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Performance Analysis Of A Fuzzy Markovian Queueing System With A Priori Impatience By The L-R Method

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Abstract:
This article aims to analyze the performance of the fuzzy Markovian queueing system FM/FM/1 with a priori impatience using the L-R method. Kolmogorov's differential equations, also known as equilibrium or balance equations, will allow us to obtain the fuzzy state probabilities of the system in steady state, considered as probabilistic indicators of the system's performance. The recursive method will help us achieve this. We will then use these fuzzy state probabilities to determine the system's performance indicators. By performance indicators, we cite the fuzzy server utilization rate, the fuzzy system throughput, the fuzzy average number of customers in the system, the fuzzy average number of customers in the queue, the fuzzy average dwell time, and the fuzzy average time in the queue. The originality of this article lies in the fact that we evaluate the performance of the system using the mathematical approach of fuzzy numbers of the L-R type. This is an approach based solely on the arithmetic of L-R fuzzy numbers. Since the calculation of the product and the quotient is made possible by the use of two approximations, the tangent approximation and the secant approximation, in this article, only the secant approximation will be used because it provides the same results as other methods regarding both supports and modes. This approach appears short and efficient when the fuzzy variables defining the system are fuzzy numbers of the same L-R type, and it provides exact values of the supports and modes of the performance indicators. A numerical application is discussed to illustrate the validity and practicability of this approach.

Key Word: *Fuzzy Markovian Queueing System, Performance, A Priori Impatience, L-R Method, Fuzzy State Probabilities, Recursive Method*

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I. Introduction

Nowadays, with the rapid development of science and technology, machines are gradually replacing human workers. Examples include ATMs, robots at intersections, etc.

Queueing theory and Markov processes have laid a solid foundation for the management of service operation and production systems and have attracted widespread attention. However, in practice, classical models are difficult to implement on a large scale in real production systems due to their lack of flexibility.

Therefore, researching a new queueing model that takes customer impatience into account within the system is of paramount importance. This theory was developed to provide models for predicting the behavior of systems attempting to provide service for randomly occurring requests [1].

Queues are phenomena we encounter daily [2]. Much research has been conducted on queueing systems to improve service quality, that is, to better manage customers (if they are people waiting) or requests (if they are processing computer data).

In studying queueing systems, the main objective is to understand the situations that arise in their management [3]. Queue systems with impatient customers are studied along several dimensions. The concept of impatience was proposed by [4], [5] and [6]. To characterize impatient customer behavior, in the literature, there are three terminologies used in the queueing system, namely, balking, defined as the decision not to join the queue at all; reneging, defined as joining a queue but leaving without being served; and recall or return or feedback, defined as a customer dissatisfied with the quality of service who decides to leave the queue to request or complete the service after a random time [3]. The queueing model proposed by this article is applicable to a variety of fields, including manufacturing, production, communication, transportation and networks, restaurants, bank or retail counters, ATMs, parks, and others.

In this article, hesitation behavior, otherwise known as a priori impatience, is considered. Ancker, [7] conducted a brief investigation of consumer impatient behavior in various queueing situations. The analysis of the M/M/1 model of finite queue length with customers who refuse and give up is carried out by [8]. Customer impatience behavior is carried out by [9] in multi-server Markov queues where impatience depends on the state. The M/M/2/N queue with refusal, refusal, and Bernoulli feedback is analyzed by [10], where the customer who refuses can remain in the system. In the following, we use only triangular fuzzy numbers and L-R fuzzy numbers.

Furthermore, impatience is the most important characteristic when customers want to receive a service that requires them to queue. In fact, in real life, we always feel anxious and impatient when waiting for a service. Unfortunately, many researchers ignore the impatience factor when studying queueing systems in order to accurately model reality.

In many realistic situations, statistical information is represented in linguistic language with qualitative data such as high, medium, low, and others, rather than by a probability distribution. Thus, fuzzy queueing systems that can adapt to the ambiguities of human language and logic in the real world are more practical than classical queueing systems. In turn, fuzzy Markovian queueing systems with impatient customers are more practical than ordinary Markovian queueing systems with impatient customers. In classical queueing theory, for example, it is generally assumed that the time between two successive arrivals and the service time follow a specific probability distribution, whereas in the real world, this type of information is expressed in words [11] and [12]

In this paper, we propose the use of a scientific approach called "L – R method". It is an approach for calculating the performance indicators of a fuzzy queueing system. It is essentially based on the arithmetic of fuzzy numbers of type L – R. Since multiplication and division are open operations on fuzzy numbers of type L – R, the calculation of the product and the quotient of these fuzzy numbers is made possible by the use of two approximations called tangent approximation and secant approximation. Among the two, the secant approximation seems the best because it gives the same results as those obtained by other methods with regard to both supports and modes. Thus, in this work, the L – R method will always be restricted to the secant approximation because it gives exact values of the performance indicators. However, the L – R method is designed on the basis of expressing these performance indicators using supports and modal values. This is a short and efficient method when the fuzzy variables that define the system are fuzzy numbers of the same type L-R. The most commonly used fuzzy numbers for this type of study are triangular and trapezoidal fuzzy numbers [13], but we will only use triangular fuzzy numbers.

Our fundamental problem in this article revolves around the following questions:

Can the mathematical approach of L-R fuzzy numbers facilitate the performance analysis of the Markovian fuzzy queueing system FM/FM/1 with a priori impatience?

What are the performance indicators that can optimize management while reducing waiting times in this system? How can they be calculated?

Provisional answers to these questions can be formulated as follows:

First, the mathematical approach of L-R fuzzy numbers and its arithmetic would be indispensable for analyzing the performance of this system simply because no parametric nonlinear program is required in its application, not to mention its weaknesses related to the fact that it is only applicable if the parameters of the queueing system are fuzzy numbers of the same L-R type.

Second, optimizing management by reducing waiting times in the system would be possible by evaluating performance indicators using mathematical tools such as convergent geometric series, fuzzy Kolmogorov equations, the fuzzy Little formula, the recursive method, fuzzy Markov chains, the L-R fuzzy number approach and its arithmetic.

The remainder of this article is organized as follows: Section 2 presents some preliminaries necessary for L-R fuzzy number arithmetic. Section 3 reviews the L-R method. Section 4 analyzes the performance of the FM/FM/1 queueing system with a priori impatience. Section 5 discusses a numerical application. Section 6 concludes the study.

II. Material And Methods

Preliminaries

Fuzzy Subset

Definition 1: [14], [15] and [13], consider X a reference set or a universe. A fuzzy subset $\tilde{A}$ (Fuzzy set) of X is defined by the membership function $\mu_{\tilde{A}}(x) : X \to [0,1]$. We denote

$$\tilde{\boldsymbol{A}} = \{(\boldsymbol{x}, \boldsymbol{\mu}_{\tilde{\boldsymbol{A}}}(\boldsymbol{x}) \;/\; \boldsymbol{x} \in \boldsymbol{X})\} \qquad (1)$$

Definition 2: [16], [17], [18] and [19], Consider $\tilde{A}$ a fuzzy subset in the universe X. The support $Supp(\tilde{A})$, the height $h(\tilde{A})$, the kernel $Noy(\tilde{A})$ and the alpha-cut $\tilde{A}_\alpha$ are numbers defined respectively $\forall\, \alpha \in [0,1]$ by:

$$\boldsymbol{Supp}(\tilde{\boldsymbol{A}}) = \{\boldsymbol{x} \in \boldsymbol{X} \;/\; \boldsymbol{\mu}_{\tilde{\boldsymbol{A}}}(\boldsymbol{x}) > \boldsymbol{0}\} \qquad (2)$$

$$\boldsymbol{h}(\tilde{\boldsymbol{A}}) = \boldsymbol{Sup}\{\, \boldsymbol{\mu}_{\tilde{\boldsymbol{A}}}(\boldsymbol{x}) \;/\; \boldsymbol{x} \in \boldsymbol{X}\} \qquad (3)$$

$$\boldsymbol{Noy}(\tilde{\boldsymbol{A}}) = \{\boldsymbol{x} \in \boldsymbol{A} / \boldsymbol{\mu}_{\tilde{\boldsymbol{A}}}(\boldsymbol{x}) = \boldsymbol{1}\} \qquad (4)$$

$$\tilde{\boldsymbol{A}}_{\boldsymbol{\alpha}} = \{\boldsymbol{x} \in \boldsymbol{X} \;/\; \boldsymbol{\mu}_{\tilde{\boldsymbol{A}}}(\boldsymbol{x}) \geq \boldsymbol{\alpha}\} \qquad (5)$$

According to [20], the membership function of a fuzzy subset $\tilde{A}$ can be expressed in terms of the characteristic functions of its alpha-cuts by the relation

$$\boldsymbol{\mu}_{\tilde{\boldsymbol{A}}}(\boldsymbol{x}) = \underset{\boldsymbol{\alpha} \in [\boldsymbol{0},\boldsymbol{1}]}{\mathbf{Sup}}\ \mathbf{min}\{\boldsymbol{\alpha}, \boldsymbol{\mu}_{\tilde{\boldsymbol{A}}_{\boldsymbol{\alpha}}}(\boldsymbol{x})\} \qquad (6)$$

where

$$\mu_{\tilde{A}_\alpha}(x) = \begin{cases} 1 & si \quad x \in \tilde{A}_\alpha \\ 0 & sinon \end{cases}$$

Definition 3: [21], [17], [19] and [22]; A fuzzy subset $\tilde{A}$ is said to be normal or normalized if and only if $h(\tilde{A}) = 1$ and is convex if and only if

$$\forall \boldsymbol{x_1}, \boldsymbol{x_2} \in \boldsymbol{X}\ \boldsymbol{et}\ \boldsymbol{\lambda} \in [\boldsymbol{0}; \boldsymbol{1}],\ \ \boldsymbol{\mu}_{\tilde{\boldsymbol{A}}}[\boldsymbol{\lambda x_1} + (\boldsymbol{1} - \boldsymbol{\lambda})\boldsymbol{x_2}] \geq \boldsymbol{min}[\boldsymbol{\mu}_{\tilde{\boldsymbol{A}}}(\boldsymbol{x_1}), \boldsymbol{\mu}_{\tilde{\boldsymbol{A}}}(\boldsymbol{x_2})] \qquad (7)$$

Fuzzy Number

Definition 4: [18], A fuzzy subset $\tilde{A}$ defined over the universe $\mathbb{R}$ of real numbers is a fuzzy number if its membership function $\mu_{\tilde{A}}(x) : \mathbb{R} \to [0,1]$ satisfies the following conditions:

1. $\mu_{\tilde{A}}$ is convex;
2. $\mu_{\tilde{A}}$ is normal, i.e., there exists an $x \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x) = 1$;
3. $\tilde{A}$ is upper semicontinuous;
4. $Supp(\tilde{A})$ is bounded in $\mathbb{R}$.

Definition 5: [23], A fuzzy number $\tilde{A}$ is said to be positive (resp. negative) if, and only if, $\mu_{\tilde{A}}(x) = 0, \forall x < 0$ *(resp.* $\mu_{\tilde{A}}(x) = 0, \forall x > 0$*)*. This is denoted $\tilde{A} > 0$ *(resp.* $\tilde{A} < 0$*)*;

If $\tilde{A}$ is a fuzzy interval, any real number m such that $\mu_{\tilde{A}}(m) = 1$ is said to be a modal value or mode or even a mean value of $\tilde{A}$. In this case, $Noy(\tilde{A})$ is the set of modal values of A˜.

Definition 6: [20], Consider $\tilde{A}$ and $\tilde{B}$ two fuzzy numbers. $\tilde{A}$ is strictly less than $\tilde{B}$ if, and only if,

$$\forall\ x\ \in supp(\tilde{A}), \forall\ y \in supp(\tilde{B}); \qquad x < y$$

We denote

$$\tilde{\boldsymbol{A}} < \tilde{\boldsymbol{B}} \Leftrightarrow \boldsymbol{sup}\{\boldsymbol{supp}(\tilde{\boldsymbol{A}})\} < \boldsymbol{inf}\{\boldsymbol{supp}(\tilde{\boldsymbol{B}})\} \qquad (8)$$

Definition 7: [24] and [25], A fuzzy number $\tilde{A}$, with membership function $\mu_{\tilde{A}}$, is said to be ***triangular***, if there exist three real numbers $a < b < c$ such that:

$$\boldsymbol{\mu}_{\tilde{\boldsymbol{A}}}(\boldsymbol{x}) = \begin{cases} (\boldsymbol{x} - \boldsymbol{a})/(\boldsymbol{b} - \boldsymbol{a}) & \boldsymbol{if} \quad \boldsymbol{a} \leq \boldsymbol{x} \leq \boldsymbol{b} \\ (\boldsymbol{c} - \boldsymbol{x})/(\boldsymbol{c} - \boldsymbol{b}) & \boldsymbol{if} \quad \boldsymbol{b} < \boldsymbol{x} \leq \boldsymbol{c} \\ \boldsymbol{0} & \boldsymbol{Otherwise} \end{cases} \qquad (9)$$

where b is the unique modal value of $\tilde{A}$; that is, $\mu_{\tilde{A}}(b) = 1$.

Definition 8: [26], [27] and [14], A fuzzy number Ã is said to be of type $L - R$ if, and only if, there exist three real numbers $m,\ a > 0,\ b > 0$ and two positive, continuous and decreasing functions L and R of $\mathbb{R}$ in the unit interval $[0,1]$ such that:

$$\boldsymbol{L}(\boldsymbol{0}) = \boldsymbol{R}(\boldsymbol{0}) = \boldsymbol{1} \qquad (10)$$

$$\boldsymbol{L}(\boldsymbol{1})\ \boldsymbol{ou}\ \boldsymbol{L}(\boldsymbol{x}) > \boldsymbol{0}, \qquad \forall\, \boldsymbol{x} \in \mathbb{R}\ \ \boldsymbol{with}\ \ \lim_{x \to +\infty} \boldsymbol{L}(\boldsymbol{x}) = \boldsymbol{0} \qquad (11)$$

$$\boldsymbol{R}(\boldsymbol{1})\ \boldsymbol{ou}\ \boldsymbol{R}(\boldsymbol{x}) > \boldsymbol{0}, \qquad \forall\, \boldsymbol{x} \in \mathbb{R}\ \ \boldsymbol{with}\ \ \lim_{x \to +\infty} \boldsymbol{R}(\boldsymbol{x}) = \boldsymbol{0} \qquad (12)$$

$$\mu_{\tilde{A}}(x) = \begin{cases} L[(m-x)/a] & if \ \ x \in [m-a, m] \\ R[(x-m)/b] & if \ \ x \in [m, m+b] \\ 0 & Otherwise \end{cases} \qquad (13)$$

where m is the modal value or mode of $\tilde{A}$; a and b are called left and right spreads or deviations of $\tilde{A}$, respectively.
The fuzzy number $\tilde{A}$ of $L-R$ type is often specified using the notation $\tilde{A} = \langle m, a, b \rangle_{LR}$;
$\tilde{A} = \langle m, a, b \rangle_{LR}$ is called the $L-R$ representation or the $L-R$ notation or the $L-R$ form of $\tilde{A}$.
The family of fuzzy numbers of $L-R$ type is denoted $\tilde{\mathcal{F}}_{LR}(\mathbb{R})$.

Arithmetic on Fuzzy Numbers
Arithmetic on Intervals
Definition 9: [28], consider two ordinary closed intervals $[a_1, a_2]$ and $[b_1, b_2]$ of $\mathbb{R}$. We define the arithmetic operation $*$ on these two intervals by the following relation:

$$[a_1 \ ; \ a_2] * [b_1 \ ; \ b_2] = \{a * b / a_1 \leq a \leq a_2 \ \ et \ \ b_1 \leq b \leq b_2\} \qquad (14)$$

where division is possible only if zero does not belong to $[b_1, b_2]$.
In the case of the operations $+; -; \times \ and \ \div$; the equality (14) simplifies as follows:

$$[a_1 \ ; \ a_2] + [b_1 \ ; \ b_2] = [a_1 + b_1; \ a_2 + b_2] \qquad (15)$$

$$[a_1 \ ; \ a_2] - [b_1 \ ; \ b_2] = [a_1 - b_2; \ a_2 - b_1] \qquad (16)$$

$$\lambda[a_1 \ ; \ a_2] = \begin{cases} [\lambda a_1 \ ; \ \lambda a_2] & si \ \ \lambda \geq 0 \\ [\lambda a_2 \ ; \ \lambda a_1] & si \ \ \lambda < 0 \end{cases} \qquad (17)$$

$$[a_1 \ ; \ a_2] \times [b_1 \ ; \ b_2] = [min\, P; \ max\, P] \qquad (18)$$

$$where \ \ P = (a_1 b_1; \ a_1 b_2; \ a_2 b_1 \ ; \ a_2 b_2)$$

$$[a_1 \ ; \ a_2]^{-1} = \left[min\left(\frac{1}{a_2}; \frac{1}{a_1}\right) \ ; \ max\left(\frac{1}{a_2}; \frac{1}{a_1}\right)\right] \qquad (19)$$

$$\frac{[a_1 \ ; \ a_2]}{[b_1 \ ; \ b_2]} = [min\, H \, ; max\, H] \quad where \quad H = \left(\frac{a_1}{b_1}; \frac{a_1}{b_2}; \frac{a_2}{b_1}; \frac{a_2}{b_2}\right) \qquad (20)$$

Furthermore, if the above intervals are defined in the set of positive real numbers $\mathbb{R}^{+}$, the multiplication, division and inverse interval operations can still be written as follows:

$$[a_1 \ ; \ a_2] \times [b_1 \ ; \ b_2] = [a_1 b_1 \ ; \ a_2 b_2] \qquad (21)$$

$$\frac{[a_1 \ ; \ a_2]}{[b_1 \ ; \ b_2]} = \left[\frac{a_1}{b_2}; \frac{a_2}{b_1}\right] \qquad (22)$$

$$[a_1 \ ; \ a_2]^{-1} = \left[\frac{1}{b_2}; \frac{1}{b_1}\right] \qquad (23)$$

Arithmetic of alpha-cuts
Definition 10: [16], Consider $\tilde{A}$ and $\tilde{B}$ two fuzzy subsets of the respective $\alpha - cuts$ $\tilde{A}_\alpha = [A_\alpha^L, A_\alpha^U]$ and $\tilde{B}_\alpha = [B_\alpha^L, B_\alpha^U]$; with $\alpha \in [0 \ ; \ 1]$. The four operations $\oplus; \ominus; \odot \ and \ \oslash$ is carried out on $\tilde{A}_\alpha$ and $\tilde{B}_\alpha$, according to Hanss, M., (2005) and Klir, J. G., (1995), by passing to their $\alpha - cuts$ in the following way:

$$\left(\tilde{A} \oplus \tilde{B}\right)_\alpha = \tilde{A}_\alpha + \tilde{B}_\alpha = [A_\alpha^L, A_\alpha^U] + [B_\alpha^L, B_\alpha^U] \qquad (24)$$

$$\left(\tilde{A} \ominus \tilde{B}\right)_\alpha = \tilde{A}_\alpha - \tilde{B}_\alpha = [A_\alpha^L, A_\alpha^U] - [B_\alpha^L, B_\alpha^U] \qquad (25)$$

$$\left(\tilde{A} \odot \tilde{B}\right)_\alpha = \tilde{A}_\alpha \cdot \tilde{B}_\alpha = [A_\alpha^L, A_\alpha^U] \cdot [B_\alpha^L, B_\alpha^U] \qquad (26)$$

$$\left(\tilde{A} \oslash \tilde{B}\right)_\alpha = \frac{\tilde{A}_\alpha}{\tilde{B}_\alpha} = \frac{[A_\alpha^L, A_\alpha^U]}{[B_\alpha^L, B_\alpha^U]} \qquad (27)$$

Remark 1: *The alpha – cuts of relations (24); (25); (26) and (27) are determined by using interval arithmetic formulated in relations (15); (16); (18) and (20).*

Alpha-Cut and Interval Arithmetic
Definition 11: [29] and [30], Performing fuzzy arithmetic using alpha-cut and interval arithmetic involves successively using:
- Relations (15); (16); (18) and (20) for defuzzification;
- Relations (24); (25); (26) and (27) for ordinary computations on closed real intervals;
- Relation (6) for fuzzification.

Arithmetic of $L-R$ Fuzzy Numbers
1°) Addition and Subtraction
Definition 12: [14] and [16], showed that if $\tilde{A} = \langle m, a, b \rangle_{LR}$ and $\tilde{B} = \langle n, c, d \rangle_{LR}$ are two fuzzy numbers of the same $L-R$ type, then their sum and difference are fuzzy numbers of the $L-R$ type given respectively by:

$$\widetilde{A} \oplus \widetilde{B} = \langle m + n, a + c, b + d \rangle_{LR} \qquad (28)$$
$$\widetilde{A} \ominus \widetilde{B} = \langle m - n, a + d, b + c \rangle_{LR} \qquad (29)$$

2°) Multiplication of fuzzy numbers of the $L - R$ type

Definition 13: Secant approximation [14] and [16], consider two numbers fuzzy numbers $\tilde{A}$ and $\tilde{B}$ of the same $L - R$ type,

$$\tilde{A} = \langle m, a, b \rangle_{LR} \quad \text{and} \quad \tilde{B} = \langle n, c, d \rangle_{LR}$$

Dubois and Prade arrived at the global formula for the multiplication of fuzzy numbers of $L - R$ type $(\tilde{A} > 0;\ \tilde{B} > 0)$:

$$\langle m, a, b \rangle_{LR} \odot \langle n, c, d \rangle_{LR} \simeq \langle mn; mc + na - ac;\ md + nb + bd \rangle_{LR} \qquad (30)$$

3°) Division of $L - R$ fuzzy numbers

Definition 14: Secant approximation of the inverse [16]

The secant approximation of the inverse of a positive $L - R$ fuzzy number, $\tilde{B} = \langle n, c, d \rangle_{LR}$ is defined by

$$\widetilde{B}^{-1} \simeq \langle 1/n;\ d/n(n + d);\ c/n(n - c) \rangle_{RL} \qquad (31)$$

If $\tilde{A} = \langle m, a, b \rangle_{LR}$ and $\tilde{B} = \langle n, c, d \rangle_{LR}$ are two fuzzy numbers of the same $L - R$ type, Hanss defined the secant approximation of the quotient of these numbers as follows:

$$\frac{\widetilde{A}}{\widetilde{B}} = \langle \frac{m}{n}; \frac{md}{n(n+d)} + \frac{a}{n} - \frac{ad}{n(n+d)}; \frac{mc}{n(n-c)} + \frac{b}{n} + \frac{bc}{n(n-c)} \rangle_{LR} \qquad (32)$$

III. Left-Right Method (L − R Method)

Description

Consider a fuzzy Markov queueing system whose rates are fuzzy numbers of the same $L - R$ type, denoted: $\tilde{v}_1;\ \tilde{v}_2;\ \cdots;\tilde{v}_n$; and that the performance indicator that we wish to calculate is designated by $\tilde{\zeta}$. In the classical model, this indicator and these rates are denoted respectively by ζ and v_1 ; v_2; …; v_n.

The formula for ζ in the classical model is given by:

$$\zeta = g(v_1; v_2;\ \dots; v_n) \qquad (33)$$

where g is a function with n real variables defined using the fundamental operations $+$; $-$; $\div$ and $\times$ in $\mathbb{R}$; while in the fuzzy model, this same formula is written:

$$\tilde{\zeta} = \tilde{g}(\tilde{v}_1;\ \tilde{v}_2;\ \cdots; \tilde{v}_n) \qquad (34)$$

where $\tilde{g}$ is a function with n fuzzy variables, defined using the fuzzy operations $\oplus$; $\ominus$; $\otimes$ and $\oslash$ in $\tilde{\mathcal{F}}(\mathbb{R})$.

To determine $\tilde{\zeta}$ using the L– R method, we proceed as follows:

Procedure:

Step 1: Determine the L– R writings of all fuzzy rates and substitute them in (34).

Step 2: Apply L– R fuzzy number arithmetic to the expression found in the previous step. In this case, only formulas suitably chosen from relations (28); (29); (30) and (32) help us. Regarding the formulas for multiplication and division, we only need those from the secant approximation, which lead to the exact modal values and supports of the desired performance indicators [Mukeba, J. P., (2015)] and [Alonge, W. J., & al., (2023)].

Step 3: According to this arithmetic, the final expression to be obtained is a fuzzy function approximately equal to the L – R fuzzy function, given by

$$\tilde{\zeta} = \langle m;\ \varphi;\ \omega \rangle_{LR} \qquad (35)$$

where m is the modal value of $\tilde{\zeta}$, and φ and ω represent its left and right deviations, respectively. The support of $\tilde{\zeta}$ in this case is the interval:

$$Supp(\tilde{\zeta}) =]m - \varphi;\ m + \omega[\qquad (36)$$

Step 4: ***(Final Result)***: The desired performance indicator is a fuzzy number whose support is the open interval with bounds $m - \varphi$ and $m + \omega$; and with a modal value m. In other words, this performance indicator is approximately between $m - \varphi$ and $m + \omega$. Its most frequent value is the mode m.

IV. Performance Of The FM / FM / 1 System With A Priori Impatience

System Description

The fuzzy FM / FM / 1 queueing system with a priori impatience is a single-server fuzzy Markovian queueing system. It is modeled by a homogeneous life and death process with fuzzy birth rates $\tilde{\lambda}_k$ and fuzzy death rates $\tilde{\mu}_k$ defined respectively by

$$\tilde{\lambda}_k = \tilde{\lambda};\ \tilde{\lambda} > 0 \quad and \quad \tilde{\mu}_k = k\tilde{\mu} \qquad for\ k = 0; 1;\ \dots$$

Model Assumptions (or Characteristics)

The FM / FM / 1 fuzzy queueing system with a priori impatience has the same characteristics as the M / M / 1 model. However, the state probabilities p_k are replaced by the fuzzy state probabilities $\tilde{p}_k$.

Steady-State Analysis
Theorem 1:
Let $\tilde{p}_k$, be the steady-state fuzzy state probabilities of the $FM\ /\ FM\ /\ 1$ system with a priori impatience, with respective rates $\tilde{\lambda}_k = \tilde{\lambda}$ and $\tilde{\mu}_k = k\tilde{\mu} \quad for\ \ k = 0; 1;\ \dots$ Then:

$$\tilde{\boldsymbol{p}}_{\boldsymbol{k}} = \frac{\tilde{\boldsymbol{\rho}}^{\boldsymbol{k}}}{\boldsymbol{k}!}\boldsymbol{e}^{-\tilde{\boldsymbol{\rho}}} \quad for\ \ k = 0; 1;\ \dots \qquad and \qquad \tilde{\boldsymbol{p}}_{\boldsymbol{0}} = \boldsymbol{e}^{-\tilde{\boldsymbol{\rho}}}$$

Proof: The Kolmogorov fuzzy differential equations of a steady-state life and death process are given by:

$$\begin{cases} \boldsymbol{0} = \tilde{\boldsymbol{\lambda}}_{\boldsymbol{k-1}}\tilde{\boldsymbol{p}}_{\boldsymbol{k-1}} - (\tilde{\boldsymbol{\lambda}}_{\boldsymbol{k}} + \tilde{\boldsymbol{\mu}}_{\boldsymbol{k}})\tilde{\boldsymbol{p}}_{\boldsymbol{k}} + \tilde{\boldsymbol{\mu}}_{\boldsymbol{k+1}}\tilde{\boldsymbol{p}}_{\boldsymbol{k+1}} & (37) \\ \boldsymbol{0} = -\tilde{\boldsymbol{\lambda}}_{\boldsymbol{0}}\boldsymbol{p}_{\boldsymbol{0}} + \tilde{\boldsymbol{\mu}}_{\boldsymbol{1}}\boldsymbol{p}_{\boldsymbol{1}} & (38) \end{cases}$$

Replacing $\tilde{\lambda}_k$ and $\tilde{\mu}_k$ by their values in (37) and (38), we obtain the following system of equations:

$$\begin{cases} \boldsymbol{0} = \tilde{\boldsymbol{\lambda}}\tilde{\boldsymbol{p}}_{\boldsymbol{k-1}} - (\tilde{\boldsymbol{\lambda}} + \boldsymbol{k}\tilde{\boldsymbol{\mu}})\tilde{\boldsymbol{p}}_{\boldsymbol{k}} + (\boldsymbol{k}+\boldsymbol{1})\tilde{\boldsymbol{\mu}}\tilde{\boldsymbol{p}}_{\boldsymbol{k+1}} & (39) \\ \boldsymbol{0} = \tilde{\boldsymbol{\mu}}\tilde{\boldsymbol{p}}_{\boldsymbol{1}} - \tilde{\boldsymbol{\lambda}}\tilde{\boldsymbol{p}}_{\boldsymbol{0}} & (40) \end{cases}$$

(39) and (40) determine a recurrent system by which:

$$\tilde{\boldsymbol{p}}_{\boldsymbol{1}} = \frac{\tilde{\boldsymbol{\lambda}}}{\tilde{\boldsymbol{\mu}}}\tilde{\boldsymbol{p}}_{\boldsymbol{0}} \qquad (41)$$

$$\tilde{\boldsymbol{p}}_{\boldsymbol{2}} = \frac{\boldsymbol{1}}{\boldsymbol{2}}\left(\frac{\tilde{\boldsymbol{\lambda}}}{\tilde{\boldsymbol{\mu}}}\right)^{\boldsymbol{2}}\tilde{\boldsymbol{p}}_{\boldsymbol{0}} \qquad (42)$$

$$\vdots$$

$$\tilde{\boldsymbol{p}}_{\boldsymbol{k}} = \frac{\boldsymbol{1}}{\boldsymbol{k}!}\left(\frac{\tilde{\boldsymbol{\lambda}}}{\tilde{\boldsymbol{\mu}}}\right)^{\boldsymbol{k}}\tilde{\boldsymbol{p}}_{\boldsymbol{0}} \qquad (43)$$

Let

$$\tilde{\boldsymbol{p}}_{\boldsymbol{k}} = \frac{\tilde{\boldsymbol{\rho}}^{\boldsymbol{k}}}{\boldsymbol{k}!}\tilde{\boldsymbol{p}}_{\boldsymbol{0}} \qquad with\ \ \tilde{\boldsymbol{\rho}} = \frac{\tilde{\boldsymbol{\lambda}}}{\tilde{\boldsymbol{\mu}}} \qquad (44)$$

Introducing the normalization condition $\sum_{k=0}^{+\infty} p_k = 1$, a condition associated with relation (44), gives us:

$$\sum_{k=0}^{+\infty}\frac{\tilde{\rho}^k}{k!}\tilde{p}_0 = 1 \quad \Rightarrow \quad \tilde{p}_0 = \frac{1}{\sum_{k=0}^{+\infty}\frac{\tilde{\rho}^k}{k!}} = \frac{1}{e^{\tilde{\rho}}} = e^{-\tilde{\rho}}$$

Thus,

$$\tilde{\boldsymbol{p}}_{\boldsymbol{0}} = \boldsymbol{e}^{-\tilde{\boldsymbol{\rho}}} \qquad (45)$$

Hence

$$\tilde{\boldsymbol{p}}_{\boldsymbol{k}} = \frac{\tilde{\boldsymbol{\rho}}^{\boldsymbol{k}}}{\boldsymbol{k}!}\boldsymbol{e}^{-\tilde{\boldsymbol{\rho}}} \quad pour\ \ k = 0; 1;\ \dots \qquad (46) \qquad \blacksquare$$

Remark 2: If < 1 , *then* $\frac{\tilde{\lambda}}{\tilde{\mu}} < 1 \quad i.e\ \tilde{\lambda} < \tilde{\mu}$. We have, $\boldsymbol{oy}(\tilde{\boldsymbol{\lambda}}) < \boldsymbol{Noy}(\tilde{\boldsymbol{\mu}})$: ***Fuzzy ergodicity or stability condition***.

Performance indicators of the $FM\ /\ FM\ /\ \mathbf{1}$ system with a priori impatience
Proposition 1: (Fuzzy server utilization rate: $\tilde{\mathcal{U}}$)
Let $\tilde{\mathcal{U}}$ be the fuzzy server utilization rate. Then;

$$\tilde{\boldsymbol{\mathcal{U}}} = \frac{\tilde{\boldsymbol{\mu}} - \tilde{\boldsymbol{\mu}}\boldsymbol{e}^{-\tilde{\boldsymbol{\rho}}}}{\tilde{\boldsymbol{\mu}}} \qquad (47)$$

Proof: Since the fuzzy server utilization rate is defined as the proportion of time during which the server is busy over a time interval, we then have:
$\tilde{\mathcal{U}} = 1 - \tilde{p}_0$, or according to (45), $\tilde{p}_0 = e^{-\tilde{\rho}}$ which gives: $\tilde{\mathcal{U}} = 1 - e^{-\tilde{\rho}}$. Multiplying the second member of this last equation by $\tilde{\mu}/\tilde{\mu}$, we obtain,

$$\tilde{\mathcal{U}} = \left(1 - e^{-\tilde{\rho}}\right)\frac{\tilde{\mu}}{\tilde{\mu}} = \frac{\tilde{\mu} - \tilde{\mu}e^{-\tilde{\rho}}}{\tilde{\mu}} \quad with\ \ \tilde{\boldsymbol{\rho}} = \frac{\tilde{\boldsymbol{\lambda}}}{\tilde{\boldsymbol{\mu}}} \quad \boldsymbol{fuzzy\ traffic\ intensity}$$

Hence

$$\tilde{\mathcal{U}} = \frac{\tilde{\mu} - \tilde{\mu}e^{-\tilde{\rho}}}{\tilde{\mu}} \qquad \blacksquare$$

Proposition 2: (Fuzzy flow rate of the system: $\tilde{d}$)
Let $\tilde{d}$ be the fuzzy flow rate of the system. Then;

$$\boldsymbol{\widetilde{d} = \widetilde{\mu} - \widetilde{\mu}e^{-\widetilde{\rho}}} \qquad (48)$$

Proof: Generally speaking, the flow rate of a system is calculated either by the input into the system (**input flow rate $\widetilde{d}_e$**), or by the output from the system (***output flow rate*** $\boldsymbol{\widetilde{d}_s}$). In the case of stable systems, these two flow rates are equal and we have: $\boldsymbol{\widetilde{d} = \widetilde{d}_e = \widetilde{d}_s}$**.** This is also the case with the system under study, for which the service is performed with a rate $\tilde{\mu}$ in each state where the system contains at least one customer, we therefore have:

$$\tilde{d} = \left(\widetilde{\mathbb{P}}[non - empty\ queue]\right) \times \tilde{\mu} = \sum_{k=0}^{+\infty} \tilde{p}_k \times \tilde{\mu} = \left(1 - e^{-\tilde{\rho}}\right) \times \tilde{\mu} = \tilde{\mu} - \tilde{\mu}e^{-\tilde{\rho}}$$

Hence

$$\tilde{d} = \tilde{\mu} - \tilde{\mu}e^{-\tilde{\rho}} \qquad \blacksquare$$

Proposition 3: (Fuzzy average number of customers in the system: $\boldsymbol{\widetilde{N}_S}$)

Let $\widetilde{N}_S$ be the fuzzy average number of customers in the system. Then;

$$\boldsymbol{\widetilde{N}_S = \widetilde{\rho}} \qquad (49)$$

Proof: Since the fuzzy process $\{\tilde{X}(t) : t \geq 0\}$ denotes the fuzzy number of customers in the system at a date t, the fuzzy average number of customers in the system in the steady state is the fuzzy mathematical expectation of the fuzzy random variable $\widetilde{X}$.

We have:

$$\widetilde{N}_S = \tilde{E}\left(\tilde{X}\right) = \widetilde{\mathbb{P}}[\tilde{X} \leq k] = \sum_{k=0}^{+\infty} k \times \tilde{p}_k = \sum_{k=0}^{+\infty} k \times \frac{\tilde{\rho}^k}{k!} \times e^{-\tilde{\rho}} = e^{-\tilde{\rho}} \sum_{k=0}^{+\infty} k \times \frac{\tilde{\rho}^k}{k!}$$
$$= e^{-\tilde{\rho}} \sum_{k=0}^{+\infty} \frac{\tilde{\rho}^k}{(k-1)!} = e^{-\tilde{\rho}} \times \tilde{\rho} \times e^{\tilde{\rho}} = \tilde{\rho}$$

Therefore

$$\widetilde{N}_S = \tilde{\rho} \qquad \blacksquare$$

Proposition 4: (Fuzzy average number of customers in the queue: $\boldsymbol{\widetilde{N}_f}$)

Let $\widetilde{N}_f$ be the average number of customers in the queue. Then;

$$\boldsymbol{\widetilde{N}_f = \frac{\widetilde{\lambda} + \widetilde{\mu}e^{-\widetilde{\rho}} - \widetilde{\mu}}{\widetilde{\mu}}} \qquad (50)$$

Proof: Since the system under study is single-server, the presence of k customers in the system implies that there are $k-1$ in the queue. Then $\widetilde{N}_f$, is the theoretical fuzzy mean of the fuzzy random variable $\tilde{X}$, with $\tilde{X}$ taking all values between 0 and $k-1$.

$$\widetilde{N}_f = \tilde{E}\left(\tilde{X}\right) = \widetilde{\mathbb{P}}[\tilde{X} \leq k-1] = \sum_{k=1}^{+\infty} (k-1) \times \tilde{p}_k = \sum_{k=1}^{+\infty} k\, \tilde{p}_k - \sum_{k=1}^{+\infty} \tilde{p}_k$$
$$= \sum_{k=0}^{+\infty} k\, \tilde{p}_k - \left(\sum_{k=0}^{+\infty} \tilde{p}_k - \tilde{p}_0\right)$$

Introducing the fuzzy normalization condition $\sum_{k=1}^{+\infty} \widetilde{p}_k = 1$ and the relation $\sum_{k=0}^{+\infty} k \times \widetilde{p}_k$ i.e. (49) into this relation, we find:

$$\widetilde{N}_f = \widetilde{N}_S - (1 - \tilde{p}_0) = \tilde{\rho} - \left(1 - e^{-\tilde{\rho}}\right) = \tilde{\rho} + e^{-\tilde{\rho}} - 1 = \frac{\tilde{\lambda}}{\tilde{\mu}} + e^{-\tilde{\rho}} - 1 = \frac{\tilde{\lambda} + \tilde{\mu}e^{-\tilde{\rho}} - \tilde{\mu}}{\tilde{\mu}}$$

Therefore

$$\widetilde{N}_f = \frac{\tilde{\lambda} + \tilde{\mu}e^{-\tilde{\rho}} - \tilde{\mu}}{\tilde{\mu}} \qquad \blacksquare$$

Proposition 5: (Fuzzy average residence time: $\boldsymbol{\tilde{t}_S}$)

Let $\tilde{t}_S$ be the fuzzy average residence time. Then ;

$$\boldsymbol{\tilde{t}_S = \frac{\tilde{\lambda}}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}}} \qquad (51)$$

Proof: Little's fuzzy formula is given by: $\boldsymbol{\widetilde{N}_S = \tilde{t}_S \times \widetilde{d}}$ (52)

From (52), we draw $\tilde{t}_S$, we have:

$$\tilde{t}_S = \frac{\widetilde{N}_S}{\tilde{d}} = \frac{\tilde{\rho}}{\tilde{\mu}(1 - e^{-\tilde{\rho}})} = \frac{\frac{\tilde{\lambda}}{\tilde{\mu}}}{\tilde{\mu}(1 - e^{-\tilde{\rho}})} = \frac{\tilde{\lambda}}{\tilde{\mu}^2(1 - e^{-\tilde{\rho}})} = \frac{\tilde{\lambda}}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}}$$

Hence

$$\tilde{t}_S = \frac{\tilde{\lambda}}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}} \qquad \blacksquare$$

Proposition 6: (Average fuzzy time in the queue: $\tilde{t}_f$)

Let $\tilde{t}_f$ be the average fuzzy waiting time of customers in the queue. Then;

$$\boldsymbol{\tilde{t}_f = \frac{\tilde{\lambda} + \tilde{\mu} e^{-\tilde{\rho}} - \tilde{\mu}}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}}} \qquad (53)$$

Proof: The average fuzzy time spent in the server by each client is $1/\tilde{\mu}$, it is easy to conclude that the average fuzzy time spent in the $FM\ /\ FM\ /\ 1$ queue is:

$$\tilde{t}_f = \tilde{t}_S - \frac{1}{\tilde{\mu}} = \frac{\tilde{\lambda}}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}} - \frac{1}{\tilde{\mu}} = \frac{\tilde{\lambda} - \left(\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}\right)}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}} = \frac{\tilde{\lambda} + \tilde{\mu} e^{-\tilde{\rho}} - \tilde{\mu}}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}}$$

Therefore

$$\tilde{t}_f = \frac{\tilde{\lambda} + \tilde{\mu} e^{-\tilde{\rho}} - \tilde{\mu}}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}} \qquad \blacksquare$$

V. Numerical Application

Problem:

In a one-stop commercial bank with infinite capacity and Poissonian arrival and exponential service times, the average time between arrivals is about 10 minutes and the average time per service is about 8 minutes. The bank operates as a waiting system with a priori impatience.

Questions:

b.1. Show that the system is ergodic;

b.2. Determine, in steady state, the fuzzy performance indicators of this system.

Solution:

Hypothetically, this is a fuzzy queueing system with a priori impatience, of the form $FM\ /\ FM\ /\ 1$ with a single server and infinite capacity. In the classical model, the arrival rate is $\lambda = 1/10$ minute, or $\lambda = 60/10 = 6$ arrivals per hour, and the service rate is $\mu = 1/8$ minute, or $\mu = 60/8 = 7.5$ departures per hour. Since the arrival rate is about 10 minutes, or about 6 arrivals per hour, and the average rate per service is about 8 minutes, or about 7.5 departures per hour, this suggests that these are fuzzy rates. Assuming that these fuzzy rates $\tilde{\lambda}$ and $\tilde{\mu}$ are triangular fuzzy numbers, we can write:

$$\tilde{\lambda} = (5/\,6/\,7\,) \qquad and \qquad \tilde{\mu} = (6.5/\,7.5/\,8.5)$$

Thus, in a fuzzy model, where the rates $\tilde{\lambda}$ and $\tilde{\mu}$ are fuzzy variables, the performance indicators $\tilde{\mathcal{U}}$; $\tilde{d}$; $\tilde{N}_S$; $\tilde{N}_f$; $\tilde{t}_S$ and $\tilde{t}_f$ also become fuzzy numbers [see formulas (47); (48); (49); (50); (51) and (53)].

Resolution by the L – R method

To evaluate the performance indicators by the L – R method, we proceed as follows:

1°) Determine the L – R forms of rates $\tilde{\lambda} = (5/\ 6\ /\ 7)$ and $\tilde{\mu} = (6.5\ /\ 7.5\ /\ 8.5)$; we have:

$$\boldsymbol{\tilde{\lambda} = \langle 6\,;1\,;1\rangle_{LR}} \quad (54) \qquad \boldsymbol{\tilde{\mu} = \langle 7.5\,;1\,;1\rangle_{LR}} \quad (55)$$

2°) Let us show that the system is ergodic:

We have: $Noy(\tilde{\lambda}) < Noy(\tilde{\mu})$ or $6 < 7$, hence ***the system is ergodic*** or ***stable*** and the calculation of the performance indicators is possible.

3°) (54) and (55) in (47); (48); (49); (50); (51) and (53) and apply the formulas of fuzzy number arithmetic of type $L - R$, suitably chosen from formulas (28); (29); (30) and (32), we obtain successively:

Let us first carry out the required auxiliary calculations:

$$*\quad \tilde{\rho} = \frac{\tilde{\lambda}}{\tilde{\mu}} = \frac{\langle 6\,;1\,;1\rangle_{LR}}{\langle 7.5\,;1\,;1\rangle_{LR}} = \langle \frac{6}{7.5}\,;\,\frac{6}{63.75} + \frac{1}{7.5} - \frac{1}{63.75}\,;\,\frac{6}{48.75} + \frac{1}{7.5} + \frac{1}{48.75}\rangle_{LR}$$

$$\boldsymbol{\tilde{\rho} = \langle 0.8\,;0.212\,;\,0.276\rangle_{LR}} \qquad (56)$$

$$*\quad \boldsymbol{e^{-\tilde{\rho}}} = e^{-0.8} = \boldsymbol{0.45} \qquad (57)$$

$$*\quad \boldsymbol{\langle 7.5\,;1\,;1\rangle_{LR} e^{-\tilde{\rho}}} = \langle 7.5\,;1\,;1\rangle_{LR} \times 0.45 = \boldsymbol{\langle 3.375\,;0.45\,;\,0.45\rangle_{LR}} \qquad (58)$$

$$*\quad \boldsymbol{\langle 7.5\,;1\,;1\rangle_{LR} - \langle 7.5\,;1\,;1\rangle_{LR} e^{-\tilde{\rho}}} = \langle 7.5\,;1\,;1\rangle_{LR} - \langle 3.375\,;0.45\,;\,0.45\rangle_{LR}$$

$$= \boldsymbol{\langle 4.125\,;1.45\,;\,1.45\rangle_{LR}} \qquad (59)$$

$$*\quad \boldsymbol{\langle 6\,;1\,;1\rangle_{LR} + \langle 7.5\,;1\,;1\rangle_{LR} e^{-\tilde{\rho}} - \langle 7.5\,;1\,;1\rangle_{LR}}$$

$$= \langle 6\,;1\,;1\rangle_{LR} + \langle 3.375\,;0.45\,;\,0.45\rangle_{LR} - \langle 7.5\,;1\,;1\rangle_{LR}$$

$$= \langle 9.375\,;1.45\,;\,1.45\rangle_{LR} - \langle 7.5\,;1\,;1\rangle_{LR}$$

$$= \langle \mathbf{1.875}\,;\mathbf{2.45}\,;\ \mathbf{2.45}\rangle_{LR} \quad (60)$$

$$*\quad (\langle \mathbf{7.5}\,;\mathbf{1}\,;\mathbf{1}\rangle_{LR})^2 = \langle 7.5\,;1\,;1\rangle_{LR} \times \langle 7.5\,;1\,;1\rangle_{LR}$$

$$= \langle 56.25\,;7.5+7.5-1\,;\ 7.5+7.5+1\rangle_{LR}$$

$$= \langle \mathbf{56.25}\,;\mathbf{14}\,;\ \mathbf{16}\rangle_{LR} \quad (61)$$

$$*\quad (\langle \mathbf{7.5}\,;\mathbf{1}\,;\mathbf{1}\rangle_{LR})^2 e^{-\tilde{\rho}} = \langle 56.25\,;14\,;\ 16\rangle_{LR} \times 0.45 = \langle \mathbf{25.3125}\,;\mathbf{6.3}\,;\ \mathbf{7.2}\rangle_{LR} \quad (62)$$

$$*\quad (\langle \mathbf{7.5}\,;\mathbf{1}\,;\mathbf{1}\rangle_{LR})^2 - (\langle \mathbf{7.5}\,;\mathbf{1}\,;\mathbf{1}\rangle_{LR})^2 e^{-\tilde{\rho}} = \langle 56.25\,;14\,;\ 16\rangle_{LR} - \langle 25.3125\,;6.3\,;\ 7.2\rangle_{LR}$$

$$= \langle \mathbf{30.9375}\,;\mathbf{21.2}\,;\ \mathbf{22.3}\rangle_{LR} \quad (63)$$

a) Calculating $\tilde{\mathcal{U}}$

(59) and (55) in (47), we obtain:

$$\tilde{\mathcal{U}} = \frac{\langle 4.125\,;1.45\,;\ 1.45\rangle_{LR}}{\langle 7.5\,;1\,;1\rangle_{LR}} = \left\langle \frac{4.125}{7.5}\,;\frac{4.125}{63.75}+\frac{1.45}{7.5}-\frac{1.45}{63.75}\,;\ \frac{4.125}{48.75}+\frac{1.45}{7.5}+\frac{1.45}{48.75}\right\rangle_{LR}$$

$$= \langle \mathbf{0.55}\,;\mathbf{0.235}\,;\ \mathbf{0.308}\rangle_{LR}$$

b) Calculation of $\tilde{d}$

(59) in (48), we have:

$$\tilde{d} = \langle \mathbf{4.125}\,;\mathbf{1.45}\,;\ \mathbf{1.45}\rangle_{LR}$$

c) Calculation of $\tilde{N}_S$

(56) in (49); we have:

$$\tilde{N}_S = \langle \mathbf{0.8}\,;\mathbf{0.212}\,;\ \mathbf{0.277}\rangle_{LR}$$

d) Calculating $\tilde{N}_f$

(60) and (55) in (50), we obtain:

$$\tilde{N}_f = \frac{\langle 1.875\,;2.45\,;\ 2.45\rangle_{LR}}{\langle 7.5\,;1\,;1\rangle_{LR}} = \left\langle \frac{1.875}{7.5}\,;\frac{1.875}{63.75}+\frac{2.45}{7.5}-\frac{2.45}{63.75}\,;\ \frac{1.875}{48.75}+\frac{2.45}{7.5}+\frac{2.45}{48.75}\right\rangle_{LR}$$

$$= \langle \mathbf{0.25}\,;\mathbf{0.338}\,;\ \mathbf{0.415}\rangle_{LR}$$

e) Calculating $\tilde{t}_S$

(54) and (63) in (51), we obtain:

$$\tilde{t}_S = \frac{\langle 6\,;1\,;\ 1\rangle_{LR}}{\langle 30.9375\,;21.2\,;\ 22.3\rangle_{LR}}$$

$$= \left\langle \frac{6}{30.9375}\,;\frac{133.8}{1647.035}+\frac{1}{30.9375}-\frac{22.3}{1647.035}\,;\ \frac{127.2}{301.254}+\frac{1}{30.9375}+\frac{21.2}{301.254}\right\rangle_{LR}$$

$$= \langle \mathbf{0.193}\,;\mathbf{0.099}\,;\ \mathbf{0.526}\rangle_{LR}$$

f) Calculating $\tilde{t}_f$

(60) and (63) in (53), we obtain:

$$\tilde{t}_f = \frac{\langle 1.875\,;2.45\,;\ 2.45\rangle_{LR}}{\langle 30.9375\,;21.2\,;\ 22.3\rangle_{LR}}$$

$$= \left\langle \frac{1.875}{30.9375}\,;\frac{41.8125}{1647.035}+\frac{2.45}{30.9375}-\frac{54.635}{1647.035}\,;\ \frac{39.75}{301.254}+\frac{2.45}{30.9375}+\frac{51.94}{301.254}\right\rangle_{LR}$$

$$= \langle \mathbf{0.060}\,;\mathbf{0.001}\,;\ \mathbf{0.384}\rangle_{LR}$$

Thus, according to relation (35), the kernels (or modes) of $\tilde{\mathcal{U}}$; $\tilde{d}$; $\tilde{N}_S$; $\tilde{N}_f$; $\tilde{t}_S$ and $\tilde{t}_f$ are respectively 0.55; 4.125; 0.8; 0.25; 0.193 and 0.060.

According to relation (36), their supports are the following open intervals:

$$Supp(\tilde{\mathcal{U}}) =]0.55-0.23\ ;\ 0.55+0.308[\ =]0.315\ ;\ 0.858[$$

$$Supp(\tilde{d}) =]4.125-1.45\ ;\ 4.125+1.45[\ =]2.675\ ;\ 5.575[$$

$$Supp(\tilde{N}_S) =]0.8-0.212\ ;\ 0.8+0.276[\ =]0.588\ ;\ 1.076[$$

$$Supp(\tilde{N}_f) =]0.25-0.338\ ;\ 0.25+0.415[\ =]-0.088\ ;\ 0.665[$$

$$Supp(\tilde{t}_S) =]0.193-0.099\ ;\ 0.193+0.526[\ =]0.094\ ;\ 0.719[$$

$$Supp(\tilde{t}_f) =]0.060-0.001\ ;\ 0.060+0.384[\ =]-0.059\ ;\ 0.444[$$

Variants introduced by the $L-R$ method

In steady state, the $L-R$ method introduces some variations in the treatment of queueing systems with a priori impatience as follows:

- The performance indicators are fuzzy numbers of the same $L-R$ type and not average values as for ordinary queueing systems with a priori impatience;
- The $L-R$ notations of these indicators contain real numbers of the $L-R$ type denoted $\langle m\,;a\,;b\rangle_{LR}$;
- The supports of these indicators are open intervals where the bounds are real numbers;
- The modes of these indicators are real numbers that correspond exactly to average values for ordinary queueing systems with a priori impatience.

VI. Result

Performance of the FM / FM / 1 System with A Priori Impatience

System Description

The fuzzy $FM\ /\ FM\ /\ 1$ queueing system with a priori impatience is a single-server fuzzy Markovian queueing system. It is modeled by a homogeneous life and death process with fuzzy birth rates $\tilde{\lambda}_k$ and fuzzy death rates $\tilde{\mu}_k$ defined respectively by

$$\tilde{\lambda}_k = \tilde{\lambda}\,;\ \tilde{\lambda} > 0 \quad and \quad \tilde{\mu}_k = k\tilde{\mu} \qquad for\ \ k = 0; 1;\ \ldots$$

Model Assumptions (or Characteristics)

The $FM\ /\ FM\ /\ 1$ fuzzy queueing system with a priori impatience has the same characteristics as the $M\ /\ M\ /\ 1$ model. However, the state probabilities p_k are replaced by the fuzzy state probabilities $\tilde{p}_k$.

Steady-State Analysis

Theorem 1:

Let $\tilde{p}_k$, be the steady-state fuzzy state probabilities of the $FM\ /\ FM\ /\ 1$ system with a priori impatience, with respective rates $\tilde{\lambda}_k = \tilde{\lambda}$ and $\tilde{\mu}_k = k\tilde{\mu}$ $\ for\ \ k = 0; 1;\ \ldots$ Then:

$$\boldsymbol{\tilde{p}_k = \frac{\tilde{\rho}^k}{k!}e^{-\tilde{\rho}}} \quad for\ \ k = 0; 1;\ \ldots \qquad and \qquad \boldsymbol{\tilde{p}_0 = e^{-\tilde{\rho}}}$$

Proof: The Kolmogorov fuzzy differential equations of a steady-state life and death process are given by:

$$\begin{cases} \boldsymbol{0 = \tilde{\lambda}_{k-1}\tilde{p}_{k-1} - (\tilde{\lambda}_k + \tilde{\mu}_k)\tilde{p}_k + \tilde{\mu}_{k+1}\tilde{p}_{k+1}} & (37) \\ \boldsymbol{0 = -\tilde{\lambda}_0 p_0 + \tilde{\mu}_1 p_1} & (38) \end{cases}$$

Replacing $\tilde{\lambda}_k$ and $\tilde{\mu}_k$ by their values in (37) and (38), we obtain the following system of equations:

$$\begin{cases} \boldsymbol{0 = \tilde{\lambda}\tilde{p}_{k-1} - (\tilde{\lambda} + k\tilde{\mu})\tilde{p}_k + (k+1)\tilde{\mu}\tilde{p}_{k+1}} & (39) \\ \boldsymbol{0 = \tilde{\mu}\tilde{p}_1 - \tilde{\lambda}\tilde{p}_0} & (40) \end{cases}$$

(39) and (40) determine a recurrent system by which:

$$\boldsymbol{\tilde{p}_1 = \frac{\tilde{\lambda}}{\tilde{\mu}}\tilde{p}_0} \qquad (41)$$

$$\boldsymbol{\tilde{p}_2 = \frac{1}{2}\left(\frac{\tilde{\lambda}}{\tilde{\mu}}\right)^2 \tilde{p}_0} \qquad (42)$$

$$\vdots$$

$$\boldsymbol{\tilde{p}_k = \frac{1}{k!}\left(\frac{\tilde{\lambda}}{\tilde{\mu}}\right)^k \tilde{p}_0} \qquad (43)$$

Let

$$\boldsymbol{\tilde{p}_k = \frac{\tilde{\rho}^k}{k!}\tilde{p}_0} \qquad with\ \boldsymbol{\tilde{\rho} = \frac{\tilde{\lambda}}{\tilde{\mu}}} \qquad (44)$$

Introducing the normalization condition $\sum_{k=0}^{+\infty} p_k = 1$, a condition associated with relation (44), gives us:

$$\sum_{k=0}^{+\infty}\frac{\tilde{\rho}^k}{k!}\tilde{p}_0 = 1 \quad \Rightarrow \quad \tilde{p}_0 = \frac{1}{\sum_{k=0}^{+\infty}\frac{\tilde{\rho}^k}{k!}} = \frac{1}{e^{\tilde{\rho}}} = e^{-\tilde{\rho}}$$

Thus,

$$\boldsymbol{\tilde{p}_0 = e^{-\tilde{\rho}}} \qquad (45)$$

Hence

$$\boldsymbol{\tilde{p}_k = \frac{\tilde{\rho}^k}{k!}e^{-\tilde{\rho}}} \quad pour\ \ k = 0; 1;\ \ldots \qquad (46) \qquad \blacksquare$$

Remark 2: If < 1 , *then* $\frac{\tilde{\lambda}}{\tilde{\mu}} < 1$ $\ i.e\ \tilde{\lambda} < \tilde{\mu}$. We have, $\boldsymbol{oy(\tilde{\lambda}) < Noy(\tilde{\mu})}$: ***Fuzzy ergodicity or stability condition***.

Performance indicators of the $FM\ /\ FM\ /\ 1$ system with a priori impatience

Proposition 1: (Fuzzy server utilization rate: $\tilde{\mathcal{U}}$)

Let $\tilde{\mathcal{U}}$ be the fuzzy server utilization rate. Then;

$$\boldsymbol{\tilde{\mathcal{U}} = \frac{\tilde{\mu} - \tilde{\mu}e^{-\tilde{\rho}}}{\tilde{\mu}}} \qquad (47)$$

Proof: Since the fuzzy server utilization rate is defined as the proportion of time during which the server is busy over a time interval, we then have:
$\tilde{\mathcal{U}} = 1 - \tilde{p}_0$, or according to (45), $\tilde{p}_0 = e^{-\tilde{\rho}}$ which gives: $\tilde{\mathcal{U}} = 1 - e^{-\tilde{\rho}}$. Multiplying the second member of this last equation by $\tilde{\mu}/\tilde{\mu}$, we obtain,

$$\tilde{\mathcal{U}} = \left(1 - e^{-\tilde{\rho}}\right)\frac{\tilde{\mu}}{\tilde{\mu}} = \frac{\tilde{\mu} - \tilde{\mu}e^{-\tilde{\rho}}}{\tilde{\mu}} \quad with\ \tilde{\boldsymbol{\rho}} = \frac{\tilde{\boldsymbol{\lambda}}}{\tilde{\boldsymbol{\mu}}} \quad \boldsymbol{fuzzy\ traffic\ intensity}$$

Hence

$$\tilde{\mathcal{U}} = \frac{\tilde{\mu} - \tilde{\mu}e^{-\tilde{\rho}}}{\tilde{\mu}} \qquad \blacksquare$$

Proposition 2: (Fuzzy flow rate of the system: $\tilde{d}$)
Let $\tilde{d}$ be the fuzzy flow rate of the system. Then;

$$\tilde{\boldsymbol{d}} = \tilde{\boldsymbol{\mu}} - \tilde{\boldsymbol{\mu}}\boldsymbol{e}^{-\tilde{\rho}} \qquad (48)$$

Proof: Generally speaking, the flow rate of a system is calculated either by the input into the system (**input flow rate $\tilde{d}_e$**), or by the output from the system (***output flow rate $\tilde{d}_s$***). In the case of stable systems, these two flow rates are equal and we have: $\tilde{\boldsymbol{d}} = \tilde{\boldsymbol{d}}_e = \tilde{\boldsymbol{d}}_s$**.** This is also the case with the system under study, for which the service is performed with a rate μ˜ in each state where the system contains at least one customer, we therefore have:

$$\tilde{d} = \left(\tilde{\mathbb{P}}[non - empty\ queue]\right) \times \tilde{\mu} = \sum_{k=0}^{+\infty} \tilde{p}_k \times \tilde{\mu} = \left(1 - e^{-\tilde{\rho}}\right) \times \tilde{\mu} = \tilde{\mu} - \tilde{\mu}e^{-\tilde{\rho}}$$

Hence

$$\tilde{d} = \tilde{\mu} - \tilde{\mu}e^{-\tilde{\rho}} \qquad \blacksquare$$

Proposition 3: (Fuzzy average number of customers in the system: $\tilde{N}_S$)
Let $\tilde{N}_S$ be the fuzzy average number of customers in the system. Then;

$$\tilde{\boldsymbol{N}}_{\boldsymbol{S}} = \tilde{\boldsymbol{\rho}} \qquad (49)$$

Proof: Since the fuzzy process $\{\tilde{X}(t) : t \geq 0\}$ denotes the fuzzy number of customers in the system at a date t, the fuzzy average number of customers in the system in the steady state is the fuzzy mathematical expectation of the fuzzy random variable $\tilde{X}$.
We have:

$$\tilde{N}_S = \tilde{E}(\tilde{X}) = \tilde{\mathbb{P}}[\tilde{X} \leq k] = \sum_{k=0}^{+\infty} k \times \tilde{p}_k = \sum_{k=0}^{+\infty} k \times \frac{\tilde{\rho}^k}{k!} \times e^{-\tilde{\rho}} = e^{-\tilde{\rho}} \sum_{k=0}^{+\infty} k \times \frac{\tilde{\rho}^k}{k!}$$
$$= e^{-\tilde{\rho}} \sum_{k=0}^{+\infty} \frac{\tilde{\rho}^k}{(k-1)!} = e^{-\tilde{\rho}} \times \tilde{\rho} \times e^{\tilde{\rho}} = \tilde{\rho}$$

Therefore

$$\tilde{N}_S = \tilde{\rho} \qquad \blacksquare$$

Proposition 4: (Fuzzy average number of customers in the queue: $\tilde{N}_f$)
Let $\tilde{N}_f$ be the average number of customers in the queue. Then;

$$\tilde{\boldsymbol{N}}_{\boldsymbol{f}} = \frac{\tilde{\boldsymbol{\lambda}} + \tilde{\boldsymbol{\mu}}\boldsymbol{e}^{-\tilde{\rho}} - \tilde{\boldsymbol{\mu}}}{\tilde{\boldsymbol{\mu}}} \qquad (50)$$

Proof: Since the system under study is single-server, the presence of k customers in the system implies that there are $k-1$ in the queue. Then $\tilde{N}_f$, is the theoretical fuzzy mean of the fuzzy random variable $\tilde{X}$, with $\tilde{X}$ taking all values between 0 and $k-1$.

$$\tilde{N}_f = \tilde{E}(\tilde{X}) = \tilde{\mathbb{P}}[\tilde{X} \leq k-1] = \sum_{k=1}^{+\infty} (k-1) \times \tilde{p}_k = \sum_{k=1}^{+\infty} k\,\tilde{p}_k - \sum_{k=1}^{+\infty} \tilde{p}_k$$
$$= \sum_{k=0}^{+\infty} k\,\tilde{p}_k - \left(\sum_{k=0}^{+\infty} \tilde{p}_k - \tilde{p}_0\right)$$

Introducing the fuzzy normalization condition $\sum_{k=1}^{+\infty} \tilde{p}_k = 1$ and the relation $\sum_{k=0}^{+\infty} k \times \tilde{p}_k$ i.e. (49) into this relation, we find:

$$\tilde{N}_f = \tilde{N}_S - (1 - \tilde{p}_0) = \tilde{\rho} - \left(1 - e^{-\tilde{\rho}}\right) = \tilde{\rho} + e^{-\tilde{\rho}} - 1 = \frac{\tilde{\lambda}}{\tilde{\mu}} + e^{-\tilde{\rho}} - 1 = \frac{\tilde{\lambda} + \tilde{\mu}e^{-\tilde{\rho}} - \tilde{\mu}}{\tilde{\mu}}$$

Therefore

$$\tilde{N}_f = \frac{\tilde{\lambda} + \tilde{\mu}e^{-\tilde{\rho}} - \tilde{\mu}}{\tilde{\mu}} \qquad \blacksquare$$

Proposition 5: (Fuzzy average residence time: $\tilde{t}_S$)

Let $\tilde{t}_S$ be the fuzzy average residence time. Then ;

$$\tilde{\boldsymbol{t}}_{\boldsymbol{S}} = \frac{\tilde{\boldsymbol{\lambda}}}{\tilde{\boldsymbol{\mu}}^2 - \tilde{\boldsymbol{\mu}}^2 \boldsymbol{e}^{-\tilde{\rho}}} \qquad (51)$$

Proof: Little's fuzzy formula is given by: $\tilde{\boldsymbol{N}}_{\boldsymbol{S}} = \tilde{\boldsymbol{t}}_{\boldsymbol{S}} \times \tilde{\boldsymbol{d}}$ (52)

From (52), we draw $\tilde{t}_S$, we have:

$$\tilde{t}_S = \frac{\tilde{N}_S}{\tilde{d}} = \frac{\tilde{\rho}}{\tilde{\mu}(1 - e^{-\tilde{\rho}})} = \frac{\frac{\tilde{\lambda}}{\tilde{\mu}}}{\tilde{\mu}(1 - e^{-\tilde{\rho}})} = \frac{\tilde{\lambda}}{\tilde{\mu}^2(1 - e^{-\tilde{\rho}})} = \frac{\tilde{\lambda}}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}}$$

Hence

$$\tilde{t}_S = \frac{\tilde{\lambda}}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}} \qquad \blacksquare$$

Proposition 6: (Average fuzzy time in the queue: $\tilde{t}_f$)

Let $\tilde{t}_f$ be the average fuzzy waiting time of customers in the queue. Then;

$$\tilde{\boldsymbol{t}}_{\boldsymbol{f}} = \frac{\tilde{\boldsymbol{\lambda}} + \tilde{\boldsymbol{\mu}}\boldsymbol{e}^{-\tilde{\rho}} - \tilde{\boldsymbol{\mu}}}{\tilde{\boldsymbol{\mu}}^2 - \tilde{\boldsymbol{\mu}}^2 \boldsymbol{e}^{-\tilde{\rho}}} \qquad (53)$$

Proof: The average fuzzy time spent in the server by each client is $1/\tilde{\mu}$, it is easy to conclude that the average fuzzy time spent in the FM / FM / 1 queue is:

$$\tilde{t}_f = \tilde{t}_S - \frac{1}{\tilde{\mu}} = \frac{\tilde{\lambda}}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}} - \frac{1}{\tilde{\mu}} = \frac{\tilde{\lambda} - \left(\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}\right)}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}} = \frac{\tilde{\lambda} + \tilde{\mu}e^{-\tilde{\rho}} - \tilde{\mu}}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}}$$

Therefore

$$\tilde{t}_f = \frac{\tilde{\lambda} + \tilde{\mu}e^{-\tilde{\rho}} - \tilde{\mu}}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}} \qquad \blacksquare$$

VII. Numerical Application

Problem:

In a one-stop commercial bank with infinite capacity and Poissonian arrival and exponential service times, the average time between arrivals is about 10 minutes and the average time per service is about 8 minutes. The bank operates as a waiting system with a priori impatience.

Questions:

b.1. Show that the system is ergodic;

b.2. Determine, in steady state, the fuzzy performance indicators of this system.

Solution:

Hypothetically, this is a fuzzy queueing system with a priori impatience, of the form FM / FM / 1 with a single server and infinite capacity. In the classical model, the arrival rate is $\lambda = 1/10$ minute, or $\lambda = 60/10 = 6$ arrivals per hour, and the service rate is $\mu = 1/8$ minute, or $\mu = 60/8 = 7.5$ departures per hour. Since the arrival rate is about 10 minutes, or about 6 arrivals per hour, and the average rate per service is about 8 minutes, or about 7.5 departures per hour, this suggests that these are fuzzy rates. Assuming that these fuzzy rates $\tilde{\lambda}$ and $\tilde{\mu}$ are triangular fuzzy numbers, we can write:

$$\tilde{\lambda} = (5/\,6/\,7\,) \qquad and \qquad \tilde{\mu} = (6.5/\,7.5/\,8.5)$$

Thus, in a fuzzy model, where the rates $\tilde{\lambda}$ and $\tilde{\mu}$ are fuzzy variables, the performance indicators $\tilde{\mathcal{U}}$; $\tilde{d}$; $\tilde{N}_S$; $\tilde{N}_f$; $\tilde{t}_S$ and $\tilde{t}_f$ also become fuzzy numbers [see formulas (47); (48); (49); (50); (51) and (53)].

Resolution by the L – R method

To evaluate the performance indicators by the L – R method, we proceed as follows:

1°) Determine the L – R forms of rates $\tilde{\lambda} = (5/\,6\,/\,7)$ and $\tilde{\mu} = (6.5\,/\,7.5\,/\,8.5)$; we have:

$$\tilde{\boldsymbol{\lambda}} = \langle \boldsymbol{6}\,;\boldsymbol{1}\,;\boldsymbol{1}\rangle_{\boldsymbol{LR}} \quad (54) \qquad \tilde{\boldsymbol{\mu}} = \langle \boldsymbol{7.5}\,;\boldsymbol{1}\,;\boldsymbol{1}\rangle_{\boldsymbol{LR}} \quad (55)$$

2°) Let us show that the system is ergodic:

We have: $Noy(\tilde{\lambda}) < Noy(\tilde{\mu})$ or $6 < 7$, hence ***the system is ergodic*** or ***stable*** and the calculation of the performance indicators is possible.

3°) (54) and (55) in (47); (48); (49); (50); (51) and (53) and apply the formulas of fuzzy number arithmetic of type $L-R$, suitably chosen from formulas (28); (29); (30) and (32), we obtain successively:

Let us first carry out the required auxiliary calculations:

$$* \quad \tilde{\rho} = \frac{\tilde{\lambda}}{\tilde{\mu}} = \frac{\langle 6\,;1\,;1\rangle_{LR}}{\langle 7.5\,;1\,;1\rangle_{LR}} = \langle \frac{6}{7.5}\,;\, \frac{6}{63.75} + \frac{1}{7.5} - \frac{1}{63.75}\,;\, \frac{6}{48.75} + \frac{1}{7.5} + \frac{1}{48.75} \rangle_{LR}$$

$$\boldsymbol{\tilde{\rho} = \langle 0.8\,;0.212\,;\,0.276\rangle_{LR}} \qquad (56)$$

$$* \quad \boldsymbol{e^{-\tilde{\rho}}} = e^{-0.8} = \boldsymbol{0.45} \qquad (57)$$

$$* \quad \boldsymbol{\langle 7.5\,;1\,;1\rangle_{LR} e^{-\tilde{\rho}}} = \langle 7.5\,;1\,;1\rangle_{LR} \times 0.45 = \boldsymbol{\langle 3.375\,;0.45\,;\,0.45\rangle_{LR}} \qquad (58)$$

$$* \quad \boldsymbol{\langle 7.5\,;1\,;1\rangle_{LR} - \langle 7.5\,;1\,;1\rangle_{LR} e^{-\tilde{\rho}}} = \langle 7.5\,;1\,;1\rangle_{LR} - \langle 3.375\,;0.45\,;\,0.45\rangle_{LR}$$

$$= \boldsymbol{\langle 4.125\,;1.45\,;\,1.45\rangle_{LR}} \qquad (59)$$

$$* \quad \boldsymbol{\langle 6\,;1\,;1\rangle_{LR} + \langle 7.5\,;1\,;1\rangle_{LR} e^{-\tilde{\rho}} - \langle 7.5\,;1\,;1\rangle_{LR}}$$

$$= \langle 6\,;1\,;1\rangle_{LR} + \langle 3.375\,;0.45\,;\,0.45\rangle_{LR} - \langle 7.5\,;1\,;1\rangle_{LR}$$

$$= \langle 9.375\,;1.45\,;\,1.45\rangle_{LR} - \langle 7.5\,;1\,;1\rangle_{LR}$$

$$= \boldsymbol{\langle 1.875\,;2.45\,;\,2.45\rangle_{LR}} \qquad (60)$$

$$* \quad (\boldsymbol{\langle 7.5\,;1\,;1\rangle_{LR}})^{\boldsymbol{2}} = \langle 7.5\,;1\,;1\rangle_{LR} \times \langle 7.5\,;1\,;1\rangle_{LR}$$

$$= \langle 56.25\,;7.5+7.5-1\,;\,7.5+7.5+1\rangle_{LR}$$

$$= \boldsymbol{\langle 56.25\,;14\,;\,16\rangle_{LR}} \qquad (61)$$

$$* \quad (\boldsymbol{\langle 7.5\,;1\,;1\rangle_{LR}})^{\boldsymbol{2}} \boldsymbol{e^{-\tilde{\rho}}} = \langle 56.25\,;14\,;\,16\rangle_{LR} \times 0.45 = \boldsymbol{\langle 25.3125\,;6.3\,;\,7.2\rangle_{LR}} \qquad (62)$$

$$* \quad (\boldsymbol{\langle 7.5\,;1\,;1\rangle_{LR}})^{\boldsymbol{2}} - (\boldsymbol{\langle 7.5\,;1\,;1\rangle_{LR}})^{\boldsymbol{2}} \boldsymbol{e^{-\tilde{\rho}}} = \langle 56.25\,;14\,;\,16\rangle_{LR} - \langle 25.3125\,;6.3\,;\,7.2\rangle_{LR}$$

$$= \boldsymbol{\langle 30.9375\,;21.2\,;\,22.3\rangle_{LR}} \qquad (63)$$

a) Calculating $\tilde{\mathcal{U}}$

(59) and (55) in (47), we obtain:

$$\tilde{\mathcal{U}} = \frac{\langle 4.125\,;1.45\,;\,1.45\rangle_{LR}}{\langle 7.5\,;1\,;1\rangle_{LR}} = \left\langle \frac{4.125}{7.5}\,; \frac{4.125}{63.75} + \frac{1.45}{7.5} - \frac{1.45}{63.75}\,;\, \frac{4.125}{48.75} + \frac{1.45}{7.5} + \frac{1.45}{48.75} \right\rangle_{LR}$$

$$= \boldsymbol{\langle 0.55\,;0.235\,;\,0.308\rangle_{LR}}$$

b) Calculation of $\tilde{d}$

(59) in (48), we have:

$$\boldsymbol{\tilde{d} = \langle 4.125\,;1.45\,;\,1.45\rangle_{LR}}$$

c) Calculation of $\tilde{N}_S$

(56) in (49); we have:

$$\boldsymbol{\tilde{N}_S = \langle 0.8\,;0.212\,;\,0.277\rangle_{LR}}$$

d) Calculating $\tilde{N}_f$

(60) and (55) in (50), we obtain:

$$\boldsymbol{\tilde{N}_f} = \frac{\langle 1.875\,;2.45\,;\,2.45\rangle_{LR}}{\langle 7.5\,;1\,;1\rangle_{LR}} = \left\langle \frac{1.875}{7.5}\,; \frac{1.875}{63.75} + \frac{2.45}{7.5} - \frac{2.45}{63.75}\,;\, \frac{1.875}{48.75} + \frac{2.45}{7.5} + \frac{2.45}{48.75} \right\rangle_{LR}$$

$$= \boldsymbol{\langle 0.25\,;0.338\,;\,0.415\rangle_{LR}}$$

e) Calculating $\tilde{t}_S$

(54) and (63) in (51), we obtain:

$$\boldsymbol{\tilde{t}_S} = \frac{\langle 6\,;1\,;\,1\rangle_{LR}}{\langle 30.9375\,;21.2\,;\,22.3\rangle_{LR}}$$

$$= \left\langle \frac{6}{30.9375}\,; \frac{133.8}{1647.035} + \frac{1}{30.9375} - \frac{22.3}{1647.035}\,;\, \frac{127.2}{301.254} + \frac{1}{30.9375} + \frac{21.2}{301.254} \right\rangle_{LR}$$

$$= \boldsymbol{\langle 0.193\,;0.099\,;\,0.526\rangle_{LR}}$$

f) Calculating $\tilde{t}_f$

(60) and (63) in (53), we obtain:

$$\boldsymbol{\tilde{t}_f} = \frac{\langle 1.875\,;2.45\,;\,2.45\rangle_{LR}}{\langle 30.9375\,;21.2\,;\,22.3\rangle_{LR}}$$

$$= \left\langle \frac{1.875}{30.9375}\,; \frac{41.8125}{1647.035} + \frac{2.45}{30.9375} - \frac{54.635}{1647.035}\,;\, \frac{39.75}{301.254} + \frac{2.45}{30.9375} + \frac{51.94}{301.254} \right\rangle_{LR}$$

$$= \boldsymbol{\langle 0.060\,;0.001\,;\,0.384\rangle_{LR}}$$

Thus, according to relation (35), the kernels (or modes) of $\tilde{\mathcal{U}}$; $\tilde{d}$; $\tilde{N}_S$; $\tilde{N}_f$; $\tilde{t}_S$ and $\tilde{t}_f$ are respectively 0.55; 4.125; 0.8; 0.25; 0.193 and 0.060.

According to relation (36), their supports are the following open intervals:

$$Supp(\tilde{\mathcal{U}}) =]0.55 - 0.23\ ;\ 0.55 + 0.308[=]0.315\ ;\ 0.858[$$

$$Supp(\tilde{d}) =]4.125 - 1.45\ ;\ 4.125 + 1.45[=]2.675\ ;\ 5.575[$$

$$Supp(\tilde{N}_S) =]0.8 - 0.212\ ;\ 0.8 + 0.276[=]0.588\ ;\ 1.076[$$

$$Supp(\tilde{N}_f) =]0.25 - 0.338 \; ; \; 0.25 + 0.415[=]-0.088 \; ; \; 0.665[$$
$$Supp(\tilde{t}_S) =]0.193 - 0.099 \; ; \; 0.193 + 0.526[=]0.094 \; ; \; 0.719[$$
$$Supp(\tilde{t}_f) =]0.060 - 0.001 \; ; \; 0.060 + 0.384[=]-0.059 \; ; \; 0.444[$$

Variants introduced by the $L - R$ method

In steady state, the $L - R$ method introduces some variations in the treatment of queueing systems with a priori impatience as follows:

- The performance indicators are fuzzy numbers of the same $L - R$ type and not average values as for ordinary queueing systems with a priori impatience;
- The $L– R$ notations of these indicators contain real numbers of the $L - R$ type denoted $\langle m \, ; a \, ; b \rangle_{LR}$;
- The supports of these indicators are open intervals where the bounds are real numbers;
- The modes of these indicators are real numbers that correspond exactly to average values for ordinary queueing systems with a priori impatience.

VIII. Conclusion

The aim of this article was to analyze the performance of the Markovian fuzzy queueing system FM/FM/1 with a priori impatience, in steady state, using the L-R method.

To achieve this, we reviewed some concepts of fuzzy set theory that can facilitate the study of this system in order to evaluate the performance indicators. In this article, we only used triangular fuzzy numbers and L-R fuzzy numbers. The main tool we relied on is the mathematical approach to L-R fuzzy numbers. This approach is based on the arithmetic of L-R fuzzy numbers, using two approximations, the tangent approximation and the secant approximation, for calculating the product and the quotient. However, in this study, we only used the secant approximation simply because it yields the same results as those provided by other approaches in terms of supports and modal values. A numerical application is discussed to illustrate the validity and practicability of this approach.

From this numerical application, it emerged that all the performance indicator values similar to those calculated using the classical model were obtained. Each indicator is characterized by its support and its mode. The fuzzy state of these performance indicators shows that a Markovian queueing system with impatient customers in which imprecise information is introduced maintains this imprecision throughout its process. Compared to the results obtained in a classical model, those obtained in a fuzzy model are more informative and more realistic.

References

[1]. Gross, D., Shortle John, F., Thompson James, M. And Harris Carl, M., (2008); Fundamentals Of Queueing Theory, Fifth Edition. John Wiley And Sons, Inc.
[2]. Fayama, F., Zongo, A. And Nitiema, P.C., (2024); Performance Evaluation Of Queuing Systems With Balking, Reneging, Feedback, Impatient Customersand Downed Servers. Far East Journal Of Theoretical Sstatistics, Vol. 68(2), Pp. 181 – 198. Https: //Doi.Org / 10.17654 / 0972086324011
[3]. Kangzou, W., Li, N. And Jiang, Z., (2010) ; Queueing System With Impatient Customers. IEEE Https: //WWW.Researchgate.Net/Publication/251948079
[4]. Haight, F.A., (1957); Queuing With Balking, I.,Biometrika, Vol. 44, Pp. 360 – 369
[5]. Barrer, D.Y., (1957) ; Queueing With Impatient Customers And Ordered Service. Operations Research, Vol. 5, Pp. 650 – 656
[6]. Barrer, D.Y., (1957); Queueing With Impatient Customers And Indifferent Clerks. Operations Research, Vol. 5, Pp. 644 – 649
[7]. Saikia, G., Medhi, P. And Choudhury, A., (2020); Analyzing Impatiencein Multiserver Markovian Queues. International Journal Of Supply And Operations Management, Vol. 7(4), Pp. 310 – 321. Https: //Doi.Org/10.22034/IJSOM.2020.4.2.
[8]. Bouchentouf, A.A., Kadi, M. And Rabhi, A., (2014) ; Analysis Of Two Heterogeneous Server Queueing Model With Balking, Reneging And Feedback. Mathematical Sciences And Applications E – Notes, Vol.2(2),Pp. 10 – 21
[9]. Jolai, F., Asadzadeh, S.M., Ghodsi, R. And Bagheri – Marani, Sh., (2016); A Multi – Objective Fuzzy Queuing Priority Assignment Model. Applied Mathematical Modeling ; Vol. 40(21), Pp. 9500 – 9513
[10]. Ehsanifar, M., Hamta, N. And Hemesy, M., (2017); A Simulation Approach To Evaluate Performance Indices Of Fuzzy Exponential Queuing System (An M / M / C Model In Balking Case Study). Journal Of Industrial Engineering And Management Studies (JIEMS), Vol. 4(2), Pp. 35 – 51
[11]. Alonge W'Omatete, J., Mabela Makengo, M.R., Lama Okenge, D., Mfumu Mbikudi, C. And Mangata, B., (2023); Computing The Perfomance Parameters Of Fuzzy Markovian Queueing System FM / FM / 1 In Transient Regime By Flexibe Alpha – Cuts Method. Journal Of Computing Research And Innovation (JCRINN), Vol. 8(1), Pp. 17 – 31
[12]. Dubois, D. And Prade, H., (1980); Fuzzy Sets And Systems: Theory And Applications. Academic Press, Inc.
[13]. Adia Leti Mawa, B., Mabela Makengo, R., Mukeba Kanyinda, J. P. And Boluma Mangata, B., (2022); Analysis Of The Performance Measures Of A Non –Markovian Fuzzy Queue Via Fuzzy Laplace Transforms Method; Journal Of Computing Research And Introvation (JCRINN), Vol. 7(2), Pp. 304 – 415
[14]. Hanss, M., (2005); Applied Fuzzy Arithmetic, An Introduction With Engineering Applications. Spriger – Verlag Berlin Heidelberg.
[15]. Celkyilmaz, A. And Turksen, I.B., (2009); Modeling Uncertainty With Fuzzy Logic. Springer –Verlag Berlin Heidelberg
[16]. Ancker Jr, C.J. And Gafarian, A.V., (1963); Somme Queuing Problems With Balking And Reneging, I. Operations Research, Vol. 11(1), Pp. 88 – 100
[17]. Ancker Jr, C.J. And Gafarian, A.V., (1963); Somme Queuing Problems With Balking And Reneging, II. Operations Research, Vol. 11(6), Pp. 928 – 937. Https: // Doi. Org / 10.1287 / Opre.11.6.928

[18]. Bede, B., (2013) Mathematics Of Fuzzy Sets And Fuzzy Logic. Spriger Verlag, Berlin Heidelberg
[19]. Wang Li, X., (1997); A Course In Fuzzy Systems And Control. International Edition QA76.87 W 35 Prentice –Hall International, Inc.
[20]. Mukeba Kanyinda, J. P., (2020); Analysis Of Retrial Fuzzy Queue With Single Server Subject To Breakdown And Repairs By Flexible Alpha –Cuts Method; Journal Of Pure And Applied Mathematics: Advances And Applications, Vol. 22(1), Pp. 41 – 58
[21]. Mathew, S., Mordeson John, N. And Malik Davender S., (2018); Fuzzy Graph Theory. Springer International Publishing AG.
[22]. Garg, H., (2016); Arithmetic Operations On Generalized Parabolic Fuzzy Numbers And Its Application. The National Academy Of Sciences, India.
[23]. Kwang, H. L., (2005); First Course On Fuzzy Theory And Applications. Spriger –Verlag Berlin Heidelberg.
[24]. Jain, A. And Sharma, A., (2020); Membership Function Formulation Methods For Fuzzy Logic Systems: A Comprehension Review. Journal Of Critical Reviews Vol. 7 (19), Pp. 2394 – 5125.
[25]. Syropoulos, A. And Grammenos, T., (2020); A Modern Introduction To Fuzzy Mathematics. John Wiley & Sons, Inc.
[26]. Mahama, S., Chutia, R. And Baruah Hemanta, K., (2010); Fuzzy Arithmetic Without Using The Method Of A-Cuts . International Journal Of Hatest Trends In Computing; Vol. 1 Isue 2; E – ISSN: 2045 – 5364
[27]. Stefanini, L., Sorini, L. And Guerra, M. L., (2006); Parametric Representation Of Fuzzy Numbers And Application To Fuzzy Calculus. Fuzzy Sets And Systems; Vol. 257, Pp. 2433 – 2455
[28]. Buckley James, J., (2004); Fuzzy Probabilities And Fuzzy Sets For Web Planning. Springer – Verlag Berlin Heidelberg.
[29]. Siler, W. And Buckley, J. J., (2005); Fuzzy Expert Systems And Fuzzy Reasoning. John Wiley & Sons, Inc. All Right Reserved
[30]. Kaufmann, A.,(1973); Introduction À La Théorie Des Sous – Ensembles Flous (Fuzzy Sets Theory) ; 1. Eléments Théoriques De Base. Masson Et Cie, Paris..
[31]. Dubois, D. And Prade, H. (1979); On Fuzzy Random –Valued Optimization. America Journal Of Operations Research, Vol. 2, Pp. 327 – 348
[32]. Haight, F.A., (1959); Queueing With Reneging. Metrika, Vol. 2, Pp. 186 – 197; Http: // Doi. Org / 10.1016 / J.Apm.2009.03.019
[33]. Klir George, J. And Yuan, B., (1995); Fuzzy Sets And Fuzzy Logic. Theory And Applications; Prentice Hall PTR Upper Saddle River, New Jersey 07458
[34]. Mukeba Kanyinda, J.P., Mabela Makengo, M.R. Et Ulungu, B., (2015) ; Computing Fuzzy Queueing Performance Measures By L-R Method. Journal Of Fuzzy Sets Valued Analysis; Vol. 1, Pp. 57 – 67, March