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Odd-Even Congruence Labeling Of Star Related Graphs

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Abstract:
Introduction: *In the field of graph theory, graph labeling is one of the fastest growing area. A graph labeling is defined as an assignment of integers to the nodes or edges or both based on certain conditions. Many types of graph labeling techniques have been studied by several authors.*
Objectives: *The application of labeling of graphs has diverse fields in the study of the data base management, secret sharing schemes, physical cosmology, debug circuit, X-ray, crystallography, astronomy, radar and much more.*
Methods: *This paper is based on Odd-even congruence labeling of different types of graphs. The vertices and edges are labeled by an assignment of natural numbers. It is based on modular arithmetic property known as congruence graph labelling of a graph. In congruence graph labelling vertices are assigned by odd integers and edges are assigned by even integers with the property of congruence labelling.*
Results: *This labeling method has been identified on bistar graph, path union of star graph, path union of bistar graph, subdivision of star graph.*
Keywords: *Labeling, Congruence labeling, Odd-even congruence labeling.*

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I. Introduction

In this paper all graphs G considered as finite, undirected, connected and without loops. The vertices and edges of a graph G denoted by $V(G)$ and $E(G)$ respectively. The cardinality of vertex set is denoted by $|V(G)|$ and the edge set is denoted by $|E(G)|$ are called the order and size of the graph G. For standard terminology of Graph Theory we used [1]. For all detailed survey of graph labeling, we refer [2]. While studying graph theory, one that has gained a lot of popularity during the last 62 years is the concept of labeling of graphs due to its wide range of applications. A labeling of a graph G is one-to-one mapping that carries the set of graph elements onto a set of numbers, called labels. In 1967, Rosa [5] published a pioneering paper on graph labeling problems. Thereafter many types of graph labeling methods have been studied by several authors.

Odd-even congruence labeling introduced by G.Thamizhendhi and K.Kanakambika [6] and they proved behaviour of several graphs like bipartite graph, comb graph, star graph, graph acquired by connecting two copies of even cycle C_r by a path P_t , shadow graph of the path P_t and the tensor product of $K_{1,t}$ & P_2 as Odd-even congruence graph. They defined an odd-even congruence graph, if vertex and edge set are assigned by distinct odd and even integers respectively, further $f(u_p) \equiv f(u_q) \left(mod\ g(e)\right)$, u_p and u_q are adjacent vertices in G. In this paper we investigate the existence of Odd-even congruence labeling for vertex switching graph, friendship graph, shell graph, generalized butterfly graph, fan graph, $P_2 + mK_1$ graph.

Definition 2.1 A star graph is defined as a simple graph with one central vertex connecting to all other vertices called leaves where these leaves are not connected to each other.

Definition 2.2 A bistar is a graph defined by joining the Centre vertices of two copies of by taking two copies of $K_{1,n}$ by an edge and it is denoted by $B_{n,n}$.The vertex set of $B_{n,n}$is $V(B_{n,n}) = \{v_1, v_2, v_3, , \dots \dots \dots \dots, v_n, v, u, u_1, u_2, u_3, \dots \dots \dots, u_n\}$ where v, u are apex vertices and $v_1, v_2, v_3, , \dots \dots \dots \dots, v_n, u_1, u_2, u_3, \dots \dots \dots, u_n$ are pendent vertices. The edge set of $B_{n,n}$is $E(B_{n,n}) = \{vv_1, vv_2, vv_3, , \dots \dots \dots \dots, vv_n, vu, uu_1, uu_2, uu_3, \dots \dots \dots, uu_n\}$.

Definition 2.3 The path union of a star graph denoted as $P(n. K_{1,m}^v)$ where v is a root vertex of $K_{1,m}$.

Definition 2.4 The path union of a bistar graph denoted as $P(n. B_{m,m}^v)$ where v is a root vertex of $B_{m,m}$.

Theorem 3.1 The Bistar graph $B_{n,n}$ is an Odd-even congruence graph.

Proof:
Suppose $V(B_{n,n}) = \{v_i, u_i : 1 \leq i \leq n\} \cup \{v, u\}$ be the vertex set and

$$E(B_{n,n}) = \{vv_i, uu_i : 1 \leq i \leq n\} \cup \{vu\}$$

Then $|V(B_{n,n})| = 2n + 2$
and $|E(B_{n,n})| = 2n + 1$
We have $d = \min\{(2n + 2), 2n + 1\}$

$$= 2n + 1$$

Define vertex labelling
$f: V(B_{n,n}) \rightarrow \{1,3,7,11,15, \ldots \ldots \ldots \ldots ..\}$ as
$f(v_i) = 4i - 1$, $1 \leq i \leq n$
$f(v) = 1$
$f(u) = 5$
$f(u_i) = 4i + 9$, $1 \leq i \leq n$
The induced edge labeling defined as
$f^* : E(B_{n,n}) \rightarrow \{2,6,10, \ldots \ldots \ldots \ldots \ldots \ldots ..\}$ such that
$f^*(vv_i) = 4n - 2$, $1 \leq i \leq n$

$$f^*(vu) = 4$$

$f^*(uu_i) = 4i + 4$, $1 \leq i \leq n$
We observe that
$f^*(vv_i)$ divides $|f(v) - f(v_i)|$
$f^*(vu)$ divides $|f(v) - f(u)|$ and
$f^*(uu_i)$ divides $|f(u) - f(u_i)|$
Hence bistar graph $B_{n,n}$ admits odd-even congruence labelling.

Example: 3.2.
Consider a bistar graph $B_{n,n}$ where $n = 4$.

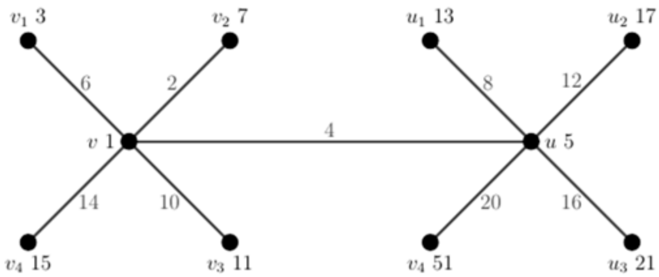

Fig - 1

Fig:1 shows that graph $B_{n,n}$ for $n = 4$ admits odd-even congruence labelling.

Example: 3.3.
Consider a bistar graph $B_{n,n}$ where $n = 5$.

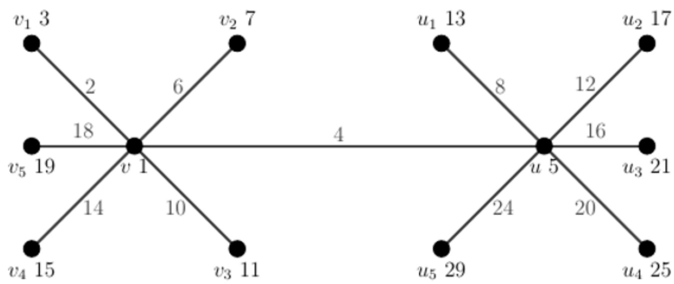

Fig - 2

Fig:2 shows that graph $B_{n,n}$ for $n = 5$ admits odd-even congruence labelling.

Theorem 3.4. The path union of star graph $P(n.K_{1,m}^{v})$ where v is a root vertex of $K_{1,m}$ is an Odd-even congruence graph.

Proof:
Suppose the vertex set of the path union of star graph $P(n.K_{1,m}^{v})$ be defined as

$$V\left(P(n.K_{1,m}^{v})\right) = \{v_i, v_i^j : 1 \le i \le n, 1 \le j \le m\}$$

and the edge set of the path union of star graph $P(n.K_{1,m}^{v})$ be defined as
$E\left(P(n.K_{1,m}^{v})\right) = \{v_i v_{i+1}, v_i v_i^j, v_n v_n^j : 1 \le i \le n-1, 1 \le j \le m\}$ such that $x_i = v_i v_{i+1}$ and $y_i^j = v_i v_i^j$ respectively.
The vertex labelling defined as
$v_1 = 1$
$f(v_{i+1}) = f(v_i) + 12i \quad 1 \le i \le n-1$
$v_1^1 = 3$
$f(v_1^{j+1}) = f(v_1^j) + 2 \quad 1 \le j \le m$
$f(v_{i+1}^1) = f(v_i^1) + 12(i+1) \quad 1 \le j \le n-1$
$f(v_i^{j+1}) = f(v_i^j) + 2 \quad 1 \le j \le m-1, 1 \le i \le n-1$
The edge labelling defined as
$f(x_i) = 12i \quad 1 \le i \le n-1$
$f(y_1^j) = 2j \quad 1 \le j \le m$
$f(y_{i+1}^j) = x_i + 2j \quad 1 \le j \le m$
It is clear that $f(x_i)$ divides $|f(v_{i+1}) - f(v_i)|$
$f(y_1^j)$ divides $|f(v_1^j) - f(v_1)| \quad 1 \le j \le m$
$f(y_{i+1}^j)$ divides $|f(v_{i+1}^j) - f(v_{i+1})| \quad 1 \le i \le n-1$
Hence, we conclude that the path union of star graph $P(n.K_{1,m}^{v})$ where v is a root vertex of $K_{1,m}$ is an Odd-even congruence graph.

Example: 3.5.
Consider the path union of star graph $P(n.K_{1,m}^{v})$ where v is a root vertex of $K_{1,m}$ where $n = 2$.

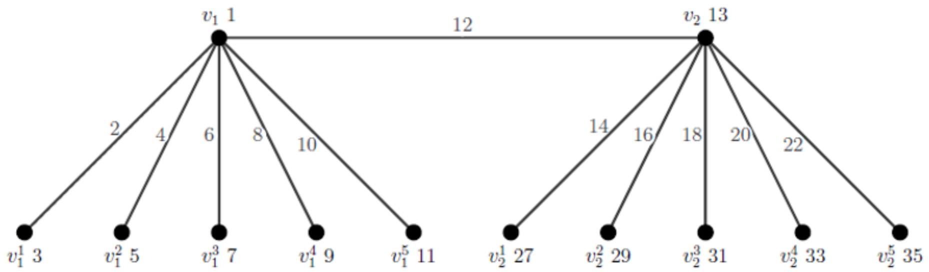

Fig.3

Fig:5.13 shows the path union of star graph $P(2.K_{1,m}^{v})$ where v is a root vertex of $K_{1,m}$ admits odd even congruence labelling.

Example: 3.6.
Consider the path union of star graph $P(n.K_{1,m}^{v})$ where v is a root vertex of $K_{1,m}$ where $n = 3$.

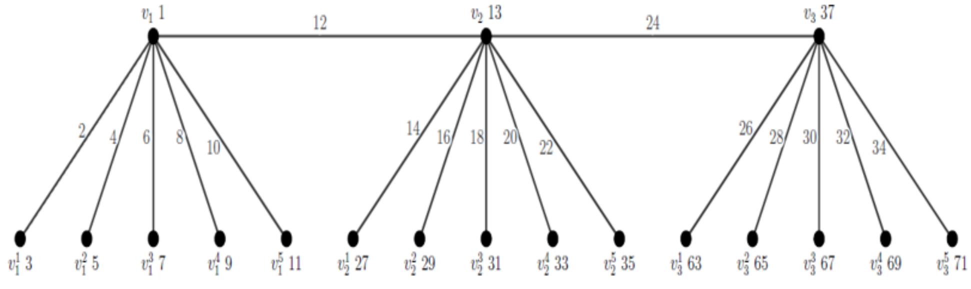

Fig.4

Fig:4 shows the path union of star graph $P(3.K_{1,m}^{v})$ where v is a root vertex of $K_{1,m}$ admits odd even congruence labelling.

Theorem 3.7. The path union of bistar graph $P(n.B_{m,m}^{v})$ where v is a root vertex of $B_{m,m}$ admits an Odd-even congruence labelling.

Proof:
Suppose the vertex set of the path union of bistar graph $P(n.B_{m,m}^{v})$ be defined as

$$V\left(P(n.B_{m,m}^{v})\right) = \{v_i, u_i, v_i^j, u_i^j : 1 \le i \le n, 1 \le j \le m\}$$

and the edge set of the path union of bistar graph $P(n.B_{m,m}^{v})$ be defined as
$E\left(P(n.B_{m,m}^{v})\right) = \{v_i v_{i+1}, v_i u_i, v_i v_i^j, u_i u_i^j, v_n u_n, v_n v_n^j, u_n u_n^j : 1 \le i \le n-1, 1 \le j \le m\}$ such that $x_i = v_i v_{i+1}$, $y_i^j = v_i v_i^j$, $z_i = v_i u_i$, $w_i^j = u_i u_i^j$ respectively.
The vertex labelling defined as
$v_1 = 1$
$f(v_{i+1}) = f(v_i) + 12i$ $\quad 1 \le i \le n-1$
$v_1^1 = 3$
$f(v_1^{j+1}) = f(v_1^j) + 2$ $\quad 1 \le j \le m$
$f(v_{i+1}^1) = f(v_i^1) + 12(i+1)$ $\quad 1 \le j \le n-1$
$f(v_i^{j+1}) = f(v_i^j) + 2$ $\quad 1 \le j \le m-1, 1 \le i \le n-1$
$u_1 = v_n^m + 2$
$f(u_{i+1}) = 2f(u_i) + 11$ $\quad 1 \le i \le n-1$
$u_i^1 = 2u_i + 1$ $\quad 1 \le i \le n$
$f(u_i^{j+1}) = f(u_i^j) + 2$ $\quad 1 \le j \le m$-1
The edge labelling defined as
$f(x_i) = 12i$ $\quad 1 \le i \le n-1$
$f(y_1^j) = 2j$ $\quad 1 \le j \le m$
$f(y_{i+1}^j) = x_i + 2j$ $\quad 1 \le j \le m$
$f(z_1) = v_n^m + 1$
$f(z_{i+1}) = 2f(z_i)$ $\quad 1 \le i \le n-1$
$f(w_i^1) = v_i + 1$ $\quad 1 \le i \le n$
$f(w_{i+1}^{j+1}) = w_i^j + 2$ $\quad 1 \le i \le n-1, \quad 1 \le j \le m-1$
It is clear that $f(x_i)$ divides $|f(v_{i+1}) - f(v_i)|$
$f(y_1^j)$ divides $|f(v_1^j) - f(v_1)|$ $\quad 1 \le j \le m$
$f(z_i)$ divides $|f(v_i) - f(u_i)|$
$f(w_i^j)$ divides $|f(u_i^j) - f(u_i)|$ $\quad 1 \le i \le n$
Hence, we conclude that the path union of bistar graph $P(n.B_{m,m}^{v})$ where v is a root vertex of $B_{m,m}$ is an Odd-even congruence graph.

Example: 3.8.
Consider the path union of bistar graph $P(n.B_{5,5}^{v})$ where v is a root vertex of $B_{m,m}$ where $n = 2$ and $m = 5$

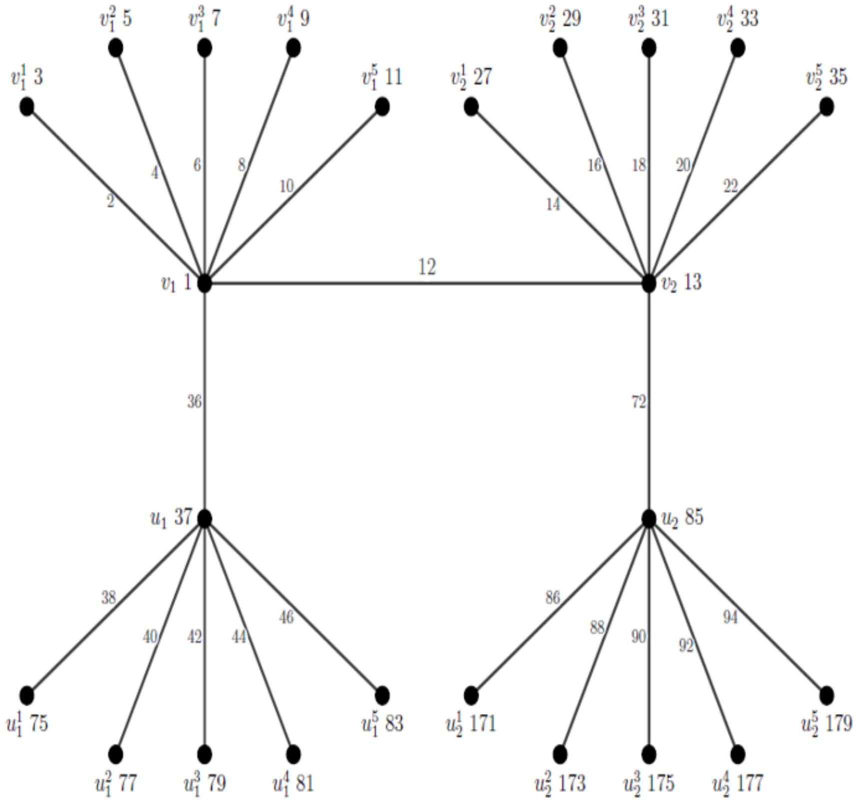

Fig. 5

Fig: 5 shows that the path union of bistar graph $P(n.B_{5,5}^{v})$ where v is a root vertex of $B_{m,m}$ where $n = 2$ and $m = 5$ admits odd-even congruence labelling.

Example: 3.9.
Consider the path union of bistar graph $P(n.B_{5,5}^{v})$ where v is a root vertex of $B_{m,m}$ where $n = 3$ and $m = 5$

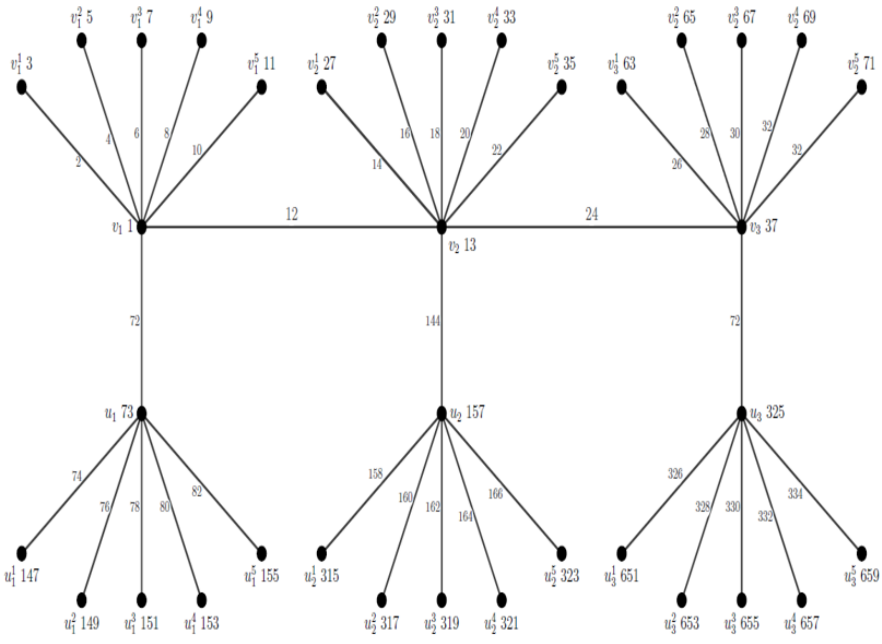

Fig.6

Fig:6 shows that the path union of bistar graph $P(n.B_{5,5}^{v})$ where v is a root vertex of $B_{m,m}$ where $n = 3$ and $m = 5$ admits odd-even congruence labelling.

Theorem:3.10. Subdivision of Star Graph $S(S_p)$ is an odd-even congruence graph.

Proof: Suppose the star image $S(S_p)$ of the vertex set be $V[S(S_p)] = \{u, u_1, u_2, u_3, \ldots \ldots \ldots, u_p, v_1, v_2, v_3, \ldots \ldots \ldots v_p\}$ and the edge set be $E[S(S_p)] = \{uu_1, uu_2, uu_3, \ldots \ldots \ldots, uu_p, u_1v_1, u_2v_2, u_3v_3, \ldots \ldots \ldots u_pv_p\}$.
Thus, $|V[S(S_p)]| = 2p + 1$ and

$$|E[S(S_p)]| = 2p$$

Then $d = \min\{2p + 1, 2p\}$

$$= 2p$$

Let $x_i = uu_p$ and $y_i = u_pv_p$
Define the bijection $f : V[S(S_p)] \to \{1,3,5,7,9, \ldots \ldots \ldots\}$ is defined as

$$f(u) = 1$$

$f(u_i) = 2i + 1$ for $1 \le i \le p$

$$f(v_1) = f(u_p) + 4$$

$f(v_{i+1}) = f(v_i) + 4$ for $1 \le i \le p - 1$
The edge labeling $g : E[S(S_p)] \to \{2,4,6,8 \ldots \ldots \ldots\}$ is defined as
$g(x_i) = 2i$ for $1 \le i \le p$

$$g(y_1) = g(x_n) + 2$$

$g(y_{i+1}) = g(y_i) + 2$ for $1 \le i \le p - 1$
Clearly
$g(x_i)$ divides $|f(u_i) - f(u)|$ for $1 \le i \le p$
$g(y_i)$ divides $|(f(v_i) - f(u_i))|$ for $1 \le i \le p$
Thus, we conclude the subdivision of Star Graph $S(S_p)$ is an odd-even congruence graph.

Example 3.11 The graph $S(S_p)$ is an odd-even congruence graph for $p = 4$.

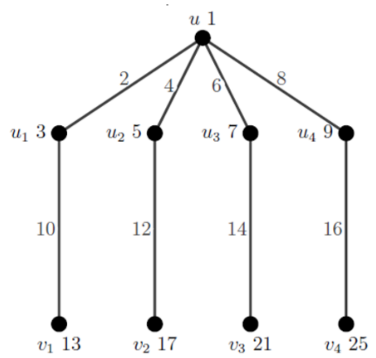

Fig-7

Fig -7, shows that $S(S_p)$ admits odd-even congruence labeling for $p = 4$
Example 3.12 The graph $S(S_p)$ is an odd-even congruence graph for $p = 5$.

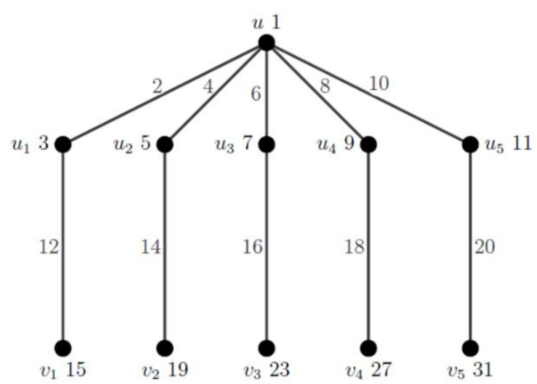

Fig-8

Fig -8, shows that $S(S_p)$ admits odd-even congruence labeling for $p = 5$

II. Conclusion

The labeling of graphs is an interesting and vast research area which is very useful and it is extended in various topics by several people. In this paper we have studied Odd-even congruence labeling behaviour of bistar graph, path union of star graph, path union of bistar graph, subdivision of star graph. To derive similar results for other graph families is an open problem.

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