IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-0661, p-ISSN: 2278-8727, Volume 21, Issue 3, Ser. 2 (May. – June. 2025), PP 01-05
www.iosrjournals.org

On Common Fixed-Point Theorems In Complete Rectangular S-Metric Spaces

T. R. Shimpi[1], S. G. Dapke[2]
Gf's Godavari College Of Engineering, Jalgaon, Maharashtra, India
Iqra's H. J. Thim College Of Arts And Science, Mehrun, Jalgaon, India

Abstract:
In this paper, we establish the some common fixed point theorems in rectangular S-metric spaces, an advanced generalization of S-metric spaces. We develop new common fixed point theorems that integrate and extend various well-known results in fixed point theory. Our findings are further supported by illustrative examples.

Keywords: *Rectangular metric spaces; S-metric spaces; Rectangular S-metric spaces*

Date of Submission: 15-05-2025 Date of Acceptance: 25-05-2025

I. Introduction And Preliminaries

Frechet [4] originally introduced the concept of a metric space. Over the course of time, mathematicians have delved into diverse extensions and generalizations of metric spaces. Initially, Gahler [5] and Dhage [3] introduced the ideas of 2-metric spaces and D-metric spaces, respectively. Subsequently, Mustafa and Sims [6] extended the theory by introducing G-metric spaces. In more recent developments, Shaban Sedghi [9, 10] introduced the novel concepts of $D^*$and S-metric spaces, providing some of their fundamental properties. Following this, Sedghi [10] focused on advancing the theory of S -metric spaces, deriving new results that have been presented in several papers. In this paper, we find some new results of rectangular S -metric spaces and prove common fixed point theorems on same spaces.

Definition 1.1. [10] Let X be a non empty set and $S: X^3 \to R^+$, a function that satisfies the following properties:
(i) $S(x,y,z) = 0$ if and only if $x = y = z$,
(ii) $S(x,y,z) \le S(x,x,a) + S(y,y,a) + S(z,z,a)$ for all $a, x, y, z \in X$ (rectangle inequality).
Then (X, S) is called S -metric space.

Definition 1.2. [1] Let X be a non empty set and $S: X^3 \to R^+$, a function satisfying the following properties:
(i) $S(x,y,z) = 0$ if and only if $x = y = z$,
(ii) $S(x,y,z) \le S(x,x,a) + S(y,y,a) + S(z,z,a)$ for all $x, y, z \in X$ and all distinct points $a \in X - \{x,y,z\}$. Then (X, S) is called a rectangular S -metric space.

Definition 1.3. [10] Let (X, S) be an S -metric space and $A \subset X$.
(i) A sequence $\{x_n\}$ in X converges to x if $S(x_n, x_n, x) \to 0$ as $n \to \infty$, that is for every $\in > 0$ there exists $n_o \in N$ such that for $n \ge n_o$, $S(x_n, x_n, x) < \in$. This case, we denote by $\lim_{n\to\infty} x_n = x$ and we say that x is the limit of $\{x_n\}$ in X.
(ii) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if for each $\in > 0$, there exists $n_o \in N$ such that $S(x_n, x_n, x_m) < \in$ for each $n, m \ge n_0$.
(iii) The S -metric space (X, S) is said to be complete if every Cauchy sequence is convergent.

Proposition 1.4. [2] Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

Lemma 1.5. [10] If (X, S) is an S -metric space, then we have $S(x,x,y) = S(y,y,x)$ for all $x, y \in X$.

Lemma 1.6. [10] Let (X, S) be an S -metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X converging to x and y respectively, that is, $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $S(x_n, x_n, y_n) \to S(x,x,y)$ as $n \to \infty$.

Lemma 1.7. [10] Let (X, S) be an S -metric space. If the sequence $\{x_n\}$ in X converges to x, then the limit x is unique.

Lemma 1.8. [10] Let (X, S) be an S -metric space. If the sequence $\{x_n\}$ in X converges to x, then $\{x_n\}$ is a Cauchy sequence.

Example 1.9. Let $X = N \cup \{0\}$ and define $S\colon X \times X \times X \to R^{+} \cup \{0\}$ by

$$S(l,m,n) = \begin{cases} 0 & if \quad l = m = n, \\ lm + mn + nl & if\ l \neq m \neq n. \end{cases}$$

Then (X, S) is a rectangular S -metric space.

II. Main Results

Theorem 2.1. Let (X, S) be a complete rectangular S -metric space and let f and g be a self mapping on X. Assume that f and g satisfies the following conditions:

$$S(fx, fx, fy) \leq h[S(fx, fx, gy) + S(fy, fy, gx)],$$

where $0 \leq h < 0.2$, also,

(i) $f(X) \subseteq g(X)$,

(ii) If $g(X)$ is complete.

Then f and g have a unique coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof: Let x_0 be any point in X. Since $fx_0 \subseteq f(X)$ and $f(X) \subseteq g(X)$, there exists a point x_1 in X such that $fx_0 = gx_1$. As $x_1 \in X$, it follows that $fx_1 \subseteq f(X)$. Thus, we can select x_2 in X such that $fx_1 = gx_2$. Repeating this process iteratively, we construct a sequence $\{x_n\}$ in X, where $x_{n+1} \in X$, satisfies $fx_n = gx_{n+1}$ for all n.

Now, consider

$$\begin{aligned} S(gx_{n+1}, gx_{n+1}, gx_n) &= S(fx_n, fx_n, fx_{n-1}) \\ &\leq h[S(fx_n, fx_n, gx_{n-1}) + S(fx_{n-1}, fx_{n-1}, gx_n)] \\ &= h[S(gx_{n+1}, gx_{n+1}, gx_{n-1}) + S(gx_n, gx_n, gx_n)] \\ &\leq hS(gx_{n+1}, gx_{n+1}, gx_{n-1}) \\ &\leq h[S(gx_{n+1}, gx_{n+1}, gx_n) + S(gx_{n+1}, gx_{n+1}, gx_n) \\ &\quad + S(gx_{n-1}, gx_{n-1}, gx_n)] \\ &= h[2S(gx_{n+1}, gx_{n+1}, gx_n) + S(gx_{n-1}, gx_{n-1}, gx_n)] \end{aligned}$$

$$\Rightarrow (1-2h)S(gx_{n+1}, gx_{n+1}, gx_n) \leq hS(gx_{n-1}, gx_{n-1}, gx_n)$$

$$\begin{aligned} \Rightarrow \quad S(gx_{n+1}, gx_{n+1}, gx_n) &\leq \frac{h}{(1-2h)} S(gx_{n-1}, gx_{n-1}, gx_n) \\ &= kS(gx_n, gx_n, gx_{n-1}) \end{aligned}$$

$$S(gx_{n+1}, gx_{n+1}, gx_n) \leq k^n S(gx_1, gx_1, gx_0).$$

For $m, n \in N$ and some $N \in N$, with $n > m$, we have

$$\begin{aligned} S(gx_n, gx_n, gx_m) &\leq S(gx_n, gx_n, gx_{n-1}) + S(gx_n, gx_n, gx_{n-1}) + S(gx_m, gx_m, gx_{n-1}) \\ &\leq k^{n-1}S(gx_1, gx_1, gx_0) + k^{n-1}S(gx_1, gx_1, gx_0) + S(gx_m, gx_m, gx_{n-1}) \\ &= 2k^{n-1}S(gx_1, gx_1, gx_0) + S(gx_m, gx_m, gx_{n-1}) \\ &\leq 2k^{n-1}z + [S(gx_m, gx_m, gx_{n-2}) + S(gx_m, gx_m, gx_{n-2}) \\ &\quad + S(gx_{n-1}, gx_{n-1}, gx_{n-2})] \\ &\leq 2k^{n-1}z + [2S(gx_m, gx_m, gx_{n-2}) + k^{n-2}S(gx_1, gx_1, gx_0)] \\ &= 2k^{n-1}z + k^{n-2}z + 2S(gx_m, gx_m, gx_{n-2}) \\ &\leq 2k^{n-1}z + k^{n-2}z + 2[2S(gx_m, gx_m, gx_{n-3}) \\ &\quad + S(gx_{n-2}, gx_{n-2}, gx_{n-3})] \\ &\leq 2k^{n-1}z + k^{n-2}z + 4S(gx_m, gx_m, gx_{n-3}) + 2k^{n-3}z \\ &\leq 2k^{n-1}z + k^{n-2}z + 2k^{n-3}z + 4[2S(gx_m, gx_m, gx_{n-4}) \\ &\quad + S(gx_{n-3}, gx_{n-3}, gx_{n-4})] \\ &\leq 2k^{n-1}z + k^{n-2}z + 2k^{n-3}z + 8S(gx_m, gx_m, gx_{n-4}) + 4k^{n-4}z \\ &= 2k^{n-1}z + k^{n-2}z + 2k^{n-3}z + 4k^{n-4}z + \cdots \\ &= 2k^{n-1}z + k^{n-2}z\left(1 + \frac{2}{k} + \frac{4}{k^2} + \frac{8}{k^3} + \cdots\right) \end{aligned}$$

$$= 2k^{n-1}z + k^{n-2}z(1 + (\tfrac{2}{k}) + \left(\tfrac{2}{k}\right)^2 + \left(\tfrac{2}{k}\right)^3 + \cdots)$$
$$= 2k^{n-1}z + k^{n-2}z(1 - \tfrac{2}{k})^{-1},$$

where $z = S(gx_1, gx_1, gx_0)$, as $n \to \infty$ and since $k < 1$, we have,

$$\lim_{n\to\infty} S(gx_n, gx_n, gx_m) = 0.$$

Thus, $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, there exist q in $g(X)$ such that $gx_n \to$ q, as $n \to \infty$. Consequently, we can find p in X such that $gp = q$. Thus,

$$S(fx_n, fx_n, fp) \leq h[S(fx_n, fx_n, gp,) + S(fp, fp, gx_n)$$
$$= h[S(gx_{n+1}, gx_{n+1}, gp,) + S(fp, fp, gx_n)$$
$$= h[S(q, q, q) + S(fp, fp, q)]$$

Letting $n \to \infty$,

$$S(q, q, fp) \leq hS(q, q, fp).$$

Since $S(q,q,q) = 0$ and $h < 1$, this is true if $S(q,q,fp) = 0 \Rightarrow fp = q$. Therefore, $fp = gp = q$. Hence, q is the point of coincidence of f and g. Now, we show that f and g have a unique point of coincidence. For this, assume that there exists another point $w \in X$ such that $fw = gw = w$.

Now

$$S(q, q, w) = S(fp, fp, fw)$$
$$\leq h[S(fp, fp, gw) + S(fw, fw, gp)]$$
$$= h[S(q, q, w) + S(w, w, q)]$$
$$\leq h[S(q, q, w) + S(q, q, w)$$
$$= 2hS(q, q, w).$$

Since $h \in [0, 0.2)$. It is true if $S(q, q, w) = 0$. So $q = w$. Hence, it is prove that, f and g have a unique point of coincidence. Since, f and g are weakly compatible, so by Preposition 1.4, f and g have a unique common fixed point in X.

Theorem 2.2. Let (X, S) be a complete rectangular S-metric space. Suppose that mapping $f: X \to X$ satisfies the following condition:

$$S(fx, fx, fy) \leq \alpha_1 S(x, x, y) + \alpha_2 S(x, x, fx) + \alpha_3 S(y, y, fy) + \alpha_4 S(x, x, fy) + \alpha_5 S(y, y, fx),$$

for all $x, y \in X$, where $\alpha i \geq 0$ for each $i \in \{1, 2, 3, 4, 5\}$ and $\alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_4 + \alpha_5 < 1$. Then f has a unique fixed point in X.

Proof: Let x_0 be an arbitrary point of X, and define x_n by $x_{2n+1} = fx_{2n}$, $x_{2n+2} = fx_{2n+1}$, $n = 0, 1, 2, \cdots$

Now,

$$S(x_{2n+1}, x_{2n+1}, x_{2n+2}) = S(fx_{2n}, fx_{2n}, fx_{2n+1})$$
$$\leq \alpha_1 S(x_{2n}, x_{2n}, x_{2n+1}) + \alpha_2 S(x_{2n}, x_{2n}, fx_{2n}) + \alpha_3 S(x_{2n+1}, x_{2n+1}, fx_{2n+1}) + \alpha_4 S(x_{2n}, x_{2n}, fx_{2n+1}) + \alpha_5 S(x_{2n+1}, x_{2n+1}, fx_{2n})$$
$$= \alpha_1 S(x_{2n}, x_{2n}, x_{2n+1}) + \alpha_2 S(x_{2n}, x_{2n}, x_{2n+1}) + \alpha_3 S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + \alpha_4 S(x_{2n}, x_{2n}, x_{2n+2}) + \alpha_5 S(x_{2n+1}, x_{2n+1}, x_{2n+1})$$
$$= \alpha_1 S(x_{2n}, x_{2n}, x_{2n+1}) + \alpha_2 S(x_{2n}, x_{2n}, x_{2n+1}) + \alpha_3 S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + \alpha_4 [S(x_{2n}, x_{2n}, x_{2n+1}) + S(x_{2n}, x_{2n}, x_{2n+1}) + S(x_{2n+2}, x_{2n+2}, x_{2n+1})]$$
$$= \alpha_1 S(x_{2n}, x_{2n}, x_{2n+1}) + \alpha_2 S(x_{2n}, x_{2n}, x_{2n+1}) + \alpha_3 S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + \alpha_4 [2S(x_{2n}, x_{2n}, x_{2n+1}) + S(x_{2n+1}, x_{2n+1}, x_{2n+2})]$$
$$= (\alpha_1 + \alpha_2 + 2\alpha_4) S(x_{2n}, x_{2n}, x_{2n+1}) + (\alpha_3 + \alpha_4) S(x_{2n+1}, x_{2n+1}, x_{2n+2})$$

$$\Rightarrow (1 - (\alpha_3 + \alpha_4))\, S(x_{2n+1}, x_{2n+1}, x_{2n+2}) \leq (\alpha_1 + \alpha_2 + 2\alpha_4) S(x_{2n}, x_{2n}, x_{2n+1})$$
$$\Rightarrow S(x_{2n+1}, x_{2n+1}, x_{2n+2}) \leq \left(\frac{\alpha_1 + \alpha_2 + 2\alpha_4}{(1 - (\alpha_3 + \alpha_4))}\right) S(x_{2n}, x_{2n}, x_{2n+1})$$
$$= \delta S(x_{2n}, x_{2n}, x_{2n+1})$$
$$\leq \delta^{2n+1} S(x_0, x_0, x_1).$$

Where $\delta = \frac{\alpha_1 + \alpha_2 + 2\alpha_4}{(1 - (\alpha_3 + \alpha_4))} < 1$. Then, For $m, n \in N$ and some $N \in N$, with $n > m$, we have

$$S(x_n,x_n,x_m) \le S(x_n,x_n,x_{n+1}) + S(x_n,x_n,x_{n+1}) + S(x_m,x_m,x_{n+1})$$
$$\le 2S(x_n,x_n,x_{n+1}) + S(x_m,x_m,x_{n+1})$$
$$\le 2\delta^n S(x_0,x_0,x_1) + [S(x_m,x_m,x_{n+2}) + S(x_m,x_m,x_{n+2}) + S(x_{n+1},x_{n+1},x_{n+2})]$$
$$= 2\delta^n z + [2S(x_m,x_m,x_{n+2}) + S(x_{n+1},x_{n+1},x_{n+2})]$$
$$\le 2\delta^n z + [2S(x_m,x_m,x_{n+2}) + \delta^{n+1} z]$$
$$\le 2\delta^n z + \delta^{n+1} z + 2\,[S(x_m,x_m,x_{n+3}) + S(x_m,x_m,x_{n+3}) + S(x_{n+2},x_{n+2},x_{n+3})]$$
$$\le 2\delta^n z + \delta^{n+1} z + 2[2S(x_m,x_m,x_{n+3}) + \delta^{n+2} z]$$
$$\le 2\delta^n z + \delta^{n+1} z + 2\delta^{n+2} z + 4[S(x_m,x_m,x_{n+4}) + S(x_m,x_m,x_{n+4}) + S(x_{n+3},x_{n+3},x_{n+4})]$$
$$\le 2\delta^n z + \delta^{n+1} z + 2\delta^{n+2} z + 4[2S(x_m,x_m,x_{n+4}) + \delta^{n+3} z]$$
$$S(x_n,x_n,x_m) \le 2\delta^n z + \delta^{n+1} z + 2\delta^{n+2} z + 4\delta^{n+3} z + 8\delta^{n+4} z + \cdots$$
$$= 2\delta^n z + \delta^{n+1} z(1 + 2\delta + 4\delta^2 + 8\delta^3 + \cdots)$$
$$= 2\delta^n z + \delta^{n+1} z(1 + (2\delta) + (2\delta)^2 + (2\delta)^3 + \cdots$$
$$S(x_n,x_n,x_m) \le 2\delta^n z + \delta^{n+1} z\left(\frac{1}{1-2\delta}\right),$$

where $z = S(x_0,x_0,x_1)$. Since $\delta < 1$, as $n \to \infty$, we have,

$$\lim_{n\to\infty} S(gx_n,gx_n,gx_m) = 0.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exist q in X such that $x_n \to$ q, as $n \to \infty$. Now,

$$S(x_n,x_n,fq) \le S(x_n,x_n,x_{n+1}) + S(x_n,x_n,x_{n+1}) + S(fq,fq,x_{n+1})$$
$$= 2S(x_n,x_n,x_{n+1}) + S(x_{n+1},x_{n+1},fq)$$
$$\le 2\delta^n z + [S(x_{n+1},x_{n+1},x_{n+2}) + S(x_{n+1},x_{n+1},x_{n+2}) + S(fq,fq,x_{n+2})]$$
$$= 2\delta^n z + [2S(x_{n+1},x_{n+1},x_{n+2}) + S(fq,fq,x_{n+2})]$$
$$\le 2\delta^n z + [2\delta^{n+1} z + S(x_{n+2},x_{n+2},fq)]$$
$$\le 2\delta^n z + 2\delta^{n+1} z + [\,S(x_{n+2},x_{n+2},x_{n+3}) + S(x_{n+2},x_{n+2},x_{n+3}) + S(fq,fq,x_{n+3})]$$
$$= 2\delta^n z + 2\delta^{n+1} z + [\,2S(x_{n+2},x_{n+2},x_{n+3}) + S(x_{n+3},x_{n+3},fq)]$$
$$\le 2\delta^n z + 2\delta^{n+1} z + 2\delta^{n+2} z + S(x_{n+3},x_{n+3},fq)$$
$$\le 2\delta^n z + 2\delta^{n+1} z + 2\delta^{n+2} z + [\,S(x_{n+3},x_{n+3},x_{n+4}) + S(x_{n+3},x_{n+3},x_{n+4}) + S(fq,fq,x_{n+4})]$$
$$= 2\delta^n z + 2\delta^{n+1} z + 2\delta^{n+2} z + [2S(x_{n+3},x_{n+3},x_{n+4}) + S(x_{n+4},x_{n+4},fq)]$$
$$\le 2\delta^n z + 2\delta^{n+1} z + 2\delta^{n+2} z + 2\delta^{n+3} z + S(x_{n+4},x_{n+4},fq)$$
$$\le 2\delta^n z + 2\delta^{n+1} z + 2\delta^{n+2} z + 2\delta^{n+3} z + \cdots$$
$$\le 2\delta^n z(1 + \delta + \delta^2 + \delta^3 + \cdots)$$
$$= 2\delta^n z\left(\frac{1}{1-\delta}\right),$$

where $z = S(x_0,x_0,x_1)$. Since $\delta < 1$, as $n \to \infty$, we have,

$$\lim_{n\to\infty} S(q,q,fq) = 0 \Rightarrow \text{fq} = \text{q}.$$

Hence, f has a fixed point in X. To prove uniqueness, suppose that if w is another fixed point of f , then

$$S(q,q,w) = S(fp,fp,fw)$$
$$\le \alpha_1 S(q,q,w) + \alpha_2 S(q,q,fq) + \alpha_3 S(w,w,fw) + \alpha_4 S(q,q,fw) + \alpha_5 S(w,w,fq)$$
$$= \alpha_1 S(q,q,w) + \alpha_2 S(q,q,q) + \alpha_3 S(w,w,w) + \alpha_4 S(q,q,w) + \alpha_5 S(w,w,q)$$
$$= \alpha_1\, S(q,q,w) + \alpha_4 S(q,q,w) + \alpha_5 S(q,q,w)$$
$$S(q,q,w) \le (\alpha_1 + \alpha_4 + \alpha_5) S(q,q,w)$$

Since, $\alpha i > 0$. This gives $S(q,q,w) = 0$. Hence, $q = w$. It is prove f has a unique fixed point in X.

References

[1] O. K. Adewale, C.Iluno, Fixed Point Theorems On Rectangular S-Metric Spaces, Scientific African, 16 (2022) 1-10.

[2] M. Abbas, G. Jungck, Common Fixed Point Results For Non-Commuting Mappings Without Continuity In Cone Metric Spaces, J. Math. Anal. Appl. 341, (2008), 416–420.

[3] B. C. Dhage, Generalized Metric Spaces Mappings With Fixed Point, Bull. Calcutta Math. Soc.84, (1992) 329–336.

[4] M. Frechet, Sur Quelques Points Du Calcul Fonctionnel, Rendiconti Del Circolo Matematico Di Palermo 22 (1906) 1–72.

[5] S. Gahler, 2-Metrische Raume Und Ihre Topologische Struktur, Mathematishe Nachrichten 26 (1963) 115–148.

[6] Z. Mustafa, B. Sims, A New Approach To Generalized Metric Spaces, J. Nonlinear Convex Anal. 7, (2006) 289–297.

[7] Z. Mustafa, R.J. Shahkoohi, V. Parvaneh, Z. Kadelburg, M.M.M. Jaradat, Ordered S_P-Metric Spaces And Some Fixed Point Theorems For Contractive Mappings With Application To Periodic Boundary Value Problems, Fixed Point Theory Appl. 2019 (2019) 20 .

[8] R. Roshan, V. Parvaneh, N. Shobkolaei, S. Sedghi, W. Shatanawi, Common Fixed Points Of Almost Generalized $(\Psi, \Phi)_S$-Contractive Mappings In Ordered B-Metric Spaces, Fixed Point Theory Appl. 2013 (2013) 159.

[9] S. Sedghi, N. Shobe And M. Ali, A Generalization Of Metric Spaces. Mathematical Communications, 17 (2012), 39–50.

[10] S. Sedghi, N. Shobe And A. Aliouche, A Generalization Of Fixed Point Theorem In S-Metric Spaces, Mat. Vesnik, 64 (2012) 258-266.

[11] T. Zamfirescu, Fixed Point Theorems In Metric Spaces, Arch. Math. Logic (Basel) 23 (1972) 292–298.