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Common Fixed Point Theorem For Compatible Mappings In G-Metric Spaces

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***Abstract**:*
In this paper, we prove common fixed point theorem for three self mappings using compatible condition in G-metric spaces.
***Key-Words**: Common fixed point, compatible mappings, G-metric spaces.*

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I. Introduction

In the metric space, fixed point theory has found its way in more general spaces. In this context Mustafa and Sims [4] generalized the concept of metric space. Based on this generalization Mustafa and Sims [3, 4, 5] and Mustafa et al. [6, 7] obtained some fixed point theorems for different mappings satisfying contractive conditions. They introduced and improved version of the generalized metric structure, which is called as G-metric spaces. In this paper, we have extended the result of Badshah et al. [1] for three self mappings.

II. Preliminaries

Definition 2.1: Let X be a non-empty set, $G: X \times X \times X \to R^+$ a function satisfying the followings conditions:
(i) $G(x, y, z) = 0$ if $x = y = z$,
(ii) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(iii) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,
(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (Symmetry in all three variables),
(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

The function G is called a generalized metric or, more specifically, a G-metric on X, and the pair (X, G) is called a G-metric spaces.

Definition 2.2: Let (X,G) be a G-metric space, $\{x_n\}$ a sequence of points in X. A sequence $\{x_n\}$ is G-convergent to x if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$; that is for each $\epsilon > 0$ there exist N such that $G(x, x_n, x_m) < \epsilon$ for all $m, n \geq N$. We say that x the limit of the sequence and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Proposition 2.3: Let (X, G) be a G-metric space. Then the following are equivalent:
(i) $\{x_n\}$ is G convergent to x,
(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,
(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$,
(iv) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Definition 2.4: Let (X, G) be G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if, for each $\epsilon > 0$ there exist an N such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$.

Proposition 2.5: In a G-metric space (X, G) the following are equivalent:
(i) The sequence $\{x_n\}$ is G-Cauchy,
(ii) for each $\epsilon > 0$ there exist an N such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$.

Proposition 2.6: Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variable.

Definition 2.7: A G- metric space $(X.G)$ is called a symmetric G-metric space if $G(x,y,y) = G(y,x,x)$ for all x,y in X.

Proposition 2.8: Every G-metric space (X,G) defines a metric space (X,d_c)
(i) $d_c(x,y) = G(x,y,y) + G(y,x,x)$ for all x,y in X.
If (X,G) is a symmetric G-metric space, then
(ii) $d_c(x,y) = 2G(x,y,y)$ for all x,y in X.
However, if (X,G) is not symmetric, then it follows from the G-metric properties that
(iii) ${}^3/_2\, G(x,y,y) \le d_c(x,y) \le 3G(x,y,y)$) for all x,y in X.

Proposition 2.9: A G-metric space (X,G) is G-complete if and only if (X,d_c) is a complete metric space.

Proposition 2.10: Let (X,G) be a G-metric space. Then, for any x,y,z,a in X it follows that:
(i) if $G(x,y,z) = 0$, then $x = y = z$,
(ii) $G(x,y,z) \le G(x,x,y) + G(x,x,z)$,
(iii) $G(x,y,y) \le 2G(y,x,x)$,
(iv) $G(x,y,z) \le 2G(x,a,z) + G(a,y,z)$,
(v) $G(x,y,z) \le {}^2/_3\{G(x,a,a) + G(y,a,a) + G(z,a,a)\}$

In 1976, Jungck [2] gave the notion of commutativity to obtain common fixed point theorems. Afterwards, in 2012, Manro et al. [3] introduced the concept of compatible maps in G-metric space.

Definition 2.11: Let f and g are mappings from G-metric space (X,G) into itself. The maps f and g are said to be compatible map if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0 \; or \; \lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) = 0$ whenever $\{x_n\}$ is sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

III. Main Result

Let (X,G) be a complete metric spaces and let f,g and t be three self mappings defined on G-metric spaces satisfying:
(i) $f(x) \cup g(x) \subseteq t(x)$, $\forall x \in X$,
(ii) one of f or t is continuous,
(iii) either (f,t) or (g,t) is compatible,
(iv) $G(fx,fy,fz) \le aG(tx,gy,gy) + bG(tx,fx,tx) + cG(tx,gz,gy) + dG(fx,gy,ty)$ (3.1)
where $a + b + c + d < 1$,
Then f,g and t will have a unique common fixed point.

Proof: Let x_0 be any arbitrary point on X and we can choose a sequence $\{y_n\}$ in X such that
$y_n = fx_n = gx_{n+1}$ and $y_{n+1} = tx_{n+2}$, $n = 0,1,2 \ldots$
From (3.1), we have

$$G(fx_n, gx_{n+1}, tx_{n+2}) \le aG(tx_n, gx_{n+1}, gx_{n+1}) + bG(tx_n, fx_n, tx_n) + cG(tx_n, gx_{n+2}, gx_{n+1}) + dG(fx_n, gx_{n+1}, tx_{n+1})$$

$$G(y_n, y_{n+1}, y_{n+2}) \le aG(y_{n-1}, y_n, y_n) + bG(y_{n-1}, y_n, y_{n-1}) + cG(y_{n-1}, y_{n+1}, y_n) + dG(y_n, y_n, y_n)$$

$$\le (a+2b)G(y_{n-1}, y_n, y_n) + cG(y_{n-1}, y_n, y_{n+1}) \qquad (3.2)$$

By rectangular inequality of G-metric space

$$G(y_{n-1}, y_n, y_{n+1}) \le G(y_{n-1}, y_n, y_n) + G(y_n, y_n, y_{n+1})$$

$$\le G(y_{n-1}, y_n, y_n) + 2G(y_n, y_{n+1}, y_{n+1}) \qquad \text{(by prop. 2.10)}$$

From inequality (3.2), we have

$$G(y_n, y_{n+1}, y_{n+2}) \le (a+2b)G(y_{n-1}, y_n, y_n) + c[G(y_{n-1}, y_n, y_n) + 2G(y_n, y_{n+1}, y_{n+1})]$$

$$\le \left(\frac{a+2b+c}{1-2c}\right) G(y_{n-1}, y_n, y_n)$$

$$\le \delta G(y_{n-1}, y_n, y_n)$$

where $\delta = \frac{a+2b+c}{1-2c} < 1$
Continuing in the same way, we have

$$G(y_n, y_{n+1}, y_{n+2}) \le \delta^n G(y_{n-1}, y_n, y_n)$$

So that for any $m > n, m, n \in N$

$$G(y_n, y_m, y_m) \leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + \cdots + G(y_{m-1}, y_m, y_m)$$
$$\leq (\delta^n + \delta^{n+1} + \cdots + \delta^{m-1}) G(y_0, y_1, y_1)$$
$$\leq \left(\frac{\delta^n}{1-\delta}\right) G(y_0, y_1, y_1)$$

Letting as $n, m \to \infty$, we have

$$\lim_{n\to\infty} G(y_n, y_m, y_m) = 0$$

Thus $\{y_n\}$ is a G-Cauchy sequence in X and since (X, G) is complete G-metric space, therefore there exist a point $u \in X$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} tx_{n+2} = u$$

Since the mapping f or t is continuous, one can assume that f is continuous, therefore

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_{n+1} = fu$$

Further, f and t are compatible, therefore

$$\lim_{n\to\infty} G(ftx_n, tfx_n, tfx_n) = 0$$

implies that $\lim_{n\to\infty} ftx_n = fu$.

Similarly as g and t are compatible therefore

$$\lim_{n\to\infty} G(gtx_n, tgx_n, tgx_n) = 0$$

implies that $\lim_{n\to\infty} gtx_n = gu$.

Consider

$$G(ffx_n, fx_n, fx_n)$$
$$\leq aG(tfx_n, gx_n, gx_n) + bG(tfx_n, ffx_n, tfx_n) + cG(tfx_n, gx_n, gx_n) + dG(ffx_n, gx_{,} tx_n)$$

Letting $n \to \infty$, we get

$$G(fu, u, u) \leq aG(fu, gu, gu) + bG(fu, fu, fu) + cG(fu, gu, gu) + dG(fu, gu, gu)$$
$$\leq (a + c + d)G(fu, gu, gu)$$

implies that

$G(fu, u, u) \leq 0$ so that $fu = u$.

Again consider,

$$G(fgx_n, fu, gu) \leq aG(tgx_n, gu, gu) + bG(fgx_n, fgx_n, tgx_n) + cG(tgx_n, gu, gu) + dG(fgx_n, gx_n, tx_n)$$

Letting $n \to \infty$, we get

$$G(gu, u, u) \leq aG(gu, gu, gu) + bG(gu, fu, tu) + cG(gu, gu, gu) + dG(fu, gu, tu)$$
$$\leq (a + d)G(fu, gu, tu)$$

Implies that $G(gu, u, u) \leq 0$ so that $gu = u$.

Similarly one can find $G(tu, u, u) \leq 0$ so that $tu = u$.

Hence $fu = gu = tu = u$. Therefore u is a common fixed point of f, g and t.

Uniqueness

Suppose that $v(\neq u)$ be another common fixed point of f, g and t. Then, $G(u, v, v) > 0$

Consider

$$G(u, v, v) = G(fu, fv, fv)$$
$$\leq aG(tu, gv, gv) + bG(tu, fu, tu) + cG(tu, gv, gv) + dG(fu, gv, tv)$$
$$\leq aG(u, v, v) + bG(u, u, u) + cG(u, v, v) + dG(u, v, v)$$
$$\leq (a + c + d)G(u, v, v)$$

$< G(u, v, v)$.

which is a contradiction, so that $u = v$.

Hence u is a unique common fixed point of f, g and t.

Example 3.1: Let $X = \{0,1,2\}$ with the G-metric on $G: X \times X \times X \to R$ defined by $G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$ for all $x, y, z \in X$. Define $f, g, t: X \to X$ by $fx = tx = 1, gx = \frac{1+x}{2}$, for all $x \in X$. Since f is continuous and $f(x) \cup g(x) \subseteq t(x)$. Also $G(fx, fy, fz) \leq aG(x, y, z)$ holds for all $x, y, x \in X$. Hence 1 is the unique common fixed point of f, g, t.

IV. Conclusion

In this paper we have proved common fixed point theorem for three self mappings in complete G-metric space using compatible condition.

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