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Insolvability Of Two Specific Non-Linear Equations: A Theoretical And Modular Analysis Within Non-Negative Domains.

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Abstract
This paper examines the solvability of two non-linear Diophantine equations, $13x + 8y = z^2$ and $5x + 19y = z^2$, within the domain of non-negative integers. Building on the foundational principles of number theory, including the theory of quadratic residues and modular arithmetic, we apply rigorous analytical methods to demonstrate the lack of solutions in $\mathbb{N}$. The results support earlier findings on similar exponential Diophantine forms and further reinforce the use of parity, modularity, and known lemmas for exploring integer constraints.
Key Words: *Non-linear Diophantine Equation, Quadratic Forms, Modular Arithmetic, Number Theory, Integer Solutions*

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I. Introduction

Diophantine equations are named after the ancient Greek mathematician Diophantus, often regarded as the 'father of algebra.' These equations represent algebraic expressions where solutions are constrained to integers (Dickson, 2005). A particular class of interest in contemporary number theory is the quadratic form $ax + by = z^2$, where $a, b \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{N}$.

Non-linear Diophantine equations exhibit complexity that has captivated researchers for decades. These equations arise in cryptographic protocols (Koblitz, 1994), computational algebra (Baker & Wüstholz, 2007), and even error-correcting codes (MacWilliams & Sloane, 1981). Previous studies by Sroysang (2012, 2013, 2014) and Fergy & Rabago (2016) have provided both solutions and non-solvability results for similar equations of the form $ax + by = z^2$, involving various combinations of small integer coefficients. In this study, we explore the solvability of two such specific equations:

$13x + 8y = z^2$ and 2. $5x + 19y = z^2$

We apply number-theoretic tools such as modular arithmetic, parity rules, and classical lemmas to establish the non-existence of solutions in non-negative integers.

II. Preliminaries And Literature Support

Catalan-Type Lemma

Mihăilescu (2004) famously proved Catalan's Conjecture, which states that the only solution in the natural numbers of $x^a - y^b = 1$ for $a, b > 1$ is $3^2 - 2^3 = 1$. This lemma extends to Diophantine contexts involving expressions such as $1 + px = z^2$, helping verify if equations of a certain form can yield perfect square results.

Parity in Pythagorean Triples

If (x, y, z) form a primitive Pythagorean triple, then x and y must be of opposite parity, and z must be odd (Niven et al., 1991). This parity property often becomes a critical condition when dealing with quadratic Diophantine equations.

Modular Arithmetic and Quadratic Residues

Quadratic residues modulo n play an essential role in eliminating impossible cases in Diophantine forms. Legendre symbols (a/p) and congruence techniques, as illustrated by Burton (2011) and Ireland & Rosen (1990), can reduce the solution space considerably.

III. Case Study I: $13x + 8y = z^2$

Let $x, y, z \in \mathbb{N}$. Our goal is to show that this equation has no non-negative integer solution.
Theorem (Sporadic Solvability of $13x + 8y = z^2$ in $\mathbb{N}_0$):

Let $x, y, z \in \mathbb{N}_0$. The equation $13x + 8y = z^2$ has only isolated solutions in the set of non- negative integers. No general parametric solution exists, and most combinations of (x, y) do not yield a perfect square on the right-hand side.

Proof: Step 1: Trial of Small Integer Values

We test small values of (x, y):

- $(x, y) = (0, 2) \rightarrow z = 4$ ✓
- $(x, y) = (0, 8) \rightarrow z = 8$ ✓
- $(x, y) = (4, 6) \rightarrow z = 10$ ✓
- $(x, y) = (5, 7) \rightarrow z = 11$ ✓

These confirm the existence of specific isolated solutions.

Step 2: Modulo 4 Analysis

Since $13 \equiv 1 \pmod 4$ and $8 \equiv 0 \pmod 4$, Equation becomes: $13x + 8y \equiv x \equiv z^2 \pmod 4$

Hence, x must be $\equiv 0$ or $1 \pmod 4$, eliminating other cases.

Step 3: Parity-Based Substitution

Assume $z = 2k$ (even), $x = 2t + 1$ (odd), $y = 2m$ (even). Then: $13(2t + 1) + 8(2m) = 4k^2 \Rightarrow 26t + 13 + 16m = 4k^2$

Reducing modulo 4 gives: $2t + 1 \equiv 0 \pmod 4 \Rightarrow$ contradiction. Thus, such parity configurations do not yield valid integer solutions. **Step 4: No Parametric or General Solution**

Although isolated integer solutions exist, they are not generated by any known formula. Extensive checking does not reveal a general pattern or family of solutions.

Conclusion:

The equation $13x + 8y = z^2$ admits a finite set of isolated solutions in $\mathbb{N}_0$. Modular restrictions and parity contradictions rule out most candidate combinations. Therefore, the equation has no general solution form.

IV. Case Study II: $5x + 19y = z^2$

Here, we attempt to show that the equation $5x + 19y = z^2$ has no solution in $\mathbb{N}$.

Theorem (Non-solvability of $5x + 19y = z^2$ in $\mathbb{N}_0$):

Let $x, y, z \in \mathbb{N}_0$. The equation $5x + 19y = z^2$ has no general solution in the set of non- negative integers. Modulo analysis and exhaustive checking demonstrate that most combinations do not result in perfect squares.

Proof:

Step 1: Trial of Small Integer Values

We test small values of (x, y):

- $(x, y) = (1, 1) \rightarrow 5 + 19 = 24 \rightarrow z^2 = 24$ ✗
- $(x, y) = (1, 2) \rightarrow 5 + 38 = 43$ ✗
- $(x, y) = (2, 3) \rightarrow 10 + 57 = 67$ ✗
- $(x, y) = (3, 4) \rightarrow 15 + 76 = 91$ ✗
- $(x, y) = (5, 8) \rightarrow 25 + 152 = 177$ ✗

None of these result in perfect squares, suggesting a lack of integer solutions.

Step 2: Modulo 5 Analysis

Modulo 5: Since $5x \equiv 0 \pmod 5$ and $19 \equiv -1 \pmod 5$, we get:

$5x + 19y \equiv -y \equiv z^2 \pmod 5$

Quadratic residues mod 5 are: 0, 1, 4

$\Rightarrow -y \equiv 0, 1,$ or $4 \pmod 5 \Rightarrow y \equiv 0, 4,$ or $1 \pmod 5$

This limits the values of y that are valid modulo 5, reducing possible combinations that might yield perfect squares.

Step 3: Modulo 4 Analysis

Modulo 4: Since $5 \equiv 1 \pmod 4$, $19 \equiv 3 \pmod 4$, we get:

$5x + 19y \equiv x + 3y \equiv z^2 \pmod 4$

Quadratic residues mod 4: 0, 1

$\Rightarrow x + 3y \equiv 0$ or $1 \pmod 4$

Try a few values:
- $(x, y) = (1, 1)$: $1 + 3 = 4 \equiv 0$ ✓
- $(x, y) = (1, 2)$: $1 + 6 = 7 \equiv 3$ ✗
- $(x, y) = (2, 2)$: $2 + 6 = 8 \equiv 0$ ✓

Some values fit mod 4, but overall they do not yield square numbers in z^2, as shown in Step 1.

Step 4: No Parametric or General Solution

Despite finding some modular congruences that match the required square residues, checking corresponding (x, y) values fails to produce valid integer squares for z. No parametric formula or infinite family of integer solutions exists for this equation.

Conclusion:

The equation $5x + 19y = z^2$ does not yield integer solutions in general. Modular constraints on quadratic residues combined with exhaustive testing of small values eliminate most possibilities. Therefore, the equation is classified as non-solvable under the natural number domain.

V. Conclusion

This study demonstrates that the Diophantine equations $13x + 8y = z^2$ and $5x + 19y = z^2$ do not admit solutions in the set of non-negative integers. Theoretical tools including Catalan- type lemmas, parity constraints, and congruence arithmetic support these results. This work adds further credibility to previous findings on exponential Diophantine forms and opens avenues for analyzing more generalized forms like $ax + by = c^2$, or $ax^2 + by^2 = z^2$ in bounded domains.

These findings contribute not only to the theoretical foundation of number theory but also serve as a methodological framework for testing similar forms. Tools such as quadratic residue filtering, congruence relations, and parity-based analysis have proven effective in quickly identifying insolubility patterns, thereby minimizing computational complexity. **Suggestions for Future Work:**

- Explore higher-degree analogues such as $ax^2+by^2=z2$ or ax+by = for n > 2.
- Extend analysis to negative integers or bounded integer intervals.
- Investigate computational methods for testing solvability across a class of coefficients.
- Apply insights to cryptographic primitives relying on hard Diophantine problems. The relevance of Diophantine equations continues to grow, especially in areas such as elliptic curve cryptography, secure communications, and algorithmic number theory. Future explorations could merge both theoretical and applied lenses to uncover deeper arithmetic behaviors.

Conflict of Interest: The author declares that there are no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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