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Difference Sequences Classes Of Interval Numbers

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Abstract:
In this article we introduce and study the notions of Δ_v^s-lacunary strongly summable, Δ_v^s- Cesàro strongly summable, Δ_v^s- statistically convergent and Δ_v^s-lacunary statistically convergent sequence of interval numbers. Consequently, we construct the sequence classes $\ell_\theta^i(\Delta_v^s)$,$\sigma_1^i(\Delta_v^s)$, $s^i(\Delta_v^s)$ and $s_\theta^i(\Delta_v^s)$ respectively and investigate the relationship among these classes.
Keywords: *Sequence of interval numbers; Difference sequence; lacunary strongly summable; Cesàro strongly summable; statistically convergent; lacunary statistically convergent.*

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I. Introduction

The concept of interval arithmetic was first introduced by Dwyer [1] in 1951. It was later developed by Moore [10], Moore and Yang [13]. Furthermore, several authors have studied various aspects of the theory and applications of interval numbers in differential equations [13], [14], [15].

The sequence of interval numbers was first introduced by Chiao [20], who defined the usual convergence. Bounded and convergent sequence spaces of interval numbers were introduced by Sengonul and Eryilmaz [18] and showed that these spaces are complete metric space.

A set consisting of closed interval of real numbers x such that $a \leq x \leq b$ is called an interval number. A real interval can also be considered as a set. Denote the set of all real valued closed intervals by ℝ. Any member of ℝis called closed interval and denoted by $\overline{x}$. Thus $\overline{x} = \{x \in \mathbb{R}: a \leq x \leq b\}$. In [20], an interval number is closed subset of real line ℝ.

Let x_l and $x_r$be the first and last points of the interval number $\overline{x}$ respectively. For $\overline{x}$, $\overline{x}_2 \in \mathbb{R}$, we have

$\overline{x}_1 = \overline{x}_2 \Leftrightarrow x_{1_l} = x_{2_l}, x_{1r} = x_{2_r}$.

$\overline{x}_1 + \overline{x}_2 = \{x \in \mathbb{R}: x_{1_l} + x_{2_l} \leq x \leq x_{1r} + x_{2_r}\}$

$\alpha\overline{x} = \{x \in \mathbb{R}: \alpha x_{1_l} \leq x \leq \alpha x_{1_r}\}$ if $\alpha \geq 0$.

$= \{x \in \mathbb{R}: \alpha x_{1_r} \leq x \leq \alpha x_{1_l}\}$ if $\alpha < 0$.

and

$\overline{x}_1 . \overline{x}_2 = \{x \in \mathbb{R}: min(x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}) \leq x \leq max(x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r})\}$.The set of all interval numbers ℝ is complete metric space under the metric defined by –

$d(\overline{x}, \overline{y}) = max\{|x_{1_l} - x_{2_l}|, |x_{1_r} - x_{2_r}|\}$ (see [18]).

Let us consider the transformation $f: \mathbb{N} \to \mathbb{R}$ by $k \to f(k) = \overline{x}$ where $\overline{x} = (\overline{x}_k)$which is known as sequence of interval numbers. $\overline{x}_k$denotes the k^{th} term of the sequence $\overline{x} = (\overline{x}_k)$. The set of all sequences of interval numbers is denoted by w^i can be found in [18].

II. Definitions And Main Results

Let X be a linear metric space. A function $p: X \to R$is called paranorm if –

(1) $p(x) \geq 0$ for all $x \in X$
(2) $p(-x) = p(x)$ for all $x \in X$
(3) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$
(4) If (λ_n)be a sequence of scalars such that $\lambda_n \to 0$as $n \to \infty$ and (x_n) be a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0 \Rightarrow x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space.

Let $\phi = (\phi_n)_n$ be a sequence of Young functions i.e. $\phi_n: \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing and convex function such that $\phi_n(x) = 0$ for $x > 0$ and $\phi_n(0) = 0$. The Musielak-Orlicz sequence space ℓ^{Φ} is given by –

$\ell^{\Phi} = \{x = (x_n)_n : \sum_n \phi_n(\lambda|x_n|) < \infty, \lambda > 0\}$. This becomes Banach space under the norm (Luxemburg)

$$|x|_{\phi} = inf\left\{\eta > 0: \sum_n \phi_n\left(\frac{|x_n|}{\eta}\right) \le 1, \eta > 0\right\}$$

Let $\phi = (\phi_k)$ be the sequence of Young functions. The space consisting of all those sequences $\overline{x} = (\overline{x}_k)$ in w^i such that $\phi\left(\frac{|\overline{x}_k|^{\frac{1}{k}}}{\eta}\right) \to 0$ as $k \to \infty$ for some $\eta > 0$ is known as class of entire sequences of interval numbers defined by sequence of Young functions and is denoted by $\overline{\Gamma}_{\phi}$. The space consisting of all those sequences $\overline{x} = (\overline{x}_k)$ in w^i such that $\sup_k\left(\phi\left(\frac{|\overline{x}_k|^{\frac{1}{k}}}{\eta}\right)\right) < \infty$ for some $\eta > 0$ is known as class of analytic sequences of interval numbers defined by sequence of Young functions and is denoted by $\overline{\Lambda}_{\phi}$.

Lemma 2.1: Let (α_k) and (β_k) be sequences of real or complex numbers and (p_k) be a bounded sequence of positive real numbers, then

$$|\alpha_k + \beta_k|^{p_k} \le D(|\alpha_k|^{p_k} + |\beta_k|^{p_k})$$

and $|\lambda|^{p_k} \le max(1, |\lambda|^H)$

where D = $max(1, |\lambda|^{H-1}), H = sup p_k$, λ is any real or complex number.

Lemma 2.2: If d is translation invariant then

(a) $d(\overline{x}_k + \overline{y}_k, \overline{0}) \le d(\overline{x}_k, \overline{0}) + d(\overline{y}_k, \overline{0})$

(b) $d(\alpha\overline{x}_k, \overline{0}) \le |\alpha| d(\overline{x}_k, \overline{0})$, $|\alpha| > 1$.

Let $\overline{x} = (\overline{x}_k)$ be sequence of interval numbers, $p = (p_k)$ be sequence of strictly positive integers, $A = (a_{nk})$ be non-negative regular matrix and $\phi = (\phi_k)$ be a sequence of Young functions, we define the following classes of sequences of interval numbers as follows :

$$\overline{\Gamma}_{\phi}(A, p, \Delta_v^s) = \left\{\overline{x} = (\overline{x}_k): \lim_{k\to\infty} \sum_k a_{nk}\left[d\left(\phi\left(\frac{|\Delta_v^s \overline{x}_k|^{\frac{1}{k}}}{\eta}, 0\right)\right)\right]^{p_k} = 0\right\}$$

$$\overline{\Lambda}_{\phi}(A, p, \Delta_v^s) = \left\{\overline{x} = (\overline{x}_k): \sup_n\left(\sum_k a_{nk}\left[d\left(\phi\left(\frac{|\Delta_v^s \overline{x}_k|^{\frac{1}{k}}}{\eta}, 0\right)\right)\right]^{p_k}\right) < \infty\right\}$$

for some $\eta > 0$. We can specialize these spaces as follows:

(a) If $A = I$, the unit matrix then –

$$\overline{\Gamma}_{\phi}(I, p, \Delta_v^s) = \left\{\overline{x} = (\overline{x}_k): \lim_{k\to\infty} \left[d\left(\phi\left(\frac{|\Delta_v^s \overline{x}_k|^{\frac{1}{k}}}{\eta}, 0\right)\right)\right]^{p_k} = 0\right\}$$

$$\overline{\Lambda}_{\phi}(I, p, \Delta_v^s) = \left\{\overline{x} = (\overline{x}_k): \sup_k\left(\left[d\left(\phi\left(\frac{|\Delta_v^s \overline{x}_k|^{\frac{1}{k}}}{\eta}, 0\right)\right)\right]^{p_k}\right) < \infty\right\}$$

(b) If we take $\phi(x) = x$ then we get –

$$\overline{\Gamma}(A, p, \Delta_v^s) = \left\{\overline{x} = (\overline{x}_k): \lim_{k\to\infty} \sum_k a_{nk}\left[d\left(\frac{|\Delta_v^s \overline{x}_k|^{\frac{1}{k}}}{\eta}, 0\right)\right]^{p_k} = 0\right\}$$

$$\overline{\Lambda}(A, p, \Delta_v^s) = \left\{\overline{x} = (\overline{x}_k): \sup_n\left(\sum_k a_{nk}\left[d\left(\frac{|\Delta_v^s \overline{x}_k|^{\frac{1}{k}}}{\eta}, 0\right)\right]^{p_k}\right) < \infty\right\}$$

(c) If $A = (a_{nk})$is Cesaro matrix of order 1 and $p_k = p$ then we have -

$$\overline{\Gamma}_\phi\left(p,\Delta_v^s\right) = \left\{\overline{x} = (\overline{x}_k): \lim_{k\to\infty} \frac{1}{n}\sum_{k=1}^{n}\left[d\left(\phi\left(\frac{|\Delta_v^s\overline{x}_k|^{\frac{1}{k}}}{\eta}, 0\right)\right)\right]^p\right\} = 0$$

$$\overline{\Lambda}_\phi\left(p,\Delta_v^s\right) = \left\{\overline{x} = (\overline{x}_k): \sup_n\left(\frac{1}{n}\sum_{k=1}^{n}\left[d\left(\phi\left(\frac{|\Delta_v^s\overline{x}_k|^{\frac{1}{k}}}{\eta}, 0\right)\right)\right]^p\right) < \infty\right\}$$

The space $\overline{\Gamma}$ is defined as follows;

$$\overline{\Gamma} = \left\{\overline{x} = (\overline{x}_k): \lim_{k\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{|\Delta_v^s\overline{x}_k|^{\frac{1}{k}}}{\eta} = 0\right\} \text{ for some } \eta > 0.$$

III. Main Results

Theorem 3.1: If d is translation invariant then the class of sequence $\overline{\Gamma}_\phi(p,\Delta_v^s)$ is closed under addition and scalar multiplication of interval numbers.

Proof: Let $\overline{\boldsymbol{x}} = (\overline{\boldsymbol{x}}_{\boldsymbol{k}}) \in \overline{\boldsymbol{\Gamma}}_{\boldsymbol{\phi}}(\boldsymbol{p},\boldsymbol{\Delta_v^s})$ and $\overline{\boldsymbol{y}} = (\overline{\boldsymbol{y}}_{\boldsymbol{k}}) \in \overline{\boldsymbol{\Gamma}}_{\boldsymbol{\phi}}(\boldsymbol{p},\boldsymbol{\Delta_v^s})$
In order to prove the result, we need to find some $\eta_3 > 0$ such that

$$\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{|\Delta_v^s(a\overline{x}_k+b\overline{x}_k)|^{\frac{1}{k}}}{\eta_3}, 0\right)\right)\right]^p \to 0 \quad \text{as } k\to\infty$$

Since $\overline{\boldsymbol{x}} = (\overline{\boldsymbol{x}}_{\boldsymbol{k}}) \in \overline{\boldsymbol{\Gamma}}_{\boldsymbol{\phi}}(\boldsymbol{p},\boldsymbol{\Delta_v^s})$ and $\overline{\boldsymbol{y}} = (\overline{\boldsymbol{y}}_{\boldsymbol{k}}) \in \overline{\boldsymbol{\Gamma}}_{\boldsymbol{\phi}}(\boldsymbol{p},\boldsymbol{\Delta_v^s})$, there exists some $\eta_1 > 0$ and $\eta_2 > 0$ such that –

$$\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{|\Delta_v^s\overline{x}_k|^{\frac{1}{k}}}{\eta_1}, 0\right)\right)\right]^p \to 0 \quad \text{as } k\to\infty \text{ and}$$

$$\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{|\Delta_v^s\overline{y}_k|^{\frac{1}{k}}}{\eta_2}, 0\right)\right)\right]^p \to 0 \quad \text{as } k\to\infty.$$

Since ϕ is non-decreasing, we have

$$\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\Delta_v^s(a\overline{x}_k+b\overline{y}_k)\right|^{\frac{1}{k}}}{\eta_3}, 0\right)\right)\right]^p \le \sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{|\Delta_v^s(a\overline{x}_k)|^{\frac{1}{k}}}{\eta_3} + \frac{\left|\Delta_v^s(b\overline{y}_k)\right|^{\frac{1}{k}}}{\eta_3}, 0\right)\right)\right]^p$$

$$\le \sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{|a|^{\frac{1}{k}}|\Delta_v^s\overline{x}_k|^{\frac{1}{k}}}{\eta_3} + \frac{|b|^{\frac{1}{k}}\left|\Delta_v^s\overline{y}_k\right|^{\frac{1}{k}}}{\eta_3}, 0\right)\right)\right]^p$$

$$\le \sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{|a||\Delta_v^s\overline{x}_k|^{\frac{1}{k}}}{\eta_3} + \frac{|b|\left|\Delta_v^s\overline{y}_k\right|^{\frac{1}{k}}}{\eta_3}, 0\right)\right)\right]^p$$

Take η_3 such that

$$\frac{1}{\eta_3} = \min\left\{\frac{1}{|a|^p}\frac{1}{\eta_1}, \frac{1}{|b|^p}\frac{1}{\eta_2}\right\}$$

Then,

$$\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{|\Delta_v^s(a\overline{x}_k+b\overline{y}_k)|^{\frac{1}{k}}}{\eta_3},0\right)\right)\right]^p \le \sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{|\overline{\Delta_v^s x}_k|^{\frac{1}{k}}}{\eta_1}+\frac{|\Delta_v^s\overline{y}_k|^{\frac{1}{k}}}{\eta_2},0\right)\right)\right]^p$$

$$\le \sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\Delta_v^s|\overline{x}_k|^{\frac{1}{k}}}{\eta_1},0\right)\right)\right]^p+\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{|\Delta_v^s\overline{y}_k|^{\frac{1}{k}}}{\eta_2},0\right)\right)\right]^p$$

Hence $\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{|\Delta_v^s(a\overline{x}_k+b\overline{y}_k)|^{\frac{1}{k}}}{\eta_3},0\right)\right)\right]^p \to 0$ as $k\to\infty$.

So $a\overline{x}_k+b\overline{x}_k\in\overline{\Gamma}_\phi(p,\Delta_v^s)$. This completes the proof.

Theorem 3.2. Let $\overline{x}=(\overline{x}_k)$ be sequence of interval numbers. The sequence class $\overline{\Gamma}_\phi(A,p,\Delta_v^s)$is complete w.r.t the topology generated by the paranorm *h* defined by –

$$h(\overline{x})=\sup_k\left(\sum_{k=1}^{n}a_{nk}\left[d\left(\phi\left(\frac{|\Delta_v^s\overline{x}_k|^{\frac{1}{k}}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}}$$

Where $M=max\left\{1,\sup_k\left(\frac{p_k}{M}\right)\right\}$.

Proof. Obviously $h(\theta)=0$and $h(-\overline{x})=h(\overline{x})$**.** It can also be easily seen that $h(\overline{x}+\overline{y})\le h(\overline{x})+h(\overline{y})$ as *d* is translation invariant.

Now for any scalar λ, we have $|\lambda|^{\frac{p_k}{M}}<max(1,sup|\lambda|)$, so that $h(\lambda\overline{x})<max(1,sup|\lambda|)$, λ fixed implies $\lambda\overline{x}\to\theta$. Now let $\lambda\to\theta$, $\overline{x}$ fixed for $s\,up|\lambda|<1$, we have

$\left(\sum_{k=1}^{n}a_{nk}\left[d\left(\phi\left(\frac{|\Delta_v^s\overline{x}_k|^{\frac{1}{k}}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}}<\varepsilon$ for some $N>N(\varepsilon)$.

Also, for $1\le n\le N$ and $\left(\sum_{k=1}^{n}a_{nk}\left[d\left(\phi\left(\frac{|\Delta_v^s\overline{x}_k|^{\frac{1}{k}}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}}<\varepsilon$ there exists *m* such that

$\left(\sum_{k=m}^{n}a_{nk}\left[d\left(\phi\left(\frac{|\lambda\Delta_v^s\overline{x}_k|^{\frac{1}{k}}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}}<\varepsilon.$

Taking λ small enough, we then find

$\left(\sum_{k=m}^{n}a_{nk}\left[d\left(\phi\left(\frac{|\lambda\Delta_v^s\overline{x}_k|^{\frac{1}{k}}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}}<2\varepsilon$ for all k.

Hence $h(\lambda\overline{x})\to 0$ as $\lambda\to 0$. So, *h* is a paranorm on $\overline{\Gamma}_\phi(A,p,\Delta_v^s)$.

To show the completeness, let $\left\{\overline{x}^{(i)}\right\}$be Cauchy sequence in $\overline{\Gamma}_\phi(A,p,\Delta_v^s)$.

Then for given $\varepsilon>0$there exists positive integer *r* such that –

$\left(\sum a_{nk}\left[d\left(\phi\left(\frac{|\Delta_v^s\overline{x}_k^i-\Delta_v^s\overline{x}_k^j|^{\frac{1}{k}}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}}<\varepsilon$ for all $j\to\infty i,j\ge r$.

Since *d* is translation invariant, so

$\left(\sum a_{nk}\left[d\left(\phi\left(\frac{\left|\Delta_v^s\overline{x}_k^i-\Delta_v^s\overline{x}_k^j\right|^{\frac{1}{k}}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} < \varepsilon$ for all $i,j \geq r$. and each *n*.

Hence

$\left[d\left(\phi\left(\frac{\left|\Delta_v^s\overline{x}_k^i-\Delta_v^s\overline{x}_k^j\right|^{\frac{1}{k}}}{\eta},0\right)\right)\right] < \varepsilon$ for all $i,j \geq r$.

Therefore $\left\{\overline{x}^{(i)}\right\}$ is a Cauchy sequence in the metric space of interval numbers which is complete and hence $\overline{x}^{(j)} \to \overline{x}$ as $j \to \infty$

Keeping $r_0 \geq r$ and letting $j \to \infty$, once can find that –

$\left(\sum a_{nk}\left[d\left(\phi\left(\frac{\left|\Delta_v^s\overline{x}_k^i-\Delta_v^s\overline{x}_k\right|^{\frac{1}{k}}}{\eta},0\right)\right)\right]\right) < \varepsilon$ for all $r_0 \geq r$.

Since *d* is translation invariant, therefore

$$\left(\sum a_{nk}\left[d\left(\phi\left(\frac{\left|\Delta_v^s\overline{x}_k^i-\Delta_v^s\overline{x}_k\right|^{\frac{1}{k}}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} < \varepsilon$$

i.e $\overline{x}^{(i)} \to \overline{x}$ in $\overline{\Gamma}_\phi(A,p,\Delta_v^s)$. It can be easily seen that $\overline{x} \in \overline{\Gamma}_\phi(A,p,\Delta_v^s)$.
Thus $\overline{\Gamma}_\phi(A,p,\Delta_v^s)$ is complete. This completes the proof.

Theorem 3.3. If $0 < inf\, p_k \leq p_k \leq 1$, then $\overline{\Gamma}_\phi(A,p) \subset \overline{\Gamma}_\phi(A)$.
Proof. Let $\overline{x} = (\overline{x}_k) \in \overline{\Gamma}_\phi(A,p)$. Since $0 < inf\, p_k \leq p_k \leq 1$, the result follows from the following inequality

$$\sum_k a_{nk}\left[d\left(\phi\left(\frac{|\overline{x}_k|^{\frac{1}{k}}}{\eta},0\right)\right)\right] \leq \sum_k a_{nk}\left[d\left(\phi\left(\frac{|\overline{x}_k|^{\frac{1}{k}}}{\eta},0\right)\right)\right]^{p_k}$$

Theorem 3.4. If $1 \leq p_k \leq sup\, p_k < \infty$, then $\overline{\Gamma}_\phi(A) \subset \overline{\Gamma}_\phi(A,p)$..
Proof. $\overline{x} = (\overline{x}_k) \in \overline{\Gamma}_\phi(A)$. Since$1 \leq p_k \leq sup\, p_k < \infty$ then for each
$0 < \varepsilon < 1$ there exists a positive integer n_0 such that

$\sum_k a_{nk}\left[d\left(\phi\left(\frac{|\overline{x}_k|^{\frac{1}{k}}}{\eta},0\right)\right)\right] \leq \varepsilon < 1$ for some $n \geq n_0$.

The result follows from the following inequality

$\sum_k a_{nk}\left[d\left(\phi\left(\frac{|\overline{x}_k|^{\frac{1}{k}}}{\eta},0\right)\right)\right]^{p_k} \leq \sum_k a_{nk}\left[d\left(\phi\left(\frac{|\overline{x}_k|^{\frac{1}{k}}}{\eta},0\right)\right)\right]$.

Theorem 3.5. Suppose $\overline{x} = (\overline{x_k})$is strongly Δ_v^s-lacunary strongly summable to
*X*0. Then

$\lim_{p\to\infty}\frac{1}{h_p}\sum_{k\in I_p} d(\Delta_v^s\overline{x}_k,\overline{x}_0)$= 0.

Now the result follows from the following inequality:

$\sum_{k\in I_p} d(\Delta_v^s\overline{x}_k,\overline{x}_0) \geq \varepsilon card\{k \leq n: d(\Delta_v^s\overline{x}_k,\overline{x}_0) \geq \varepsilon\}$

Theorem 3.6. If a sequence $\overline{x} = (\overline{x_k})$of interval numbers is$\Delta_v^s$-bounded and Δ_v^s- statistically convergent, then it is Δ_v^s- Cesàro strongly summable.

Proof. Suppose $\overline{x} = (\overline{x_k})$is Δ_v^s-bounded and Δ_v^s- statistically convergent to $\overline{x}_0$. Since $\overline{x} = (\overline{x_k})$is Δ_v^s-bounded, we can find a interval number *M* such that $d(\Delta_v^s\overline{x}_k, \overline{x}_0) \leq M$ for all *k*∈*N*

Again since $\overline{x} = (\overline{x_k})$is Δ_v^s- statistically convergent to $\overline{x}_0$, for every ε > 0

$$\lim_n \frac{1}{n} card\{k \leq n : d(\Delta_v^s\overline{x}_k, \overline{x}_0) \geq \varepsilon\} = 0,$$

Now the result follows from the following inequality

$$\frac{1}{n}\sum_{1\leq k\leq n} d(\Delta_v^s\overline{x}_k, \overline{x}_0)=$$

$$\frac{1}{n}\sum_{\substack{1\leq k\leq n \\ d(\Delta_v^s\overline{x}_k,\overline{x}_0)\geq\varepsilon}} d(\Delta_v^s\overline{x}_k, \overline{x}_0)+ \quad \frac{1}{n}\sum_{\substack{1\leq k\leq n \\ d(\Delta_v^s\overline{x}_k,\overline{x}_0)<\varepsilon}} d(\Delta_v^s\overline{x}_k, \overline{x}_0)$$

$$\leq\frac{M}{n} card\{k \leq n : d(\Delta_v^s\overline{x}_k, \overline{x}_0) \geq \varepsilon\} + \varepsilon$$

Theorem 3.7. Let θ be a lacunary sequence. Then if a sequence $\overline{x} = (\overline{x_k})is\Delta_v^s$-bounded and Δ_v^s- lacunary statistically convergent, then it is Δ_v^s- lacunary strongly summable.

Proof. Proof follows by similar arguments as applied to prove above Theorem.

Theorem 3.8. If a sequence $\overline{x} = (\overline{x_k})$is Δ_v^s- statistically convergent and lim inf$_p$ $\left(\frac{h_p}{p}\right)$> 0 then it is Δ_v^s-lacunary statistically convergent.

Proof. Assume the given conditions. For a given ε > 0, we have

$$\{k \in I_p : d(\Delta_v^s\overline{x}_k, \overline{x}_0) \geq \varepsilon\}\subset\{k \leq n : d(\Delta_v^s\overline{x}_k, \overline{x}_0) \geq \varepsilon\}$$

Hence the proof follows from the following inequality:

$$\frac{1}{p} card\{k \leq p : d(\Delta_v^s\overline{x}_k, \overline{x}_0) \geq \varepsilon\}\geq\frac{1}{p} card\{k \in I_p : d(\Delta_v^s\overline{x}_k, \overline{x}_0) \geq \varepsilon\}$$

$$=\frac{h_p}{p}\frac{1}{h_p} card\{k \in I_p : d(\Delta_v^s\overline{x}_k, \overline{x}_0) \geq \varepsilon\}$$

IV. Conclusion:

In this article, we introduced and studied new notions of convergence and summability for sequences of interval numbers using the generalized difference operator Δ_v^s. The concepts of Δ_v^s-lacunary strongly summable, Δ_v^s-Cesàro strongly summable, Δ_v^s-statistically convergent, and Δ_v^s-lacunary statistically convergent sequences were defined, and the corresponding sequence spaces $l_\theta^i(\Delta_v^s)$, $\sigma_1^i(\Delta_v^s)$, $s^i(\Delta_v^s)$, and $s_\theta^i(\Delta_v^s)$were constructed. Inclusion relations among these classes were established, generalizing several known results in summability theory. The results extend existing frameworks to interval-valued sequences and provide a basis for further research in generalized difference sequence spaces.

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