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(σ,τ) –Reverse Left Centralizer On Prime Rings

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Abstract:
In this paper, we introduced the concepts of (σ,τ)- reverse left centralizer, Jordan (σ,τ)- reverse left centralizer, Jordan triple (σ,τ)- reverse left centralizer on rings and we prove that:
Every Jordan (σ,τ) reverse left centralizer of a 2-torsion free prime ring R is a (σ,τ)- reverse left centralizer centralizer of R.
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I. Introduction:

The concept of a left (resp. right) centralizer and a Jordan left (resp. right) centralizer of rings, was first introduced in [3] and thus proved that any left (resp. right) Jordan centralizer of a 2- torsion free semi-prime ring is a left (resp. right) centralizer. While, when n = 1 [2] introduced the concept of a reverse left (resp. right) centralizer and a Jordan reverse left (resp. right) centralizer on rings, many results were founded by the researchers.

The definition of prime ring, semiprime ring and 2- torsion free ring were introduced in [1]. In this paper, we define and study the concepts of a (σ,τ)- reverse left centralizer, a Jordan (σ,τ)- reverse left centralizer and we present some properties about these concepts. One important question can be answered in this paper whether there is a relation between a concept of (σ,τ)- reverse left centralizer and Jordan (σ,τ)- reverse left centralizer within certain condition .

II. (σ,τ)- Reverse Left Centralizer On Rings:

In this section we introduced the concept of a (σ,τ) -reverse left centralizer and a Jordan (σ,τ)- reverse left centralizer on rings.

Definition (2.1):

Let t be an additive mapping of a ring R into itself, such that $t_o = id_R$ and σ , τ be two endomorphisms of R. Then t is called a (σ,τ)- reverse left centralizer if for all x ,y ∈ R
$t(xy) = t(\sigma(y))\,\tau(x)$.

Definition (2.2):

Let t be an additive mapping of a ring R into itself, such that $t_o = id_R$ and σ , τ be two endomorphisms of R. Then t is called a Jordan (σ,τ)- reverse left centralizer if for all x ∈ R
$t(x^2) = t(\sigma(x))\,\tau(x)$.

Definition (2.3):

Let t be an additive mapping of a ring R into itself, such that $t_o = id_R$ and σ , τ be two endomorphisms of R. Then t is called a Jordan triple (σ,τ)- reverse left centralizer if for all x ,y ∈ R
$t(xyx) = t(\sigma(x))\,\tau(y)\,\tau(x)$.

Lemma (2.4):
Let t be a Jordan (σ,τ)- reverse left centralizer of a ring R. Then for all x, y, z ∈ R
(i) $t(xy+yx) = t(\sigma(y))\,\tau(x) + t(\sigma(x))\,\tau(y)$.
(ii) $t(xyz+zyx) = t(\sigma(z))\,\tau(y)\,\tau(x) + t(\sigma(x))\,\tau(y)\,\tau(z)$.
(iii) In particular, if R is a 2-torsion free commutative ring. Then
$t(xyz) = t(\sigma(z))\,\tau(y)\,\tau(x)$.

Proof:

(i) $t((x+y)(x+y)) = t(\sigma(x+y))\,\tau(x+y)$
$t(\sigma(x))\,\tau(x) + t(\sigma(x))\,\tau(y) + t(\sigma(y))\,\tau(x) + t(\sigma(y))\,\tau(y)$ … (1)
On the other hand:
$t((x+y)(x+y)) = t(x^2 + xy + yx + y^2)$
$= t(x^2) + t(y^2) + t(xy + yx)$
$= t(\sigma(x))\,\tau(x) + t(\sigma(y))\,\tau(y) + t(xy + yx)$ … (2)

Comparing (1) and (2), we get:
$t(xy+yx) = t(\sigma(y))\,\tau(x) + t(\sigma(x))\,\tau(y)$

(ii) Replace x + z for x in Definition (2.3), we get:
$t((x+z)\,y\,(x+z)) = t(\sigma(x+z))\,\tau(y)\,\tau((x+z))$
$= t(\sigma(x))\,\tau(y)\,\tau(x) + t(\sigma(x))\,\tau(y)\,\tau(z) +$
$t(\sigma(z))\,\tau(y)\,\tau(x) + t(\sigma(z))\,\tau(y)\,\tau(z)$ … (1)

On the other hand:
$t((x+z)\,y\,(x+z)) = t(xyx + xyz + zyx + zyz)$
$= t(xyx) + t(zyz) + t(xyz + zyx)$
$= t(\sigma(x))\,\tau(y)\,\tau(x) + t(\sigma(z))\,\tau(y)\,\tau(z) + t(xyz + zyx)$…(2)

Comparing (1) and (2), we get:
$t(xyz+zyx) = t(\sigma(z))\,\tau(y)\,\tau(x) + t(\sigma(x))\,\tau(y)\,\tau(z)$.

(iii) By (ii) and since R is a 2-torsion free commutative ring, we get the require result.

Definition (2.5):

Let t be a (σ,τ)- reverse left centralizer of a ring R. Then for all x, y $\in$ R , we define $G: R\times R \longrightarrow R$ by :
$G(x,y) = t(xy) - t(\sigma(y))\,\tau(x))$

Lemma (2.6):

Let t be a Jordan (σ,τ)- reverse left centralizer of a ring R. Then for all x , y, z $\in$ R
(i) $G(x,y) = -G(y,x)$
(ii) $G(x+y, z) = G(x,z) + G(y,z)$
(iii) $G(x,y+z) = G(x,y) + G(x,z)$

Proof:

(i) By Lemma (2.4)(i) , we have
$t(xy+yx) = t(\sigma(y))\,\tau(x) + t(\sigma(x))\,\tau(y)$
$t(xy) - t(\sigma(y))\,\tau(x) = -(t(yx) - t(\sigma(x))\,\tau(y))$
$G(x,y) = -G(y,x)$

(ii) $G(x+y, z) = t((x+y)z) - t(\sigma(z))\,\tau(x+y)$
$= t((xz+yz)) - t(\sigma(z))\,\tau(x+y)$
$= t((xz)) - t(\sigma(z))\,\tau(x) + t((yz)) - t(\sigma(z))\,\tau(y)$
$= G(x,z) + G(y,z)$

(iii) $G(x,y+z) = t(x(y+z) - t(\sigma(y+z))\,\tau(x)$
$= t((xy+xz)) - t(\sigma(y+z))\,\tau(x)$
$= t((xy)) - t(\sigma(y))\,\tau(x) + t((xz)) - t(\sigma(z))\,\tau(x)$
$= G(x,y) + G(x,z)$

Remark (2.7):

Note that a (σ,τ)- reverse left centralizer of a ring R if and only if $G(x,y) = 0$, for all x,y $\in$ R.

Lemma (2.8):

Let t be a Jordan (σ,τ)- reverse left centralizer of a ring R. Then for all x, y, z $\in$ R $G(x,y)\,\tau(z)\,[\tau(x),\tau(y)] = 0$

Proof:

Let $w = xyzyx + yxzxy$

$t(w) = t(x(yzy)x + y(xzx)y)$

$= t(\sigma(x))\,\tau(yzy)\,\tau(x) + t(\sigma(y))\,\tau(xzx)\,\tau(y)$

$= t(\sigma(x))\,\tau(y)\,\tau(z)\,\tau(y)\,\tau(x) + t(\sigma(y))\,\tau(x)\,\tau(z)\,\tau(x)\,\tau(y)$ … (1)

On the other hand

$t(w) = t((xy)\,z\,(yx) + (yx)\,z\,(xy))$

$= t(yx)\,\tau(z)\,\tau(xy) + t(xy)\,\tau(z)\,\tau(yx)$ … (2)

Compare (1), (2), we have that

$0 = (t(yx) - t(\sigma(x))\,\tau(y))\,\tau(z)\,\tau(y)\,\tau(x) + (t(xy) - t(\sigma(y))\,\tau(x))\,\tau(z)\,\tau(x)\,\tau(y)$

$0 = G(y,x)\,\tau(z)\,\tau(y)\,\tau(x) + G(x,y)\,\tau(z)\,\tau(x)\,\tau(y)$

$0 = -G(x,y)\,\tau(z)\,\tau(y)\,\tau(x) + G(x,y)\,\tau(z)\,\tau(x)\,\tau(y)$

$0 = G(x,y)\,\tau(z)\,(\tau(x)\,\tau(y) - \tau(y)\,\tau(x))$

$G(x,y)\,\tau(z)\,[\tau(x),\tau(y)] = 0$, for all x , y , z $\in$ R

Lemma (2.9):

Let t be a Jordan (σ,τ)- reverse left centralizer of a prime ring R. Then for all x, y, z, u, v $\in$ R $G(x,y)\,\tau(z)\,[\tau(u),\tau(v)] = 0$

Proof:

Replacing x + u for x in Lemma (2.8), we have that

$G(x+u,y)\,\tau(z)\,[\tau(x+u),\tau(y)] = 0$

$G(x,y)\,\tau(z)\,[\tau(x),\tau(y)] + G(x,y)\,\tau(z)\,[\tau(u),\tau(y)] +$

$G(u,y)\,\tau(z)\,[\tau(x),\tau(y)] + G(u,y)\,\tau(z)\,[\tau(u),\tau(y)] = 0$

By Lemma (2.8) , we get

$G(x,y)\,\tau(z)\,[\tau(u),\tau(y)] + G(u,y)\,\tau(z)\,[\tau(x),\tau(y)] = 0$

Therefore, we get

$0 = G(x,y)\,\tau(z)\,[\tau(u),\tau(y)]\,\tau(z)\,G(x,y)\,\tau(z)\,[\tau(u),\tau(y)]$

$0 = -G(x,y)\,\tau(z)\,[\tau(u),\tau(y)]\,\tau(z)\,G(u,y)\,\sigma(\tau(z))\,[\tau(x),\tau(y)]$

Since R is prime ring, we get

$G(x,y)\,\tau(z)\,[\tau(u),\tau(y)] = 0$, for all x , y , z , u $\in$ R … (1)

Now, replacing y + v for y in Lemma (2.8), we have that

$G(x,y+v)\,\tau(z)\,[\tau(x),\tau(y+v)] = 0$

$G(x,y)\,\tau(z)\,[\tau(x),\tau(y)] + G(x,y)\,\tau(z)\,[\tau(x),\tau(v)] +$

$G(x,v)\,\tau(z)\,[\tau(x),\tau(y)] + G(x,v)\,\tau(z)\,[\tau(x),\tau(v)] = 0$

By Lemma (2.8), we get:

$G(x,y)\,\tau(z)\,[\tau(x),\tau(v)] + G(x,v)\,\tau(z)\,[\tau(x),\tau(y)] = 0$

Therefore, we get:

$0 = G(x,y)\,\tau(z)\,[\tau(x),\tau(v)]\,\tau(z)\,G(x,y)\,\tau(z)\,[\tau(x),\tau(v)]$

$0 = -G(x,y)\,\tau(z)\,[\tau(x),\tau(v)]\,\tau(z)\,G(x,v)\,\tau(z)\,[\tau(x),\tau(y)]$

Hence, by the primness of R:

$G(x,y)\,\tau(z)\,[\tau(x),\tau(v)] = 0$, for all x , y , z ,v $\in$ R … (2)

Finally, $G(x,y)\,\tau(z)\,[\tau(x+u),\tau(y+v)] = 0$

$G(x,y)\,\tau(z)\,[\tau(x),\tau(y)] + G(x,y)\,\tau(z)\,[\tau(x),\tau(v)] +$

$G(x,y)\,\tau(z)\,[\tau(u),\tau(y)] + G(x,y)\,\tau(z)\,[\tau(u),\tau(v)] = 0$

By (1), (2) and Lemma (2.8) , we get
$G(x,y)\ \tau(z)\ [\tau(u)\ ,\ \tau(v)] = 0$, for all $x , y , z , u ,v \in R$

Theorem (2.10):

Every Jordan (σ,τ)- reverse left centralizer of a 2-torsion free prime ring R is a (σ,τ)- reverse left centralizer of R.

Proof:

Let t be a Jordan (σ,τ)- reverse left centralizer of a 2-torsion free prime ring R .
Since R is a prime ring. Then by Lemma (2.9) , therefore either
$G(x,y)$ or $[\tau(u)\ ,\ \tau(v)] = 0$, for all $x , y, u ,v \in R$
If $[\tau(u)\ ,\ \tau(v)] \neq 0$, for all $u ,v \in R$. Then $G(x,y) = 0$, for all $x , y \in R$.
Hence by Remark (2.7), we get t is a (σ,τ)- reverse left centralizer of R.
But if $[\tau(u)\ ,\ \tau(v)] = 0$, for all $u ,v \in R$. Then R is a commutative ring
By Lemma (2.4) (i), we have that
$t(xy+yx) = 2\ t\ (xy)$
$= 2\ t\ (\sigma(y))\ \tau(x)$
Since R is a 2-torsion free ring, we get t is a (σ,τ)- reverse left centralizer of R.

Proposition (2.11):

Let t be a Jordan (σ,τ)- reverse left centralizer of a 2-torsion free ring R, such that $\sigma^2 = \sigma$, $\tau(\sigma(x)) = \tau(x)$ and $\tau(\sigma(y)) = \tau(y)$ .Then t is a Jordan triple (σ,τ)- reverse left centralizer of R .

Proof:

Replace $xy + yx$ for y in Lemma (2.4) (i) , we have that
$t\ (x(xy+yx) + (xy+yx)\ x) = t\ (\sigma(xy+yx))\ \tau(x) + t\ (\sigma(x))\ \tau(xy+yx)$
$= t\ (\sigma^2(y))\ \tau(\sigma(x))\ \tau(x) + t\ (\sigma^2(x))\ \tau(\sigma(y))\ \tau(x)$
$+\ t\ (\sigma(x))\ \tau(x)\ \tau(y) + t\ (\sigma(x))\ \tau(y)\ \tau(x)$
Since $\sigma^2 = \sigma$, $\tau(\sigma(x)) = \tau(x)$ and $\tau(\sigma(y)) = \tau(y)$, we get
$= t\ (\sigma(y))\ \tau(x)\ \tau(x) + t\ (\sigma(x))\ \tau(y)\ \tau(x)$
$+\ t\ (\sigma(x))\ \tau(x)\ \tau(y) + t\ (\sigma(x))\ \tau(y)\ \tau(x)$ ….(1)

On the other hand:
$t\ (x(xy+yx) + (xy+yx)\ x) = t\ (x^2y + xyx + xyx + yx^2)$
$= t\ (\sigma(y))\ \tau(x)\ \tau(x) + t\ (\sigma(x))\ \tau(x)\ \tau(y) + 2\ t\ (xyx)$ ….(2)

Compare (1), (2) , we get :
$2\ t(xyx) = 2\ t(\sigma(x))\ \tau(y)\ \tau(x)$

Since R is a 2-torsion free ring, we get the require result.

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