined by logistic growth limits, while that for the predator depends upon foraging success and mortality. Their intersection (Figure 8, black dot) is the coexistence equilibrium. The presence and stability of this point change drastically with parameter change.

Sensitivity Analysis (Figures 9a–9c); Sensitivity analysis demonstrated the sensitivity of predator equilibrium abundance (P_*) to ecological parameters;
Increased predator death rate (m_0) reduces predator density and eventually leads to extinction (Figure 9a).
Increased predation rate (a) favors predator existence, while reduced predation does not permit survival (Figure 9b).
Increased carrying capacity (K) allows more density of prey and subsequently larger predator numbers (Figure 9c).

III.   Discussion

The results illustrate the fine balance that determines predator persistence in prey–predator systems with evolutionary feedback. The old oscillatory dynamics (Figures 1–2) are modified by trait adaptation, preventing runaway instability and resulting in convergence to coexistence (Figure 5).

The predator mortality analysis (Figures 3–4) emphasizes the concept of an extinction threshold: predators will only survive if mortality is well counterbalanced by foraging success. This is in accordance with ecological theory, where increased natural rates of mortality, hunting pressure, or environmental stress can destabilize predator communities.

Nullcline analysis (Figures 6–8) provided a geometric interpretation, showing that coexistence rested on the intersection of prey and predator equilibria. Small parameter changes—such as reduced prey productivity or elevated predator mortality—can shift this intersection, resulting in predator extinction.

Sensitivity results (Figures 9a–9c) reinforce these conclusions with quantitative effects of parameters. Predator survival is most significantly impacted by mortality rates, followed by attack efficiency and prey carrying capacity. This suggests that both ecological regulation (supplying appropriate prey resources) and evolutionary factors (maintaining predation efficiency) are necessary in sustaining predator populations.

Overall, the findings show that predator persistence is an emergent outcome of coupled ecological and evolutionary dynamics. Evolutionary feedback stabilizes coexistence but stressors from the environment or over-mortality can force systems past survival thresholds. The implications are broad for conservation biology and in predator ecosystems subject to anthropogenic pressures such as habitat loss and hunting.

IV.   Conclusion

This study took a predator–prey–trait model and applied it to explore the interaction of ecological dynamics and evolutionary feedback. Simulations and analyses depicted that while conventional predator–prey interactions result in oscillations, the inclusion of trait evolution guarantees convergence to stable coexistence. Predator survival is, however, bounded by a clear extinction threshold that is determined primarily by basic mortality rates. Sensitivity analysis further showed that predator density depends on mortality, predation efficacy, and prey carrying capacity.

At a larger ecological scale, these results underscore that predator persistence is not only predicated upon prey abundance but also evolutionary dynamics stabilizing dynamics. But environmental stressors or human pressures amplifying predator mortality can disrupt this stability and lead to extinction. The theory established here therefore provides theoretical light and applied significance to biodiversity conservation, underlining the relevance of accounting for both ecological and evolutionary dynamics in predator–prey management.

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