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Edge Domination In Euler Totient Cayley Graph

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Abstract:
For each positive integer n, let Zn be the additive group of integers modulo n and let S be the set of all numbers less than n and relatively prime to n. The Euler totient Cayley graph G(Zn, ϕ) is defined as the graph whose vertex set V is the set Zn = { 0, 1, 2, ..., n–1} and the edge set is given by E = { (x, y) / (x –y ∈ S, or, y – x ∈ S }. In a graph G, a vertex v and an edge e in G are said to cover each other if they are incident. An edge cover of a Graph G is a set of edges covering all the vertices of G. A minimum edge cover is the one with minimum cardinality. The number of edges in a minimum edge cover of G is called the edge covering number of G is denoted by $\beta'(G)$.
In this paper we study the edge cover, edge covering number, edge dominating set, edge domination number and matching number of the Euler totient Cayley graph G(Zn, ϕ).
Keywords: *Euler totient Function, Cayley graph, Edge cover, Edge covering number, edge dominating set, edge domination number Matching number.*

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I. Introduction

Nathanson [10] was the pioneer in introducing the concepts of number theory and thus paved way for the study of a new class of graphs called Arithmetic Graphs. The theory of groups provides an interesting and powerful abstract approach to the study of symmetries of various graphs. A new class of graphs namely, Cayley Graphs can be constructed by making use of a group (X,.) and a symmetric subset S of X (a subset S of X is called a symmetric subset if s ∈ S implies $s^{-1} \in$ S). It is the graph G(X, S) whose vertex set is X and edge set {(x, y) / x,y∈X and either $xy^{-1} \in S, or, yx^{-1} \in S$ }. It is well known that [Th. 1.4.5, p 16 of 8] $G(X,S)$ is an undirected graph without loops, which is $|S^*|$ − regular having $\frac{|X||S|}{2}$ edges. We refer the reader for graph theoretic notions Bondy and Murty [5] and Harary[7] and for number theoretic notions Apostol [2]. Madhavi [9] studied Arithmetic Cayley graphs associated with quadratic residues modulo p, a prime, the Euler-Totient function $\varphi(n)$ and the divisor function $d(n)$, $n \geq 1$ an integer.

II. Euler Totient Cayley Graph And Its Properties

For any positive integer n, Let Z_n = { 0, 1, 2,…, n –1} . Then (Z_n , ⊕), where, ⊕ is addition modulo n, is an abelian group of order n. For any positive integer n, let S denote the set of all positive integers less than and relatively prime to n, that is, S = { r / 1 ≤ r < n and (r, n) = 1 }. Then | S | = ϕ(n), where ϕ is the Euler totient function. Further S is a symmetric subset of the group (Z_n, ⊕). To see this let r ∈ S. Then (r, n) =1.
Also (n – r , n) = 1 . For, if (n – r , n) = d , then d | n and d | (n – r), so that d | n – (n – r), or, d | r .
But (r, n) = 1 implies that d = 1. So n – r ∈ S. It is easy to see that S is a multiplicative subgroup of order ϕ(n) of the semi- group (Z_n^*,⊙), where $Z_n^* = Z_n - \{0\}$ and ⊙ is multiplication modulo n.

Definition 2.1: For each positive integer n, let Z_n be the additive group of integers modulo n and let S be the set of all numbers less than n and relatively prime to n. The Euler totient Cayley graph G(Z_n, ϕ) is defined as the graph whose vertex set V is the set Z_n = { 0, 1, 2, …, n–1} and the edge set is given by E = { (x, y) / (x – y ∈ S, or, y – x ∈ S } .

We present some properties of Euler totient Cayley Graphs without proofs.
The proofs can be found in [9].

(i) The graph G(Z_n, ϕ) is ϕ(n) – regular. Moreover, the number of edges in G(Z_n,ϕ) is $\frac{n\phi(n)}{2}$.

(ii) The graph G(Z_n, ϕ) is Hamiltonian and hence it is connected.
(iii) G(Z_n, ϕ) is a complete graph, if n is a prime.

(iv) For $n \geq 3$, the graph $G(Z_n, \phi)$ is Eulerian.
(v) If n is an even number, then the graph $G(Z_n, \phi)$ is a bipartite graph.
(vi) The Euler totient Cayley graph $G(Z_n,\phi)$, $n \geq 3$, can be decomposed into $\frac{\phi(n)}{2}$ edge disjoint Hamilton cycles.
(vii) The Euler Totient Cayley graphs for $G(Z_9, \phi)$, $G(Z_{10}, \phi)$ and $G(Z_{11}, \phi)$ are given below

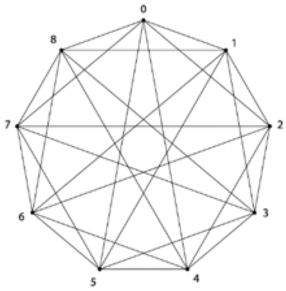


Fig.1: graph G (Z_9, ϕ)

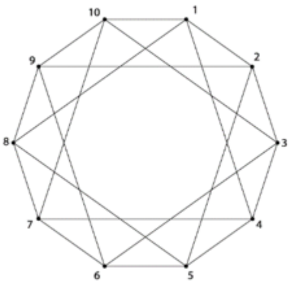


Fig.2: graph G (Z_{10}, ϕ)

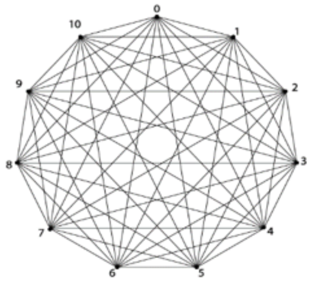


Fig.3: graph G (Z_{11}, ϕ)

III. Main Results

Definition: 3:

An **edge cover** of a graph G is a set of edges covering all the vertices of G. A **minimum edge cover** is one with minimum cardinality. The number of edges in a minimum edge cover of G is called the **edge covering number** of G and it is denoted by $\beta^1(G)$.

Theorem: 3.1:
If $n > 1$ is an integer, the minimum edge covering of the Euler totient Cayley graph $G(Z_n, \phi)$ is given by:
i) { (0,1), (2,3),……., (n–4, n–3), (n–2, n–1) } , if n is even.
ii) { (0,1), (2,3),……., (n–3, n–2), (n–1, 0) } , if n is odd.

Proof:
(i) Suppose n is an even number. Consider the set F_1 of ordered pairs of vertices given by
F_1 = { (0,1), (2,3),……, (n–4, n–3), (n–2, n–1) }.

For each ordered pair (2i, 2i+1), $0 \leq i \leq \frac{n-2}{2}$, $(2i + 1) - 2i = 1 \in S$, so that (2i, 2i+1) is an edge of $G(Z_n, \phi)$. So F_1 is a set of edges in $G(Z_n, \phi)$. Clearly, all the vertices of $G(Z_n, \phi)$ are covered by the edges in F_1 so that F_1 forms an edge covering of $G(Z_n, \phi)$. Furthermore, the end vertices of the edges in F_1 are distinct.

To show that F_1 is a minimum edge covering of $G(Z_n, \phi)$, let us consider the edge set $F_1 - \{e_i\}$, where $e_i \in F_1$. Then $e_i = (2i, 2i+1)$. Clearly, the vertices 2i, 2i+1 are not covered by the remaining edges of the edge set $F_1 - \{e_i\}$, so that $F_1 - \{e_i\}$ cannot form an edge covering of $G(Z_n, \phi)$.
Hence F_1 is the minimum edge covering of $G(Z_n, \phi)$.
Since n is even, the number of distinct pairs of distinct vertices of the form (2i, 2i+1), $0 \leq i \leq \frac{n-2}{2}$ is $\frac{n}{2}$, so that the cardinality of the set F_1 is $\frac{n}{2}$.

(ii) Suppose n is an odd number. Consider the set F_2 of ordered pairs of vertices given by F_2 = { (0,1), (2,3),….., (n–3, n–2), (n – 1, 0).

For each ordered pair (2i, 2i+1), $0 \leq i \leq \frac{n-1}{2}$ is an edge of $G(Z_n, \phi)$, since $(2i + 1) - (2i) = 1 \in S$, So F_2 is a set of edges in $G(Z_n, \phi)$. Further the edges in F_2 cover all the vertices of $G(Z_n, \phi)$, so that F_2 form an edge covering of $G(Z_n, \phi)$.

To show that F_2 is the minimum edge covering of $G(Z_n, \phi)$, let us consider the edge set $F_2 - \{ e_i\}$, where $e_i = (2i, 2i+1) \in F_2$. Clearly, the vertices 2i, 2i+1 are not covered by the edge set $F_2 - \{e_i \}$, so that F_2 is the minimum edge covering of $G(Z_n, \phi)$.

Since the n+1 vertices 0, 1, 2, …., n – 1, 0 can be paired into $\frac{n+1}{2}$ distinct pairs of vertices (2i, 2i+1),

$0 \le i \le \frac{n-1}{2}$, the cardinality of F_2 is $\frac{n+1}{2}$.

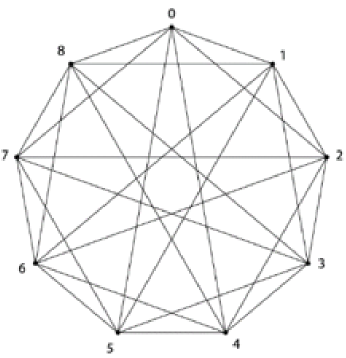
The following corollary is immediate from the above Theorem.

Corollary: 3.2:

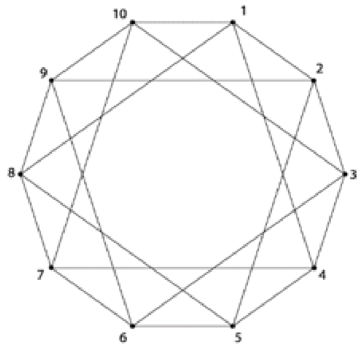
If n > 1 , the edge covering number of Euler totient Cayley graph $\beta^1(G(Z_n, \phi)$ is given by

$$\beta^1(G(Z_n, \phi) = \begin{cases} \frac{n}{2}, & \text{if n is even} \\ \frac{n+1}{2}, & \text{if n is odd.} \end{cases}$$

Example: 3.3: The minimum edge covering sets of the graphs of $G(Z_9, \phi)$, $G(Z_{10}, \phi)$ are given below



Minimum edge covering set of $G(Z_9, \phi)$



Minimum edge covering set of $G(Z_{10}, \phi)$

For the graph $G(Z_9, \phi)$, the minimum edge cover is given by { (0,1), (2,3),(4,5),(6,7),(8,0)} and the edge covering number is $5 = \frac{9+1}{2}$

For the graph $G(Z_{10}, \phi)$, the minimum edge cover is given by { (0,1), (2,3),(4,5),(6,7),(8,9)} and the edge covering number is $5 = \frac{10}{2}$

Definition 4:

A subset F of the edge set E in a graph G is called an **edge dominating set** if each edge E not in F is adjacent to at least one edge in F. The minimum cardinality among all edge dominating sets of G is called an **edge domination number** of G and is denoted by γ^1 (G).

Theorem: 4.1:

The edge dominating set of the Euler totient Cayley graph $G(Z_n, \phi)$, n > 2 is the set of edges :

(i) { (0,1), (2,3),…., (n–2, n–1) } , if n is even.

(ii) { (1,2), (3,4),…., (n–2, n–1) } , if n is odd.

Proof :

(i): Let n be even. Consider the set of ordered pairs of vertices given by

E_1 = { (0,1), (2,3),…., (n–2, n–1) }.

For each ordered pair (2i, 2i+1), $0 \le i \le \frac{n-2}{2}$, (2i+1) – (2i) = 1 $\in$ S, so that (2i, 2i+1) is an edge of $G(Z_n, \phi)$. So E_1 is the set of edges in $G(Z_n, \phi)$. Evidently no two edges in E_1 are adjacent.

Let (r, s) $\in$ E – E_1. Then r ≥ 0 and s ≠ r + 1. Here two cases will arise.

Case i:

Suppose r is even. Then r = 2t, for some integer t ≥ 0 and (r, s) = (2t, s).

Consider the edge (2t, 2t+1) is in E_1 and clearly it is adjacent with the edge (2t, s).

Case ii:

Suppose r is odd. Then r = 2t + 1, for some integer t ≥ 0 and (r, s) = (2t+1, s).

Consider then the edge (2t + 1, 2t) (which is same as (2t, 2t+1)) in E_1. Clearly this is adjacent with (2t+1, s). So E_1 is an edge dominating set of $G(Z_n, \phi)$.
Let us now show that E_1 is the minimum edge dominating set of $G(Z_n, \phi)$. To see this let us delete the edge (i , i+1) is not adjacent to any edge of the edge set E_1', since any edge (r, s) $\in E_1'$ is such that $r \neq i, i+1$ and $s \neq i, i+1$. So E_1' is not an edge dominating set of $G(Z_n, \phi)$.
Hence E_1 is the minimum edge domination set of $G(Z_n, \phi)$.

(ii) : Let n be odd. Consider the set of ordered pairs of vertices given by
$E_2 = \{ (1, 2) , (3, 4) ,\ldots, (n-2, n-1) \}$.
For each ordered pair (2i– 1, 2i) , $0 \leq i \leq \frac{n-1}{2}$, 2i – (2i–1) = 1 $\in$ S, So E_2 is the set of edges in $G(Z_n, \phi)$.
Let (r,s) $\in E - E_2$. As in (i), we may assume $r \geq 1$ and $s \neq r+1$.

case i :
Suppose r is odd. Then r = 2t +1, for some integer t > 0 and (r, s) = (2t +1, s). Then the edge (2t + 1, 2t+2) is in E_2 and this is adjacent with the edge (2t+1, s).

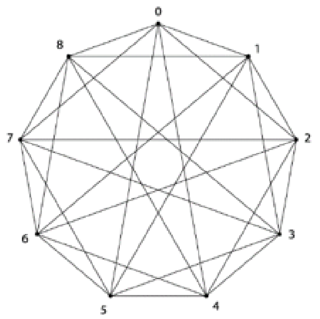
Case ii:
Suppose r is even. Let r = 2t, for some integer t > 0. Then edge (2t, s). Consider the edge (2t, 2t–1) which is same as (2t–1, 2t) in E_2 and this is adjacent with the edge (2t, s).
If r = 0, then the edge (0, s) is adjacent with (s – 1, s) in E_2 , if s is even and it is adjacent with (s, s+1) in E_2 , if s is odd. Thus, E_2 is an edge dominating set. As in (i), we can see that E_2 is the minimum edge dominating set.
The following corollary is immediate from the above Theorem.

Corollary : 4.2 :
If n > 2, the edge domination number $\gamma^1(G(Z_n, \phi))$ is given by

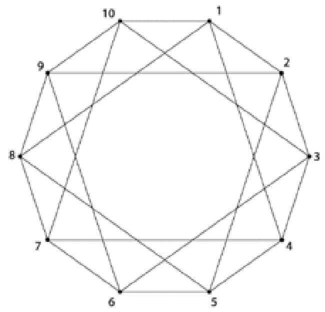
$$\gamma^1(G(Z_n, \phi)) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Proof :
If n is even, the minimum edge dominating set of $G(Z_n, \phi)$ is
$E_1 = \{ (0,1), (2,3),\ldots, (n-2, n-1) \}$
and this contains $\frac{n}{2}$ edges.
If n is odd, the minimum edge dominating set of $G(Z_n, \phi)$ is
$E_2 = \{ (1,2), (3,4),\ldots, (n-2, n-1) \}$
and this contains $\frac{n-1}{2}$ edges.

Example: 4.3: The minimum edge dominating sets of the graphs of $G(Z_9, \phi)$, $G(Z_{10}, \phi)$ are given below



Minimum edge dominating set of $G(Z_9, \phi)$



Minimum edge dominating set of $G(Z_{10}, \phi)$

For the graph $G(Z_9, \phi)$, the minimum edge dominating is given by { (1,2), (3,4),(5,6),(7,8)} and the edge domination number is $4 = \frac{9-1}{2}$

For the graph $G(Z_{10},\phi)$, the minimum edge dominating is given by { (0,1), (2,3),(4,5),(6,7),(8,9)} and the edge domination number is $5 = \frac{10}{2}$

IV. The Matching Number Of Euler Totient Cayley Graph:

Definition: 5:

A **matching** F of a graph G is a subset of E such that no two edges of F are adjacent.
A matching F is called a **perfect matching** if it covers all the vertices of the graph G.

Definition: 5. 1:

A maximal matching is a matching with maximum number of edges and the cardinality of a maximum matching is known as the **matching number.**

Theorem: 5.2 :

The matching number of Euler totient Cayley graph is

(i) $\frac{n}{2}$, if n is even.

(ii) $[\frac{n}{2}]$, if n is odd.

Proof :

(i): Let n be even. Consider the set of ordered pairs of vertices given by

F_1 = { (0,1), (2,3), …., (n–2, n–1) }. For each ordered pair (2i, 2i+1), $0 \le i \le \frac{n-2}{2}$, is an edge of $G(Z_n, \phi)$, since $(2i+1) - 2i = 1 \in S$, So F_1 is a set of edges in $G(Z_n, \phi)$. By inspection, it is easy to see that no two edges in F_1 are adjacent. So, F_1 is a matching of the graph $G(Z_n, \phi)$.

To show that F_1 is a maximal matching of $G(Z_n, \phi)$ let us consider the edge set { (0,1), (2,3),.., (n–2, n–1) } ∪ { r, s}, where $r \ge 0$ and $s \ne r+1$.

Suppose r is even. Then r = 2t , for some integer $t \ge 0$ and (r, s) = (2t, s). Then the edge (2t, 2t+1) is in F_1 and is adjacent with (2t, s).

Suppose r is odd, then r = 2t +1, for some integer $t \ge 0$ and (r,s)= (2t+1,s). Then the edge (2t +1, 2t) which is same as (2t, 2t+1) is in F_1 and is adjacent with (2t+1, s). So, the edge set { (0,1), (2,3),…., (n–2, n–1)} ∪ {r,s}, where $r \ge 0$ and $s \ne r+1$, is not a matching of the graph $G(Z_n, \phi)$.

Hence, the edge set { (0,1), (2,3), …, (n–2, n–1) } is a maximal matching of the graph $G(Z_n, \phi)$.

In this case the cardinality of a maximal matching of the graph $G(Z_n, \phi)$ is $\frac{n}{2}$.

(ii) : Let n be odd. Consider the set of ordered pairs of vertices given by

F_2 = { (0,1), (2,3), …., (n–3, n–2) }. For each ordered pair (2i, 2i+1), $0 \le i \le \frac{n-3}{2}$, is an edge of $G(Z_n, \phi)$ since $(2i+1) - 2i = 1 \in S$, So, F_2 is a set of edges in $G(Z_n, \phi)$.

By inspection, it is easy to see that no two edges in F_2 are adjacent. So, F_2 is the matching of the graph $G(Z_n, \phi)$.

To show that F_2 is a maximal matching of $G(Z_n, \phi)$ let us consider the edge set { (0,1), (2,3),…., (n–3, n–2) } ∪ {r, s} where $r \ge 0$ and $s \ne r + 1$.

Suppose r is even. Then r = 2t, for some integer $t \ge 0$ and (r, s) = (2t, s). Then the edge (2t, 2t+1) is in F_2 and is adjacent with the edge (2t, s).

Suppose r is odd. Then r = 2t +1, for some integer $t \ge 0$ and (r, s) = (2t+1, s). Then the edge (2t+1, 2t) is same as (2t, 2t+1) is in F_2 and is adjacent with (2t+1, s).

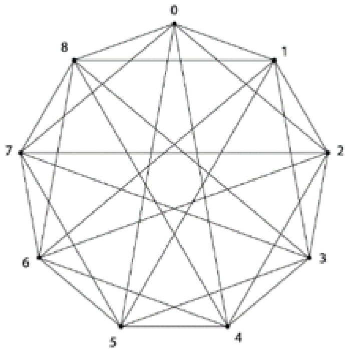
If r = n – 1 , then the edge (n-1, s) is adjacent with (s–1, s) in F_2 , if s is even and it is adjacent with (s, s+1) in F_2 , if s is odd.

So, the edge set { (0,1), (2,3),…, (n–3, n–2) } ∪ { r, s} where $r \ge 0$ and $s \ne r+1$ is not a matching of the graph $G(Z_n,\phi)$.

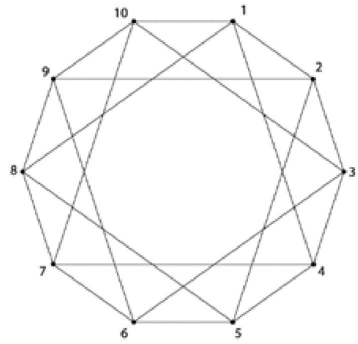
Hence the edge set { (0,1), (2,3),…., (n–3, n–2) } is a maximal matching of the graph G(Z_n, ϕ).

The cardinality of a maximal matching of the graph G(Z_n, ϕ) is [$\frac{n}{2}$].

Example: 5.3: The maximal matching sets of the graphs of G(Z_9, ϕ) , G(Z_{10}, ϕ) are given below



Maximal matching set of G(Z_9, ϕ)



Maximal matching set of G(Z_{10}, ϕ)

For the graph G(Z_9, ϕ), the Maximal matching set is given by { (0,1), (2,3),(4,5),(6,7)} and the matching number is 4 = [$\frac{9}{2}$]

For the graph G(Z_{10}, ϕ), the Maximal matching set is given by { (0,1), (2,3),(4,5),(6,7),(8,9)} and the matching number is 5 = $\frac{10}{2}$

V. Conclusion

Certain domination parameters of the Euler Totient Cayley graph relating to Edge cover, Edge domination, Matching number have been studied. Domination parameters like total edge domination number, bondage number of this graph are under study.

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