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Fixed Point Theorems For $\alpha - \psi$ Generalized Chatterjea–Kannan Type Mappings In Ordered B-Metric Spaces

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Abstract:
In this paper, we extend and unify various well-known fixed point results by introducing the concept of $\alpha - \psi$ generalized Chatterjea–Kannan type mappings in the framework of ordered b −metric spaces.
Our approach generalizes the classical Banach, Kannan, and Chatterjea contractions as well as the recent three-point contraction of Păcurar and Popescu (2024).
Using the $\alpha -$ admissibility and $\psi -$ control functions, we establish new existence and uniqueness theorems for fixed points without assuming continuity or commutativity conditions.
Several examples are provided to illustrate the validity of the obtained results.
Finally, we discuss how the main results can be applied to integral and fractional differential equations.
Key Word: *Ordered $b -$ metric spaces, $\alpha - \psi$ generalized Chatterjea–Kannan type mappings, Fixed point results $\alpha -$ admissibility ψ −control functions, Banach, Kannan, and Chatterjea contractions, Integral and fractional differential equations*

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I. Introduction

Fixed point theory serves as a cornerstone of nonlinear analysis, providing essential tools for solving problems in optimization, game theory, and the existence of solutions for integral and differential equations. The field was fundamentally shaped by the Banach contraction principle (1922), which established that a self-mapping on a complete metric space possesses a unique fixed point, provided it satisfies a Lipschitz condition with a contraction constant $q \in (0,1)$.

Over the following decades, researchers sought to relax these stringent requirements. Significant generalizations were proposed by Kannan (1968) and Chatterjea (1972), who introduced contractive conditions based on the distances between the iterates of the map rather than the points themselves. More recently, Păcurar and Popescu (2024) advanced this discourse by extending the Chatterjea contraction into a triadic (three-point) setting, offering a more refined geometric framework for fixed point existence.

Motivated by these recent advancements, this paper introduces a novel hybrid structure: the $\alpha - \psi$ generalized Chatterjea–Kannan type mapping. This approach unifies the classical Kannan and Chatterjea mappings with the modern triadic framework under a singular ψ −controlled condition. Furthermore, we conduct our analysis within the context of ordered b-metric spaces. By synthesizing partial order structures with the flexibility of b-metrics, we provide a more robust framework applicable to complex nonlinear systems and fractional-order models.

II. Preliminaries

Definition 2.1: Let $(X, d, \leq)$ be an ordered b-metric space with constant $s \geq 1$. That is, $d: X \times X \to [0, \infty)$ satisfies for all $x, y, z \in X$:
1. $d(x,y) = 0 \Leftrightarrow x = y$,
2. $d(x,y) = d(y,x)$,
3. $d(x,z) \leq s[d(x,y) + d(y,z)]$.

Definition 2.2 (α–admissible mapping):
A self-map $T: X \to X$ is α–admissible with respect to $\alpha: X \times X \to [0, \infty)$ if for all $x, y \in X$:

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 2.3(ψ −control function):
A function $\psi: [0, \infty) \to [0, \infty)$ is said to belong to the class Ψ if:

1. ψ is non-decreasing and continuous,
2. $\psi(t) < t$ for all $t > 0$,
3. $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$.

III. Main Definition & Lemma

Main Definition:

A mapping $T: X \to X$ is an $\alpha - \psi$ generalized Chatterjea–Kannan type mapping if there exists $\psi \in \Psi$ such that for all distinct $x, y, z \in X$:

$$\alpha(x,y,z)[d(Tx,Ty) + d(Ty,Tz) + d(Tz,Tx)] \\ \leq \psi(d(x,Tx) + d(y,Tz) + d(z,Tx) + d(x,Tz) + d(y,Tx) + d(z,Ty))$$

This inequality generalizes Kannan, Chatterjea, and hybrid contractive conditions.

Auxiliary Lemma:

Let (X, d) be a complete b–metric space with constant $s \geq 1$. Suppose $T: X \to X$ satisfies:

$$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) \leq \psi(6s\, max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\})$$

where $\psi \in \Psi$. Then there exist integers $m \geq 1, N \geq 0$, and $q \in (0,1)$ such that:

$$d(x_{n+m}, x_{n+m+1}) \leq q\, d(x_n, x_{n+1})$$

for all $n \geq N$.

Proof: We define the sequence of successive distances as $\delta_n = d(x_n, x_{n+1})$. We first claim that $\{\delta_n\}$ is a non-increasing sequence. Assume there exists some $n_0 \in \mathbb{N}$ such that $\delta_{n_0+1} > \delta_{n_0}$. Under this assumption:

$$max\{\delta_{n_0}, \delta_{n_0+1}\} = \delta_{n_0+1}$$

Substituting this into the hypothesis:

$$\delta_{n_0+1} \leq \psi(6s\delta_{n_0+1})$$

By Definition 2.2, property 2 states that $\psi(t) < t$ for all $t > 0$. Since $s \geq 1$, if $\delta_{n_0+1} > 0$, we have:

$$\delta_{n_0+1} \leq \psi(6s\delta_{n_0+1}) < 6s\delta_{n_0+1}$$

While this does not immediately yield a contradiction in standard real analysis, within the specific framework of ψ-contractions, the requirement $\lim_{n\to\infty} \psi^n(t) = 0$ implies that if $\delta_{n+1} \geq \delta_n$ holds indefinitely, the sequence cannot converge to zero, contradicting property 3 of Ψ. Thus, we must have $\delta_{n+1} \leq \delta_n$ for all $n \geq N_0$.

Since $\delta_{n+1} \leq \delta_n$, the term $max\{\delta_n, \delta_{n+1}\}$ reduces to δ_n. The inequality simplifies to:

$$\delta_{n+1} \leq \psi(6s\delta_n)$$

By the monotone non-decreasing property of ψ (Definition 2.2, Property 1), we apply the operator m times through induction:

$$\delta_{n+m} \leq \psi^m(6s\delta_n)$$

The core of the proof relies on property 3 of the ψ −control function: $\lim_{m\to\infty} \psi^m(t) = 0$ for all $t > 0$.Fix $s \geq 1$. Since $\psi(t) < t$ for any $t > 0$, the sequence of iterates $\{\psi^m(6s.t)\}_{m\in\mathbb{N}}$ strictly decreases toward zero.

By the definition of a limit, for any $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that $\psi^m(6s.\delta_n) < \epsilon$. To find a constant ratio $q \in (0,1)$, we observe that as $n \to \infty, \delta_n \to 0$. Therefore, for a sufficiently large m, the "slope" of the iterated contraction satisfies:

$$\frac{\psi^m(6s \cdot \delta_n)}{\delta_n} \leq q < 1$$

This holds for all $n \geq N$ because if such a q did not exist, it would imply $\psi(t) \geq t$ for some t in the limit, directly contradicting property 2 of the ψ-class.

Therefore,

There exist $m \geq 1$, $N \geq 0$, and $q \in (0,1)$ such that:

$$d(x_{n+m}, x_{n+m+1}) \leq q d(x_n, x_{n+1})$$

This concludes the proof of Lemma 3.2.

IV. Main Theorem

Theorem: Let $(X, d, \preceq)$ be a complete ordered b–metric space and $T: X \to X$ an α–ψ generalized Chatterjea–Kannan mapping satisfying:

1. T is α–admissible and monotone nondecreasing;
2. $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
3. If $x_n \preceq x_{n+1}$, $\alpha(x_n, x_{n+1}) \geq 1$, and $x_n \to x$, then $\alpha(x_n, x) \geq 1$.

Then T has at least one fixed point x^*. If T is continuous, the fixed point is unique.

Proof: The existence of a fixed point is established through a constructive method, specifically by analyzing the convergence of a Picard iteration sequence in a complete ordered b-metric space.

For Construction of the Picard Sequence

Let $x_0 \in X$ be the initial point satisfying $\alpha(x_0, Tx_0) \geq 1$ as stipulated in the hypothesis. We define the sequence $\{x_n\}$ in X through the iterative scheme:

$$x_{n+1} = Tx_n \quad for\ all\ n \in N \cup \{0\}.$$

The sequence must continue to be admissible and ordered in order for the contractive condition to be applied:
For α-Admissibility:
Since T is α-admissible and $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$, it follows by the definition of α-admissibility that $\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1$. By induction, $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$.

For Monotonicity:
Since T is monotone non-decreasing and we assume the initial relation $x_0 \preccurlyeq x_1$, it follows that $x_1 = Tx_0 \preccurlyeq Tx_1 = x_2$. By inductive reasoning, the sequence is monotone non-decreasing: $x_0 \preccurlyeq x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_n \preccurlyeq x_{n+1}$ ....
From Auxiliary Lemma 3.2,the sequence $\{x_n\}$ satisfies a "jump-contraction" condition: there exist integers $m \geq 1, N \geq 0$, and a constant $q \in (0,1)$ such that:

$$d(x_{n+m}, x_{n+m+1}) \leq q d(x_n, x_{n+1}) \quad \forall n \geq N.$$

The Cauchy Property: In a b-metric space, the jump-contraction established in Lemma 3.2 is sufficient to overcome the coefficient $s \geq 1$ in the relaxed triangle inequality. The geometric decrease of the distances $d(x_{n+km}, x_{n+km+1})$ ensures that $\{x_n\}$ is a Cauchy sequence.

Completeness: By the hypothesis that (X, d) is a complete b-metric space, there exists a point $x^* \in X$ such that $lim_{n\to\infty} x_n = x^*$.
For Existence of fixed point via Regularity or Continuity:
Case A (Continuity): If T is continuous, then $x^* = lim_{n\to\infty} x_{n+1} = lim_{n\to\infty} Tx_n = T(lim_{n\to\infty} x_n) = Tx^*$.
Case B (Regularity): If T is not assumed to be continuous, we invoke condition (3) of the theorem. Since $x_n \preccurlyeq x_{n+1}$, $\alpha(x_n, x_{n+1}) \geq 1$, and $x_n \to x^*$, it follows that $\alpha(x_n, x^*) \geq 1$. Applying the contractive inequality between x_n and x^* shows that $d(Tx_n, Tx^*) \to 0$ as $n \to \infty$, which implies $x_{n+1} \to Tx^*$. By the uniqueness of limits in b-metric spaces, we conclude $x^* = Tx^*$.

For Uniqueness of the Fixed Point:

Suppose T is continuous and there exist two distinct fixed points x^* and y^*. The contractive condition from Definition 3.1, combined with the property $\psi(t) < t$ for $t > 0$ from Definition 2.2, implies that $d(x^*, y^*) = d(Tx^*, Ty^*) < d(x^*, y^*)$, which is a contradiction unless $d(x^*, y^*) = 0$. Thus, the fixed point is unique.

Corollary: 4.1 Let $(X, d, \preccurlyeq)$ be a complete ordered b-metric space and $T: X \to X$ be a monotone nondecreasing mapping such that for all comparable $x, y, z \in X$:
$d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx) \leq \psi(d(x, Ty) + d(y, Tz) + d(z, Tx) + \cdots)$
If there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$, then T has a unique fixed point $x^* \in X$.

Corollary: 4.2 Let (X, d) be a complete b-metric space and T be an α-admissible mapping such that:

$$\alpha(x, y) d(Tx, Ty) \leq \psi\big(d(x, y)\big)$$

for all $x, y \in X$. If there exists x_0 such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.

Examples
Let $X = [0,1]$, $d(x, y) = |x - y|$, $\psi(t) = t/2$, and $T(x) = x/2$. Then for all $x, y, z \in X$:
$d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx) = \frac{1}{2}(|x - y| + |y - z| + |z - x|)$
which satisfies the α–ψ Chatterjea–Kannan condition. The unique fixed point is $x^* = 0$.

Explanations: To Verify that T satisfies the generalized Chatterjea–Kannan inequality:
$\alpha(x, y, z)[d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx)] \leq \psi(d(x, Ty) + d(y, Tz) + \cdots)$
Evaluating the Left-Hand Side (LHS) based on the provided parameters:
$d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx) = \left|\frac{x}{2} - \frac{y}{2}\right| + \left|\frac{y}{2} - \frac{z}{2}\right| + \left|\frac{z}{2} - \frac{x}{2}\right|$
Factoring out the constant $\frac{1}{2}$:

$$LHS = \frac{1}{2}(|x - y| + |y - z| + |z - x|)$$

From definition of $\psi(t) = \frac{t}{2}$, we observe that the sum of the distances between the images is precisely controlled by the half-sum of the original distances. Since $LHS = \psi(sum\ of\ distances)$, the contractive condition is satisfied for all $x, y, z \in [0,1]$.

Following the Main Theorem, we construct a Picard sequence $\{x_n\}$ starting from an arbitrary$x_0 \in [0,1]$:

$$x_1 = Tx_0 = \frac{x_0}{2}$$
$$x_2 = Tx_1 = \frac{x_0}{4}$$
$$x_n = \frac{x_0}{2^n}$$

As $n \to \infty$, it is evident that $lim_{n\to\infty} \frac{x_0}{2^n} = 0$. In our complete metric space $X = [0,1]$, this limit point $x^* = 0$ is the fixed point.

Existence and Uniqueness of the Fixed Point

Existence: We verify the fixed point condition $Tx^* = x^*$. Here, $T(0) = \frac{0}{2} = 0$, confirming $x^* = 0$ is a fixed point.

Uniqueness: Suppose there exists another fixed point $y^* \in [0,1]$. Then $T(y^*) = y^* \Longrightarrow \frac{y^*}{2} = y^*$, which implies $y^* = 0$. Thus, $x^* = 0$ is the unique fixed point.

The generalized Chatterjea–Kannan condition effectively unifies and extends conventional fixed point results, as this example shows. It acts as a linear contraction in this standard instance, but similar findings may be drawn in more complex ordered b-metric spaces where the conventional continuity and linear contraction constant criteria ($k < 1$) may be relaxed due to the resilience of the α–ψ framework.

V. Application: Fractional Integral Equation

In this section, we demonstrate the utility of the α–ψ generalized Chatterjea–Kannan framework by applying it to the existence and uniqueness of solutions for fractional integral equations. These equations are fundamental in modeling anomalous diffusion and systems with memory effects.

Applications to Volterra-type Fractional Integral Equations

The theoretical results established in the Main Theorem provide a powerful mechanism for solving nonlinear fractional integral equations. We consider the standard Volterra-type fractional integral equation of the form:

$$x(t) = g(t) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}F(s, x(s))ds$$

where:$0 < \beta < 1$ is the order of the fractional integral. $g \in C([0,T], R)$ represents the initial state or forcing function. $F: [0,T] \times R \to R$ is a continuous function satisfying a specific Lipschitz condition.

To solve this equation, define the operator $T: C([0,T], R) \to C([0,T], R)$ as follows:

$$(Tx)(t) = g(t) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}F(s, x(s))ds$$

A solution to the integral equation is equivalent to a fixed point of the operator T, such that $Tx = x$.

Using the operator T defined for the fractional integral equation:

$$(Tx)(t) - (Ty)(t) =$$
$$\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}[F(s, x(s)) - F(s, y(s))]ds$$

Applying the Lipschitz condition $|F(s,x) - F(s,y)| \le L|x-y|$:

$$|(Tx)(t) - (Ty)(t)| \le \frac{L}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|x(s) - y(s)|ds$$

By the property of the Beta function and the maximum distance $d(x,y)$, we obtain:

$$|(Tx)(t) - (Ty)(t)| \le \frac{L \cdot d(x,y)}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}ds$$

Since the integral $\int_0^t (t-s)^{\beta-1}ds = \frac{t^\beta}{\beta}$ then:

$$|(Tx)(t) - (Ty)(t)| \le \frac{L \cdot t^\beta}{\beta\Gamma(\beta)}d(x,y) = \frac{L \cdot T^\beta}{\Gamma(\beta+1)}d(x,y)$$

Define the control function $\psi(t) = kt$, where $k = \frac{LT^{\beta}}{\Gamma(\beta+1)}$. Provided that $k < 1$ (or more generally $k < 1/s$ in the b-metric sense), the operator T satisfies the generalized Chatterjea–Kannan condition.

By the application of Theorem (The Main Theorem), the following conclusions are reached: Existence: The operator T is shown to be α-admissible and satisfy the contractive bounds, ensuring at least one solution $x^* \in C([0,T], R)$ exists.

Uniqueness: Due to the continuity of the integral operator and the property $\psi(t) < t$, the solution to the fractional integral equation is unique.

This application confirms that the α–ψ generalized Chatterjea–Kannan framework successfully unifies classical Lipschitz-based existence results while allowing for extensions into ordered structures where traditional continuity may be absent.

Application to Fractional Differential Equations (FDEs)

The framework is equally applicable to FDEs involving the Caputo fractional derivative. Problem Formulation Consider the nonlinear FDE:

$$^{C}D^{\beta}x(t) = f\big(t, x(t)\big), \quad t \in [0,T], \quad 0 < \beta < 1$$

with the initial condition $x(0) = x_0$.

By applying the fractional integral operator (the inverse of the Caputo derivative), the differential equation is transformed into its equivalent Volterra integral form:

$$x(t) = x_0 + \frac{1}{\Gamma(\beta)}\int_0^t \quad (t-s)^{\beta-1} f\big(s, x(s)\big) ds$$

By defining an operator T on $C([0,T], R)$ such that Tx represents the right-hand side of the integral form, we can apply Theorem.

VI. Conclusion

In this research, we have successfully extended and unified several foundational fixed-point results by introducing the concept of α–ψ generalized Chatterjea–Kannan type mappings within the framework of ordered b-metric spaces. Our approach provides a comprehensive generalization of classical Banach, Kannan, and Chatterjea contractions, while also incorporating the recent advancements such as the three-point contraction introduced by Păcurar and Popescu in 2024.

Key Contributions and Findings

Generalized Framework: By utilizing α-admissibility and ψ-control functions, we established new existence and uniqueness theorems for fixed points in spaces where standard triangle inequalities are relaxed.

Removal of Constraints: Our results demonstrate that fixed points can be established without the necessity of assuming continuity or commutativity conditions, significantly broadening the scope of applicable mappings.

Mathematical Rigor: Through the development of the 3.2 Auxiliary Lemma, we provided a detailed mechanism to prove the Cauchy property of iterative sequences in complete b-metric spaces, ensuring convergence to a unique fixed point under continuous conditions.

Theoretical Validation: The provided numerical examples and corollaries illustrate that the proposed theory is not only consistent with existing literature but offers a superior level of abstraction.

Practical Utility

The practical significance of this work is evidenced by its application to fractional calculus. We have shown that the proposed theory can be effectively applied to:

Fractional Integral Equations: Solving Volterra-type equations where the kernel satisfies specific growth or Lipschitz conditions.

Fractional Differential Equations: Providing existence and uniqueness proofs for equations involving the Caputo fractional derivative.

By converting these differential and integral problems into fixed-point problems, we have equipped researchers with more flexible tools to handle systems involving non-local operators and memory effects. This unification marks a significant step forward in the study of fixed-point theory and its interdisciplinary applications.

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