IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-0661, p-ISSN: 2278-8727, Volume 21, Issue 5, Ser. 1 (Sept. – Oct. 2025), PP 46-51
www.iosrjournals.org

Unitary Divisor Cayley Graph And Its Basic Properties

Dr. K. Sujatha, Dr. L. Madhavi And Shaik Shahida
Associate Professor Of Mathematics, Tara Government College, Sangareddy, Sangareddy District-502001.
Professor, Department Of Applied Mathematics, Yogi Vemana University, Kadapa-516005.
Research Scholar, Department Of Applied Mathematics, Yogi Vemana University, Kadapa-516005.

Abstract:
Let be an integer and let be the set of unitary divisors of , other than . Then the set is a symmetric subset of the group, the additive abelian group of integers modulo . The Cayley graph of) associated with the above symmetric subset is called the ***unitary divisor Cayley graph*** *and it is denoted by . That is, the graph is the graph whose vertex set is and the edge set is the set of all ordered pairs of vertices such that either*
,or, .
In this paper, it is established that the graph is regular, Hamiltonian and connected. It is also studied that for what values of , the graph is Eulerian or not and bipartite or not.
Keywords: *Unitary divisor, Cayley graph, unitary divisor Cayley graph, bipartite graph, Hamilton cycle, Eulerian graph.*

Date of Submission: 11-09-2025 Date of Acceptance: 21-09-2025

I. Introduction

Nathanson [9] was the pioneer in introducing the concepts of number theory into graph theory and thus paved way for the study of a new class of graphs called Arithmetic Graphs arising by defining adjacency using various arithmetic functions. The theory of groups provides an interesting and powerful abstract approach to the study of symmetries of various graphs.

A new class of graphs namely, Cayley Graphs can be constructed by making use of a group and a symmetric subset of (a subset of is called a symmetric subset if). It is the graph , whose vertex set is and edge set . It is well known that [Th. 1.4.5, p 16 of 8] is an undirected graph without loops, which is regular having edges. The cycle structure of Cayley graphs and Unitary Cayley graphs were studied by Berrizbeitia and Guidicci [2,3] and Detzer and Guidicci [5]. Madhavi [8] studied Arithmetic Cayley graphs associated with quadratic residues modulo , a prime, the Euler-Totient function and the divisor function , an integer.

The degree of a vertex in a graph is the number of edges incident with each vertex . If degree of each vertex in is same, say , then is called regular graph. A graph is a complete graph, if every vertex is adjacent to all other vertices of the graph. A walk in a graph is an alternating sequence of vertices and edges, beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it. A walk is closed if . A closed walk in which all the edges are distinct is called a circuit. An Eulerian circuit in a graph is a circuit containing every edge of and is an Eulerian graph if it contains an Eulerian circuit.

A cycle in a graph is a sequence of distinct vertices such that are edges. It is denoted by and is called its length. A Hamilton cycle in a graph is a cycle containing every vertex of and is called a Hamiltonian graph if it contains a Hamilton cycle. A bipartite graph is a graph, whose vertex set can be partitioned into two disjoint subsets and (that is,) such that each edge has one end in and other end in .

For standard terminology and notions in graph theory, we refer Bondy and Murty [4] and Harary [7] and for number theoretic notions Apostol [1] and Eckford Cohen [6] .

II. Unitary Divisor Cayley Graph

Let be an integer. Consider the set of residue classes modulo . Since , we can as well denote . In view of this, the set is henceforth represented by , or, simply . In the abelian group , is the identity element and is the inverse of in .

Definition 2.1: Let be an integer. A divisor of which is such that , is called a **unitary divisor** of . The number of unitary divisors of is denoted by and the set of unitary divisors of is denoted by .

For example, for , the unitary divisors are and , while for , the unitary divisors are and .

In the following table the unitary divisors and their number are given for integers up to .

Unitary divisors of															

Let be the set of unitary divisors of , **other than** . The set need not be a symmetric subset of the group . For example, for , the set of unitary divisors of , other than is . Now for , its inverse in is , which is not a unitary divisor of .

However, the set is a symmetric subset of the group . Using this symmetric subset of , the unitary divisor Cayley graph is defined as follows:

Definition 2.2: Let be an integer and let . The **Unitary Divisor Cayley graph** is the graph, whose vertex set is and the edge set .

For the unitary divisor Cayley graphs are given below:

Fig: $G(Z_4, U_4)$

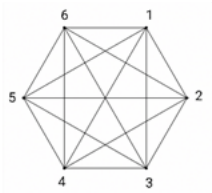


Fig: $G(Z_6, U_6)$

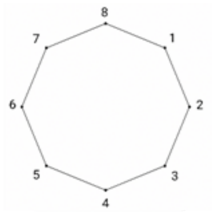


Fig: $G(Z_8, U_8)$

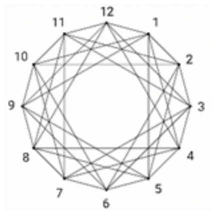


Fig: $G(Z_{12}, U_{12})$

Theorem 2.3:
(i) If , where is odd, then is odd.
(ii) If , where and is odd, then is even.
(iii) If is odd, then is even.

Proof: We know that .
Let , where is odd.

Now , since is odd. So, is a unitary divisor of and . Also , since . Pairing the elements of as , where is a unitary divisor of , except the pair , all other pairs have distinct elements of . So, the number of elements in other than is even, so that is odd.

Let , where and is odd and let be a unitary divisor of .

First we observe that must be for some odd number . For, if , where , then , since . This shows that is not a unitary divisor of , which is a contradiction. So, , for some odd number .

We claim that . For, if , then , which gives , or, . This is again a contradiction to the fact that is odd. So, the elements of can be paired into , where , so that is even.

Let be odd.

For any unitary divisor of , . This is because, if , then , so that is even and this is a contradiction to the fact that is odd.
So, the elements of can be paired into , where , so that is even.

Theorem 2.4: The graph is regular. Moreover the number of edges in is .

Proof: By the Theorem 1.4.5 (pg.16 of [8]), the Cayley graph associated with a symmetric subset of a group is -regular and contains edges. Since the graph is the Cayley graph of the group with respect to the symmetric set , it follows that the graph is regular, and contains edges.

Theorem 2.5: The graph is connected.

Proof: Clearly .
Let and be any two vertices in the graph . Then .
For definiteness, let and let .
Consider the vertices .
Since , it follows that is an edge for . This shows that

is a path connecting the vertices and . So the graph is connected.
Theorem 2.6: The graph is Hamiltonian.
Proof: Clearly , so that .
Now for any , , we have , so that is an edge of the graph .
Also . Thus

is a closed path connecting all the vertices of exactly once, so that is a Hamilton cycle of length in . Thus is Hamiltonian.

Definition 2.7: The cycle is called the outer Hamilton Cycle of the graph .

III. Properties Of The Unitary Divisor Cayley Graph

Theorem 3.1: If is a power of a prime, then the graph is the outer Hamilton Cycle .
Proof: Suppose that is a power of a prime, say .
Any divisor of other than is of the form , where .
Clearly , . So .
Hence , where .

So , which shows that is not a unitary divisor. It follows that ‘’ is the only unitary divisor of other than .

Thus . So, each vertex is of degree , and thus the graph is 2 – regular. Hence the only edges in are for and the graph is the outer Hamilton Cycle .

Example 3.2: The outer Hamilton cycle of the graph is as follows:

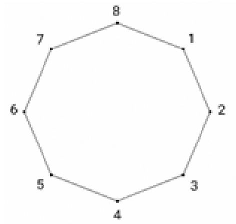


Fig: $G(Z_8, U_8)$

Theorem 3.3: The graph is a complete graph for .
Proof: If then the symmetric set and the graph is the trivial graph containing only one vertex.

Fig: $G(Z_1, U_1)$

If then the symmetric set and the graph is the graph with vertex set and the edge set . This is evidently a complete graph.

If , then the vertex set of is and . The edge set of is , which is a complete graph.

Fig: $G(Z_2, U_2)$

Fig: $G(Z_3, U_3)$

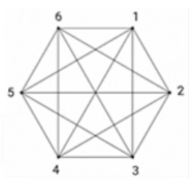


Fig: $G(Z_6, U_6)$

If , then the vertex set of is and . So the graph is a complete graph, which is given below.

Theorem 3.4: If , where is odd, then the unitary Cayley graph is **not** Eulerian.
Proof: Suppose , where is odd. Then by the Theorem 2.3 (i), is odd.

By the Theorem 2.4, the graph is - regular. So each vertex in the graph is of odd degree.
Thus by the Theorem 4.1 (pp.51&52 of [4]), the graph is **not** Eulerian.

DOI: 10.9790/5728-2105014651

Theorem 3.5:

i. If is a power of a prime, then the graph is Eulerian.
ii. If , where and is odd, then the graph is Eulerian.
iii. If is odd, then the graph is Eulerian.

Proof: By the Theorem 2.4, the graph is - regular.

i. Suppose that is a power of a prime. Then is the only unitary divisor of other than , so that . So each vertex is of degree and it is even. Hence the graph is Eulerian [4].
ii. Let , where and is odd, then by part ii of the Theorem 2.3, is even. That is, the degree of each vertex in is even, so that, the graph Eulerian [4].
iii. By part iii of the Theorem 2.3, is even for is odd. That is, the degree of each vertex in is even, so that, the graph is Eulerian[4].

Theorem 3.6: If is a power of , then the graph is a bipartite graph.

Proof: First we shall show that has no odd cycles. To see this, let be a cycle in . Then are edges in , so that for and . Since and is a prime, '' is the only unitary divisor of other than .

So, . This shows that is equal to , or, and similarly is equal to , or, . Since is a power of , it is even so that is odd. So, and must be odd for . That is, one of and is even and the other is odd for and the same is true for and .

Thus, if is even then is odd, is even and so on. Further is odd. Similarly, if is odd then is even, is odd and so on. Further is even.

This shows that half of are even and the other half are odd, so that their number is even. That is, the cycle is an even cycle and has no odd cycles.

Hence by the Theorem 1.2 (pp.14&15 of [4]), the graph is bipartite.

Example 3.7: The bipartite Graph with its bipartition , where and is as follows:

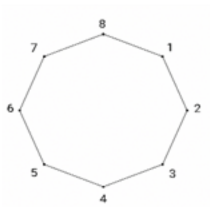
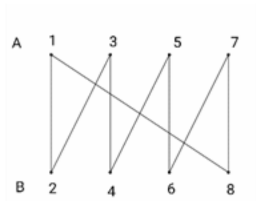


Fig: $G(Z_8, U_8)$



The Graph $G(Z_8, U_8)$ with its Bipartition (A,B)

Theorem 3.8:

i. If is odd, then the graph is not a bipartite graph.
ii. If , where is odd, then the graph is not a bipartite graph.
iii. If , where are integers and is odd, then the graph is not a bipartite graph.

Proof: For , . So is a (Hamilton) cycle of length .

i. Suppose is odd. Then the cycle is an odd cycle, so that is not bipartite by the Theorem 1.2 (pp.14&15 of [4]).
ii. Suppose , where is odd. Then , since is odd. So , so that

.

Consider the set of vertices .

Since , and are the edges in the graph . That is, is a – cycle, which is an odd cycle in . Hence is not bipartite by the Theorem 1.2 (pp.14&15 of [4]).

iii. Suppose , where are integers and is odd. is a unitary divisor of , since , as is odd.

Consider the set of vertices . Now , for , so that and are edges.

Further , a unitary divisor and hence is in .

So is a cycle of length , which is odd. Hence the graph contains an odd cycle, so that it is not bipartite again by the Theorem 1.2 (pp.14&15 of [4]).

Example 3.9: The graph given below is not a bipartite graph, since it has the outer Hamilton cycle of length , which is odd.

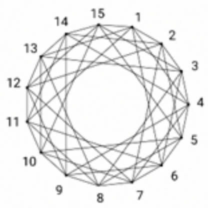


Fig: (G_{15}, U_{15})

IV. Conclusion:

The domination parameters and the metric properties of this graph are under study.

Acknowledgement: The authors thank Prof. L. Nagamuni Reddy for his suggestions during the preparation of this paper.

References

[1] Apostol, Tom M., Introduction To Analytic Number Theory, Springer International Student Edition, (1989).
[2] Berrizbeitia, P. And Giudici, R.E., Counting Pure -Cycles In Sequences Of Cayley Graphs, Discrete Math., 149(1996), 11-18.
[3] Berrizbeitia, P. And Giudici, R.E., On Cycles In Sequences Of Unitary Cayley Graphs, To Appear. (Reporte Technico No. 01-95, Universidad Simon Bolivar, Dpto. De Mathematicas) Caracas, Venezula, 1995.
[4] Bondy, J.A. And Murty, U.S.R., Graph Theory With Applications, Macmillan, London (1976).
[5] Dejter, I. And Giudici, R.E., On Unitary Cayley Graphs. JCMCC, 18, 1995, 121-124.
[6] Eckford Cohen, Arithmetical Functions Associated With The Unitary Divisor Of An Integer, Mathematische Zeitsehrift, Vol.74(1960), Pp.66-80.
[7] Harary, F., Graph Theory, Addison Wesley, Reading Mass (1969).
[8] Madhavi, L., Studies On Domination Parameters And Enumeration Of Cycles In Some Arithmetic Graphs, Ph.D., Thesis, Sri Venkateswara University, Tirupati, December 2002.
[9] Nathanson, B. Melvyn., Connected Components Of Arithmetic Graphs, Monat. Fur. Math, 29(1980).