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A Study Of Euler-Matrix Triple Product Summability Method Of The Fourier Series

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Abstract:
This research paper is related to the concept of approximation of a function belonging to the Lipschitz class by the Euler-matrix triple product summability method of the Fourier series. The Euler-matrix product summability method has been used while working in this direction. So, many known results may become particular cases of our result. Our result may be useful for future researchers.

Keywords: *Euler sum, Matrix mean, product summability method, Lipschitz class, Fourier series, Lebesgue integral.*

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I. Introduction

The studies of the degree of approximation of functions belonging to various classes by using different means have been made by the researchers like [3],[4],[7],[8],[9],[10],[11],[12],[13],[14],[15] et al. Das[2], et al have discussed the degree of approximation of a function by using (E, q)A product summability means of the Fourier series. No work seems to have been done to find the degree of approximation of a function belonging to the Lipschitz class by using Euler-matrix triple product summability means.

In this paper, we present a new theorem on the degree of approximation of functions belonging to the Lipschitz class by the Euler-matrix triple $(\mathrm{E}, \mathrm{q})(\mathrm{E}, \mathrm{q})$A. product summability method.

II. Definitions And Notations

In this section, we have given the following definitions:

Definition 2.1.

A function $f \in \mathrm{Lip}\alpha$ if

$$|f(x+t) - f(x+t)| = O(|t|^{\alpha})\ for\ 0 < \alpha \leq 1.$$

Definition 2.2.

L∞-norm of a function $f: R \to R$ is defined by $\|f\|_{\infty}$

$$\|f\|_{\infty} = sup\{|f(x)|/f: R \to R\}$$

L_p- norm is defined by

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p\right)^{\frac{1}{p}}, p \geq 1.$$

The degree of approximation of function $f: R \to R$ by a trigonometric polynomial t_n [1] is defined by

$$\|\mathrm{t_n} - \mathrm{f}\|_{\infty} = sup\{|t_n - f|: x \in R\}\ or\ \|\mathrm{t_n} - \mathrm{f}\|_{\mathrm{p}} = min\|\mathrm{t_n} - \mathrm{f}\|.$$

Let f be 2π- periodic function and Lebesgue integrable on $(-\pi, \pi)$. The Fourier series of $f(x)$ is given by

$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n cosnx + b_n sinnx)$ (1)

with n^{th} partial sums $s_n(f; x)$.

The Matrix $A = [c_{mn}]$ is regular if and only if

$$(i)\quad lim_{m\to\infty} c_{mn=0,}$$

$$(ii) lim_{m\to\infty} \sum_{n=0}^{m} c_{mn} = 1;$$

$$(iii) \exists M > 0, \sum_{m=0}^{\infty} |c_{mn}| < M. \forall m \geq 0.$$

The Fourier series (1) is said to be summable to s by a matrix (T) method, if $T_m(f;x) \to s$ as $m \to \infty$.
The matrix (A) method of the Fourier series is given by

$$t_n = \sum_{n=0}^{m} c_{m,n} s_n$$

The (E,q) means [6] of s_n is defined by-
$R_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k.$ (2)
The (E,q) Transformation of Matrix A of s_n defined by
$\eta_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} t_k = \frac{1}{(1+q)^n} \sum_{k=0}^{n} q^{n-k} \{\sum_{v=0}^{k} c_{k,v} t_v\}$ (3)
The $(E,q)(E,q)$ transform of a matrix A of s_n is defined by
$\chi_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} \frac{q^{n-k}}{(1+q)^k} \sum_{v=0}^{k} \binom{k}{v} q^{k-v} t_v$ (4)

$\chi_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} \frac{q^{n-k}}{(1+q)^k} \sum_{v=0}^{k} \binom{k}{v} q^{k-v} \{\sum_{u=0}^{v} c_{v,u} t_u\}$ (5)
If $\chi_n \to s$ as $n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ is said to be $(E,q)(E,q)A$-summable to sum s.
We use the following notations:

$$(i)\phi(x,t) = f(x+t) - f(x-t) - 2f(x).$$

$$(ii)\ K_n(t) = \frac{1}{2\pi(1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} \frac{q^{n-k}}{(1+q)^k} \sum_{v=0}^{k} \binom{k}{v} q^{k-v} \left\{ \sum_{u=0}^{v} c_{v,u} \frac{\sin\left(u+\frac{1}{2}t\right)}{\sin\frac{t}{2}} \right\} \right|$$

Further, the method $(E,q)(E,q)A$ is assumed to be regular.

III. Known Theorem

Dealing with the degree of approximation of the Fourier series by-product means, $(E,q)A$ in 2013, Padhy [4] proved The following theorem:

Theorem

Let$A = (a_{mn})_{\infty\times\infty}$ be a regular matrix. if f is a 2π- Periodic function of class $Lip\alpha$, then the degree of approximation by product $(E,q)A$- summability mean of Fourier series (1) is given by

$$||\tau_n - f||_\infty = O\left(\frac{1}{(n+1)^\alpha}\right), 0 < \alpha < 1.$$

IV. Main Theorem

The objective of this paper is to prove the following theorem.

Theorem

Let $A = (a_{mn})_{\infty\times\infty}$ be a regular matrix. if f is a 2π- Periodic function of class $Lip\alpha$ Then the degree of approximation by product $(E,q)(E,q)A$- summability means of the Fourier series (1) is given by

$$\|x_n - f\| = O\left(\frac{1}{(n+1)^\alpha}\right), 0 < \alpha < 1.$$

V. Lemmas:

For the proof of our theorem, we have required the following lemmas:

Lemma
$k_n(t) = O(r)$ for $0 \le t \le (n+1)^{-1}$.
Proof. For $0 \le t \le (n+1)^{-1}$, we have

$$|K_n(t)| = \frac{1}{2\pi(1+q)^n} \left\| \left[\sum_{k=0}^{n} \binom{n}{k} \frac{q^{n-k}}{(1+q)^k} \sum_{v=0}^{k} q^{k-v} \left\{ \sum_{u=0}^{v} c_{v,u} \frac{\sin\left(u+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\} \right] \right\|$$

First, we solve:

$$
\begin{aligned}
|\sum_{u=0}^{v} c_{v,u} \frac{\sin\left(u+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}| &\leq \sum_{u=0}^{v} c_{v,u} \frac{\left(u+\frac{1}{2}\right)}{\left(\frac{t}{\pi}\right)} \\
&= \frac{\pi}{2}\left\{\sum_{u=0}^{v} c_{v,u}(2u+1)\right\} \\
&= \frac{\pi}{2}\left\{\sum_{u=0}^{v} c_{v,u} + 2\sum_{u=0}^{v} u c_{v,u}\right\} \\
&= \frac{\pi}{2}\left\{1+2\left(c_{v,1}+2c_{v,2}+3c_{v,3}+\ldots+vc_{v,v}\right)\right\} \\
&\leq \frac{\pi}{2}\left\{1+2\left(c_{v,1}+vc_{v,2}+vc_{v,3}+\ldots+vc_{v,v}\right)\right\} \\
&= \frac{\pi}{2}\left\{1+2v\left(c_{v,1}+c_{v,2}+c_{v,3}+\ldots+c_{v,v}\right)\right\} \\
&= \frac{\pi}{2}\left\{1+2v\left(c_{v,0}+c_{v,1}+c_{v,2}+\ldots+c_{v,v}\right)-2vc_{v,0}\right\} \\
&\leq \frac{\pi}{2}\left\{1+2v\left(1-c_{v,0}\right)\right\} \\
&\leq \frac{\pi}{2}(1+2v) \\
&= O(v+1)
\end{aligned}
$$

Now, we use the above result in $K_n(t)$

$$
\begin{aligned}
|K_n(t)| &\leq \frac{1}{4\pi(1+q)^n}\left\|\left[\sum_{k=0}^{n}\binom{n}{k}\frac{q^{n-k}}{(1+q)^k}\sum_{v=0}^{k} q^{k-v}\{2v+1\}\right]\right\| \\
&\leq \frac{1}{4\pi(1+q)^n}\left|\left[\sum_{k=0}^{n}\binom{n}{k} q^{n-k}(2k+1)\right]\right| \\
&\leq \frac{1}{4}(2n+1) \\
&= O(n)
\end{aligned}
$$

Lemma

$K_n(t) = O\left(\frac{1}{t}\right)$ for $(n+1)^{-1} \leq t \leq \pi$.

Proof. For $t \in \left[\frac{1}{n+1}, \pi\right]$, $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$.

$$
\begin{aligned}
|K_n(t)| &= \frac{1}{2\pi(1+q)^n}\left\|\left[\sum_{k=0}^{n}\binom{n}{k}\frac{q^{n-k}}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v} q^{k-v}\left\{\sum_{u=0}^{v} c_{v,u}\frac{\sin\left(u+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right]\right\| \\
&\leq \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}\frac{q^{n-k}}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v} q^{k-v}\left\{\sum_{u=0}^{v}\frac{\pi a_{ku}}{t}\right\}\right| \\
&\leq \frac{M}{2(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k}\frac{q^{n-k}}{(1+q)^k}\sum_{v=0}^{k} q^{k-v}\right| \quad \text{by regularity condition} \\
&= O\left(\frac{1}{t}\right)
\end{aligned}
$$

Proof of The Main Theorem

Proof. Using the Riemann-Lebesgue theorem and considering the n^{th} partial sum of the Fourier series of $f(x)$ as $s_n(f;x)$ and following Titchmarsh [5] , We have

$$
s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,dt.
$$

Further, under the A-Transform of$s_n(f;x)$, we have

$$t_n - f(x) = \frac{1}{2\pi}\int_0^{\pi} \phi(t) \sum_{n=0}^{m} c_{mn} \frac{sin\left(n+\frac{1}{2}\right)t}{sin\frac{t}{2}} dt.$$

Then, considering τ_n as the $(E.q)$A-Transform of $s_n(f;x)$, we obtain

$$\tau_n - f(x) = \frac{1}{2\pi}\int_0^{\pi} \phi(t) \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{\sum_{v=0}^{k} c_{k,v} \frac{sin\left(v+\frac{1}{2}\right)}{sin\frac{t}{2}}\right\}$$

Directing the $(E,q)(E,q)A$ transform of $s_n(f;x)$ by χ_n , we have

$$|\chi_n - f(x)| = \frac{1}{2\pi(1+q)^n}\int_0^{\pi} \phi(t) \sum_{k=0}^{n} \binom{n}{k} \frac{q^{n-k}}{(1+q)^k} \sum_{v=0}^{k} \binom{k}{v} q^{k-v} \left\{\sum_{u=0}^{v} c_{v,u} \frac{sin\left(u+\frac{1}{2}\right)t}{sin\frac{t}{2}}\right\} dt.$$

$$= \int_0^{\pi} \phi(t) K_n(t) dt$$

$$= \left\{\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right\} \phi(t) K_n(t) dt$$

$$= I_1 + I_2, \quad 1cm\text{say}$$

Now

$$|I_1| = \frac{1}{2\pi(1+q)^n}\int_0^{\frac{1}{n+1}} \phi(t) \sum_{k=0}^{n} \binom{n}{k} \frac{q^{n-k}}{(1+q)^k} \sum_{v=0}^{k} \binom{k}{v} q^{k-v} \left\{\sum_{u=0}^{v} c_{v,u} \frac{sin\left(u+\frac{1}{2}\right)t}{sin\frac{t}{2}}\right\} dt.$$

$$\leq O(n) \int_0^{\frac{1}{n+1}} |\phi(t)| dt \quad \text{using Lemma 5.1}$$

$$= O(n) \int_0^{\frac{1}{n+1}} |t^{\alpha}| dt.$$

$$= O(n) \left[\frac{t^{\alpha+1}}{\alpha+1}\right]_0^{\frac{1}{n+1}}$$

$$= O(n) \left[\frac{1}{(\alpha+1)(n+1)^{\alpha+1}}\right].$$

$$= O\left[\frac{1}{(n+1)^{\alpha+1}}\right].$$

Next

$$|I_2| = \frac{1}{2\pi(1+q)^n}\int_{\frac{1}{n+1}}^{\pi} \phi(t) \sum_{k=0}^{n} \binom{n}{k} \frac{q^{n-k}}{(1+q)^k} \sum_{v=0}^{k} \binom{k}{v} q^{k-v} \left\{\sum_{u=0}^{v} c_{v,u} \frac{sin\left(u+\frac{1}{2}\right)t}{sin\frac{t}{2}}\right\} dt.$$

$$\leq \int_{\frac{1}{n+1}}^{\pi} |\phi(t)||K_n(t)| dt$$

$$= \int_{\frac{1}{n+1}}^{\pi} |\phi(t)| O\left(\frac{1}{t}\right) dt \quad \text{using Lemma 5.2}$$

$$\leq \int_{\frac{1}{n+1}}^{\pi} t^{\alpha-1} dt$$

$$= O\left(\frac{1}{(n+1)^{\alpha}}\right)$$

Then, from the above results, we have

$$|\chi_n - f(x)| = O\left(\frac{1}{(n+1)^{\alpha}}\right), \text{ for } 0 < \alpha < 1$$

$$\|\chi_n - f(x)\| = sup_{-\pi<x<\pi} |\chi_n - f(x)|$$

$$= O\left(\frac{1}{(n+1)^{\alpha}}\right), \quad \text{for} \quad 0 < \alpha < 1$$

Hence, the proof of the main theorem.

VI. Conclusion

The result established here is a more general form than some earlier existing results in the sense that one $(E, q) = 1$ our proposed mean is reduced to $(E, q)A$ Mean.

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