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Analyzing Gujarat's Coastline

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I. Introduction

Growing up near the Indian State of Gujarat, I am no stranger to coastlines. Every summer my family and I headed up and drove along the coastline marvelling at the intersection of sea and land. The long straight roads often followed the complex and irregular coastline of the state. One day, looking at my car's odometer I observed that the distance travelled is not the same as the value of the coastline stated on the internet for the measured piece of land. [1] I pondered, "What difference can the small insignificant bumps and edges have on the total distance I travel?" This sparked my curiosity as I searched up the net only to realise that the actual problem is much greater than the trivial issue I pondered over. On further research, I discovered that this phenomenon is known as "The Coastline Paradox" and it was first discovered in 1952 by Lewis Richardson and was mathematically talked about by Benoit Mandelbrot in 1967. [2] This also opened a new area of mathematics known as Fractal Geometry.

The Coastline Paradox

Think of a shape … not any normal shape but one that has a finite area bounded by an infinite perimeter. At first, this shape might seem implausible and it would break the laws of mathematics as we know them – mathematics is a strange discipline where there are ample situations which make absolutely no sense to the common man: Gabriel's Horn, Von Koch Snowflake etc – but such a peculiar shape is the very basis of the Coastline paradox.

The Coastline Paradox poses a very complex answer to a seemingly simple question: how long is the coastline of a country? Well, obviously the perimeter of it but the fundamental problem here lies in measuring this length. Nature does not work on circles and triangles (mostly) and it is often uneven. When you zoom in on a regular shape, you would see the complexity keep decreasing till the point you reach a straight smooth line [Figure 1]. In contrast, when you zoom in on a coastline of a country [Figure 2], you would see the irregularities keep growing, and the complexity does not reduce, even on the molecular level. Cartographers around the world have a had a problem when measuring coastlines and this has led to discrepancies between measurements of the same length; for example, the CIA Factbook says that the coastline of b is 12,429km long whereas the World Resources Institute says it is 19,717km long.

The true answer to this lies in the length of the measuring stick you use to measure the length. When you use a long measuring stick (for example, 500kms) then you would not go into the complexities of each crack and crevice. However, if you shrink your ruler down to a kilometre, you are suddenly able to capture much more fine detail, because your instrument for measurement is much more precise, and the same perimeter would seem inflated. The trend shows that the shorter the length of the measuring stick, the longer the perimeter would come out to be. One could theoretically go all the way down to the molecular level for the measuring stick, but then the length of the coast would seem to approach infinity.

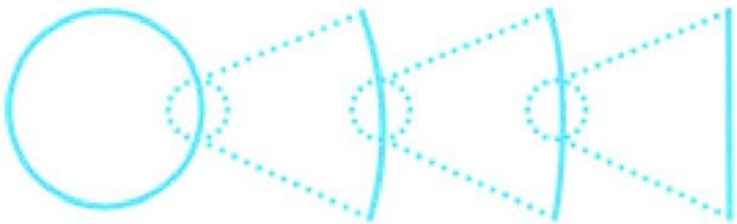


Figure 1: Showing the decreasing complexity of a Euclidian Circle as we zoom in

Figure 2 : Showing the increasing the complexity of Gujarat's Coastline as we zoom in

From this figure we can see that at no point does the complexity of the coastline decrease, no matter how much you zoom in. This is because of the constant erosion of the coastlines by waves, tsunamis and other factors. Such shapes cannot be examined using Euclidian Geometry which analyses shapes such as Figure 1. Such peculiar shapes – that have a finite area bounded by an infinite perimeter – can be explained using the concept of fractional dimensions (refer to Exploring Fractal Dimensions). A fractal is a never-ending pattern which is infinitely complex and is self-similar across different scales. It resides between our conventional dimensions geometrically. Because nature is full of fractals, fractal patterns are quite familiar. "Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line." (Mandelbrot, 1983).

Aim

Through this exploration, I aim to find the fractal dimension of Gujarat's coastline which can be helpful in measuring the perimeter of the coastline. I will use this value to determine the length of the ruler used by cartographers around the world to measure the same perimeter. I will also be investigating if a change in the perimeter has a change in the area to explore if such irregularities in measurements are present in the values of area for landmasses too.

Exploring Fractal Dimensions

Conventionally, a dimension was used to define the number of coordinates required to describe a point in that space [4]. However, with progress in Physics and Maths, more complex dimensions came up such as fractal dimensions and infinity dimensions which are used frequently in theoretical physics. For example, consider the following shape below, "The Osgood Curve" *Figure 3*.

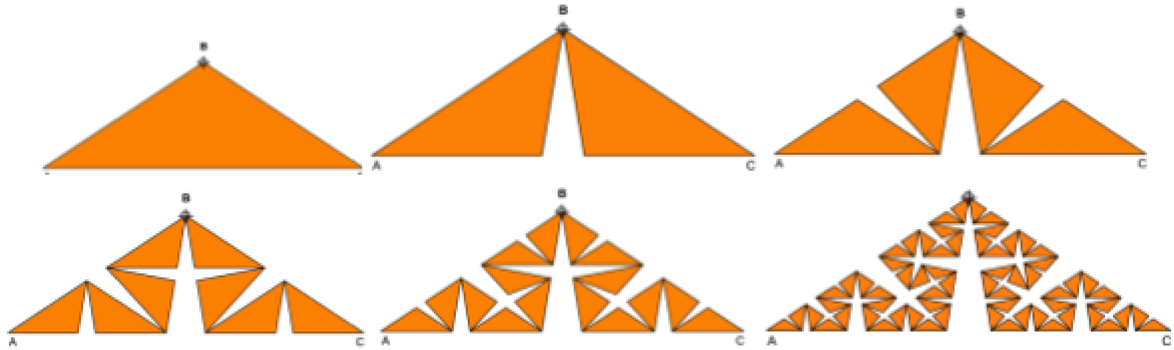


Figure 3 : Showing the various levels of the Osgood curve.
Reference: https://demonstrations.wolfram.com/KnoppsOsgoodCurveConstruction/

For instance, Knopp's construction involves recursively splitting triangles into pairs of smaller triangles, meeting at a shared vertex, by removing triangular wedges. When each level of this construction removes a reduced fraction of the area per level, rapidly enough to leave a constant fraction of the area unremoved, it produces an Osgood curve. Such a shape cannot be classified within our conventional dimensions because

1. A common understanding of dimensions is that one part of the shape will have the same dimension as the whole shape itself. Where the misconception comes often is that we do not realise that the blank white spaces are as much a part of the shape as the orange triangles *Figure 3.* It is because of this, that although the individual triangles in the figure are 2-dimensional and the line segments are 1-dimensional, the whole shape is not 1 or 2 dimensional because these do not take into account the blank spaces

As a result, a more common definition of dimension was brought up which could be used for both whole number and fractional dimensions. Imagine an object with dimension D, then divide the sides of the object by an integer M. This results in a shape with M^D copies of the original object. Let the number of copies be N. Therefore we have the equation:

$$N = \left(\frac{1}{M}\right)^D \qquad \text{N: number of copies, } \frac{1}{M} \text{ : scale factor, and D is the dimension}$$

$$\log log\, N = \log log\left(\frac{1}{M}\right)^D \qquad \{\text{Applying Log to both the sides}\}$$

$$\log log\, N = \frac{1}{M} \qquad \{\text{Applying Log of power rule}\}$$

$$D = \frac{\log log\, N}{\log log\, \frac{1}{M}} \qquad \{\text{Making D the subject}\}$$

I will be using this formula with familiar shapes and figures to test the accuracy of the equation and also how it is applied in real life.

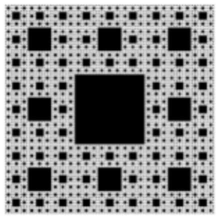
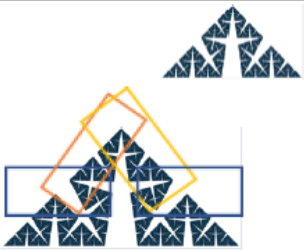
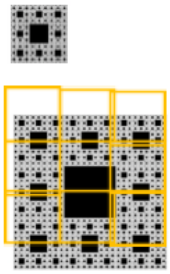
Shape	 Rectangle	 Osgood Curve	 Sierpiński carpet
Scale Factor	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$
Resultant Shape			
Number of copies	$4 = 2^2$	4	8
Dimension	$\frac{\log\log 4}{\log\log \frac{1}{\frac{1}{2}}} = \frac{\log\log 4}{\log\log 2}$ $= 2$	$\frac{\log\log 4}{\log\log \frac{1}{\frac{1}{3}}} = \frac{\log\log 4}{\log\log 3} =$	$\frac{\log\log 8}{\log\log \frac{1}{\frac{1}{3}}} = \frac{\log\log 8}{\log\log 3}$

Table 1 : Summarizing various shapes and their dimensions

All these values matched the values found on the internet for the available shapes. This suggests that the following equation is accurate. However, another challenge I faced during my investigation was applying this equation to coastlines. It is because coastlines are not self-similar and have complicated edges therefore using a scale factor and counting the resultant copies of the original object would not lead to the most accurate data set.

The Hausdorff Method

Through my exploration, I learned that the length of a coastline is subjective and varies according to the length of the measuring stick being used, however, maths as I know it is a very objective discipline and therefore I needed to find a variable which is more objective in nature. Mandelbrot's research revealed that the fractal dimension of a coastline is constant for a particular data set. This means that the fractal dimension of Gujarat would be the same and the length of the measuring sticks being used would not have a profound impact on that. With further research and to maintain the accuracy of my experiment, I will be using one of the most prominent method to measure the fractal dimension of complicated shapes: The Hausdorff Method. [5]

Devised by Felix Hausdorff in 1918, Hausdorff dimension (Hausdorff Method) is the earliest known method to measure the roughness of a shape. The length of the curve being measured is equal to the number of measuring sticks multiplied by the length of one measuring stick.

$$L(\xi) = \xi \times N(\xi)$$

ξ = length of measuring stick, $L(\xi)$ = Length of the coastline, $N(\xi)$ = Number of measuring sticks

Richardson furthered this equation by establishing a power law with this equation : $L(\xi) = A\times \xi^{\epsilon}$, where A and ϵ are constants. Mandelbrot later through repeated observations and experimentation discovered that $\epsilon = 1 - D$ where D is the fractal dimension. This led to a paradigm shift in the way complex objects were processed as now the original length equation could be used to calculate the fractal dimension of the object.

Using log laws, I can convert this equation in the form y = mx + c

$$L(\xi) = A \times \xi^{\epsilon}$$
$$\xi \times N(\xi) = A \times \xi^{\epsilon}$$
$$\xi \times N(\xi) = A \times \xi^{1-D}$$
$$N(\xi) = A \times \xi^{1-D-1} = A \times \xi^{-D}$$
$$\log log\ N(\xi) = \log log\ A\ \xi$$

On a linear graph, the following quantities correspond to:

$N(\xi)$ $, m = - D,$ $x = \log log\ (\xi)$ and A

$$\therefore D = - m$$

Below are two images (Figure 4 and Figure 5) which demonstrate how this method works.

Figure 4 : A rough irregular line which can be thought of as a coastline

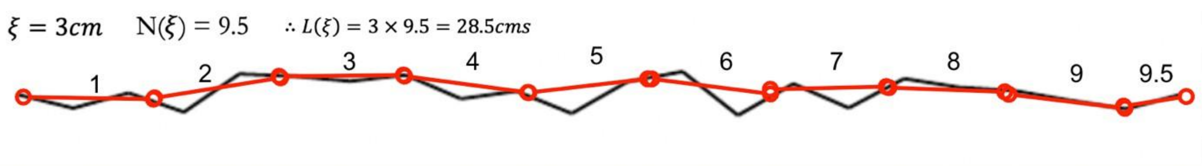


Figure 5 : Measuring the length of the irregular line using the Hausdorff Method

The new measuring stick starts where the preceding one ends. The placement of it is done by eye and this is a limitation of this exploration. This placement might differ from person to person so in the future, I can either ask many people to use the Hausdorff Method on the same piece of landmass and then calculate the resultant perimeter or use computer-assisted software to accurately place the measuring sticks.

The Coastline of Gujarat

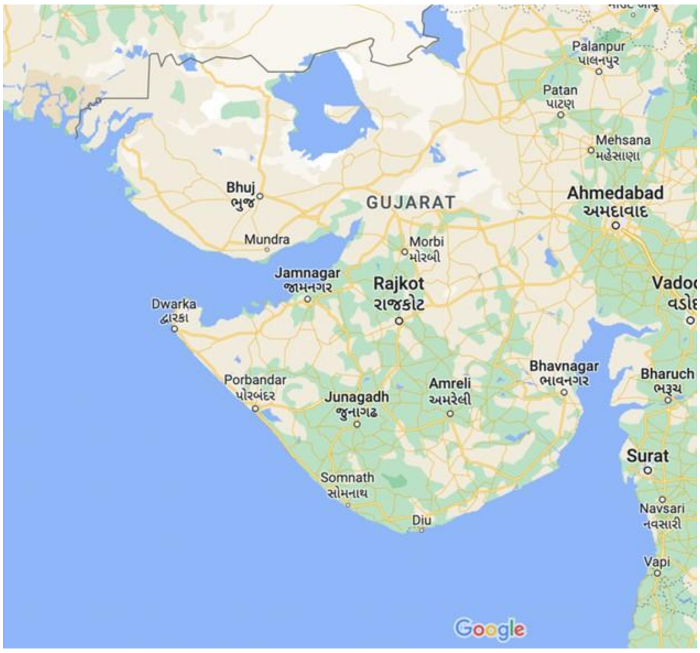


Figure 5 : Gujarat's Coastline

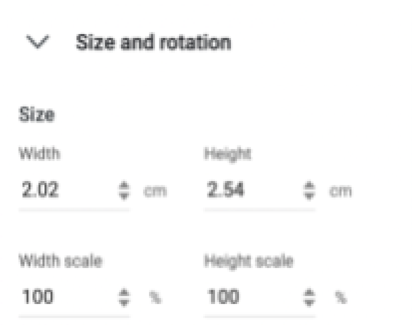


Figure 4 showing Width and Height

Fig. 6 shows the coastline of Gujarat. For my calculations, I chose 4 arbitrary values for the measuring stick: 100kms, 50kms, 25kms, and 10kms. I used the scale present on the bottom-right corner of Figure 4 which established the relationship 100kms = 3.49cm (1) . After this, I used Google drawings to draw the various measuring sticks and made sure that it fit the best by eye. This placement, however, is subjective and another individual might place the rulers differently resulting in a different fractal dimension. This was a limitation in my exploration. After drawing each line, I used the feature offered by Google and calculated the width and the height of the stick [Figure 7]. Using the basic Pythagoras equation, I measured the hypotenuse which was the length of the stick. I ensured this correlated to the scale (1) I used therefore improving the accuracy of my experiment. Below are the images of Gujarat's Coastline measured with varying lengths of the measuring stick.

Figure a (ξ) = 100kms

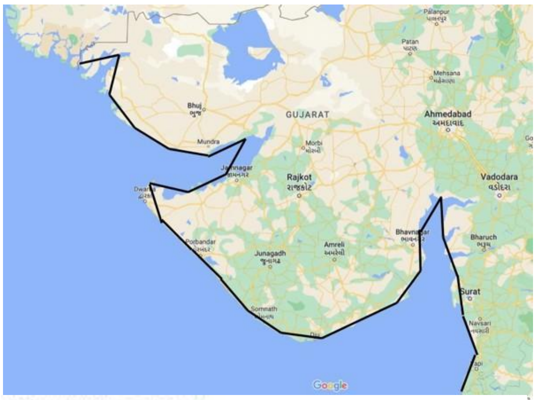


Figure b (ξ) = 50kms

Figure c (ξ) = 25kms

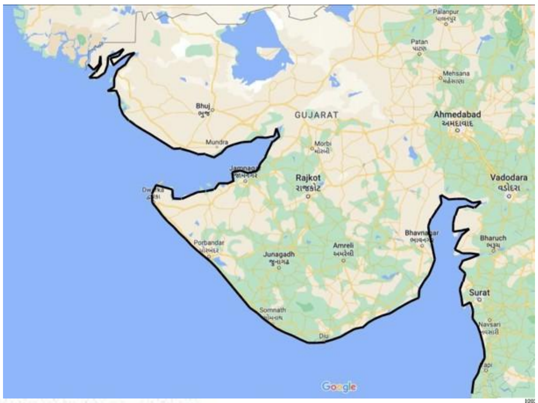
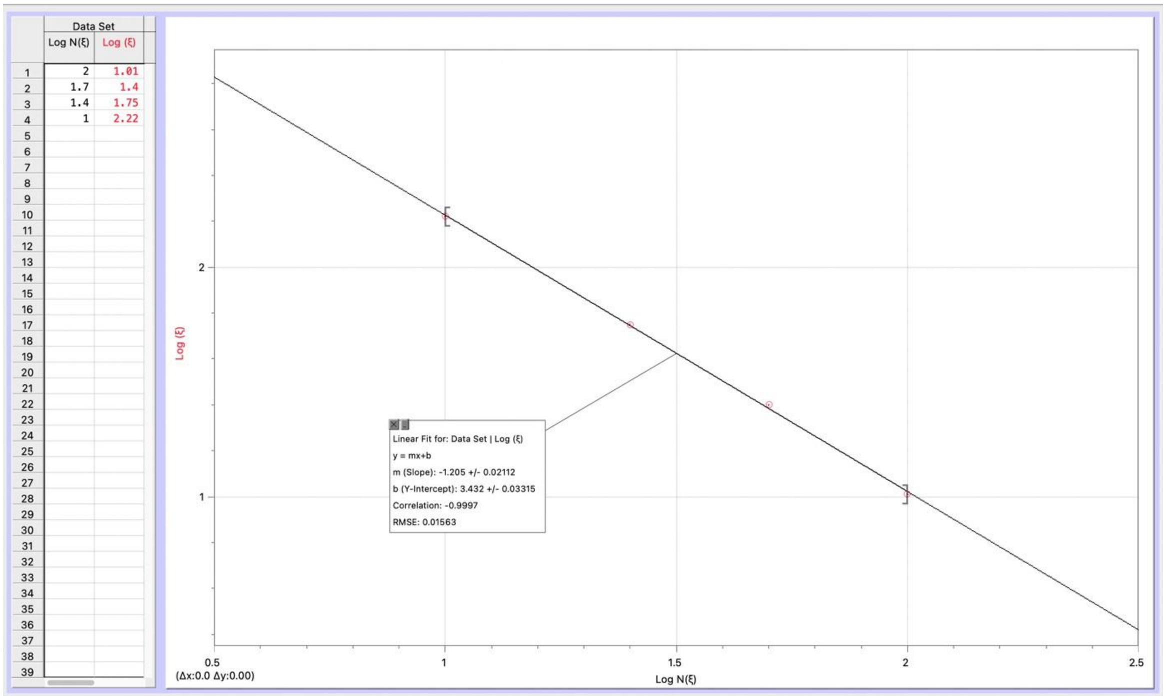


Figure d (ξ) = 10kms

A trend I noticed was, which was supported by past observations, that as I reduced the length of the measuring sticks the number of sticks I had to use increased exponentially and the resultant length increased with each subsequent decrease in the length of the measuring stick. This proved the fractal nature of the Coastline. In Table 1, I summarize the data accumulated from the Hausdorff Method.

Length of Ruler (ξ) in kms	100	50	25	10
Log (ξ)	2.00	1.70	1.40	1.00
Number of Rulers used N(ξ)	10.3	25	56.4	166.5
Log N(ξ)	1.01	1.40	1.75	2.22

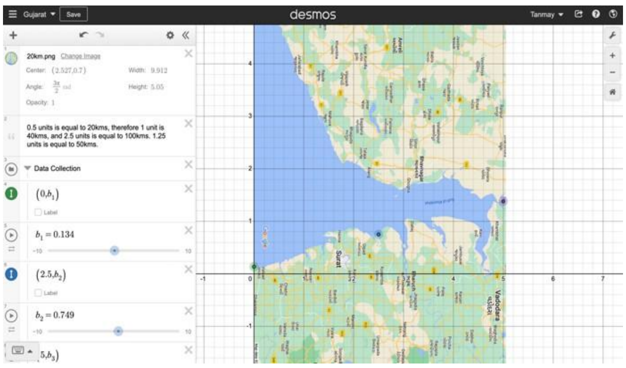
Table 2 : Showing the cumulative data for the Hausdorff Method

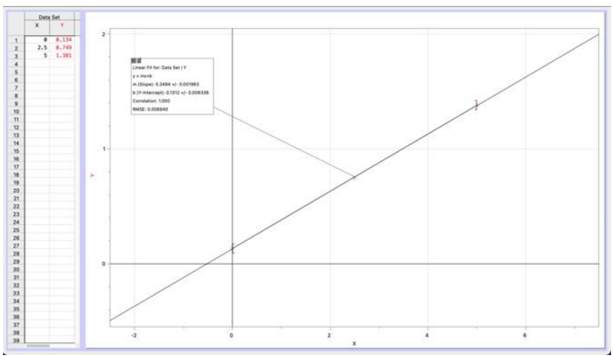


Graph 1 : Showing a linear relationship between the log of the number of rulers and the log of the length of one ruler

The gradient of the graph comes out to be -1.205. As $D =- m$, D = + 1.205. This number gives me an insight about the roughness of Gujarat's coastline. I will be using this value to measure the length of the measuring stick cartographers have used to measure the same perimeter. According to DeshGujarat [7], the perimeter of Gujarat is 1600kms and 1214.7kms. However, as explored in the Hausdorff Method, the length of the measuring stick does not have any impact on the fractal dimension.

After finding out the trend in the perimeter of the coastline, I was intrigued how little changes in the variable can have a profound change in the resultant value of the perimeter. To further my exploration, I wanted to find out if such changes would have any changes on the area of the landmass in scope. To do this I chose the coastline around Surat. After this, I scaled this landmass on Desmos such that 1 unit is 40km and 1 unit sq. is 1600km sq. Reference points were placed linearly along the coastline in distances of 100kms, 50kms, 25kms and 10kms. This would simulate the changing length of measuring stick in the context of perimeter calculation. With this, we can see if a change in perimeter would have a change in area of the landmass. After marking the points and therefore acquiring the respective coordinates, these were plotted in LoggerPro so that a best-fit line could be generated. This equation was then used to calculate area by Definite Integrals. To maintain the accuracy of this exploration the correlation factor was kept above 0.99 which showcases the high accuracy of the equations being used. Following are the respective images, graphs, and the Integral equations.





100kms

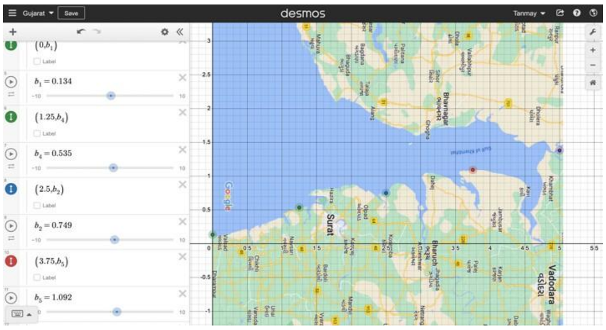
Use near equal sign, tell the justification of integration

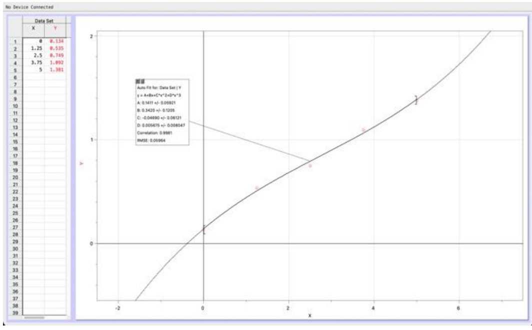
$$\int_{0}^{5} (0.2494x + 0.1312)dx$$

$$\int_{0}^{5} (0.2494x)dx + \int_{0}^{5} (0.1312)dx$$

$$\left[0.1247x^{2}\right]_{0}^{5} + \left[0.1312x\right]_{0}^{5}$$

$$= 3.7685$$





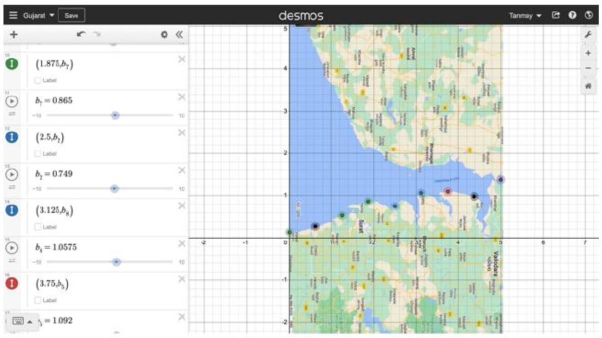
50kms

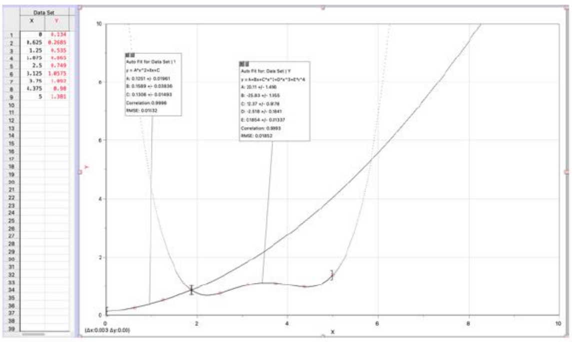
$$\int_{0}^{5} (0.005675x^{3} - 0.04690x^{2} + 0.3420x + 0.1411)\, dx$$

$$\int_{0}^{5} (0.005675x^{3})dx + \int_{0}^{5} (-\ 0.04690x^{2})dx + \int_{0}^{5} (0.3420x)dx + \int_{0}^{5} (0.1411)dx$$

$$[0.00141875x^{4}]_{0}^{5} + [-\ 0.01563x^{3}]_{0}^{5} + [0.171x^{2}]_{0}^{5} + [0.1411x]_{0}^{5}$$

$$= 3.913$$





25kms

$$\int_{0}^{1.875} \left(0.1251x^{2} + 0.1589x + 0.1306\right)dx$$

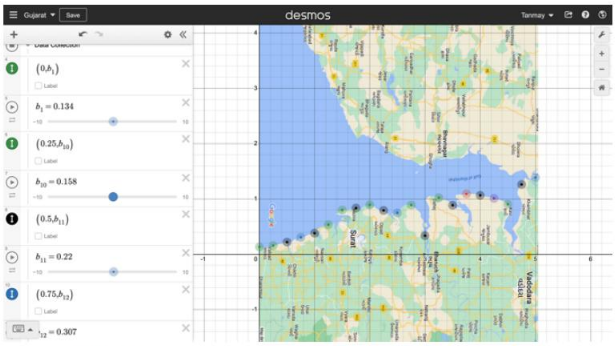
$$+ \int_{1.875}^{5} \left(0.1854x^{4} - 2.518x^{3} + 12.37x^{2} - 25.83x + 20.11\right)dx$$

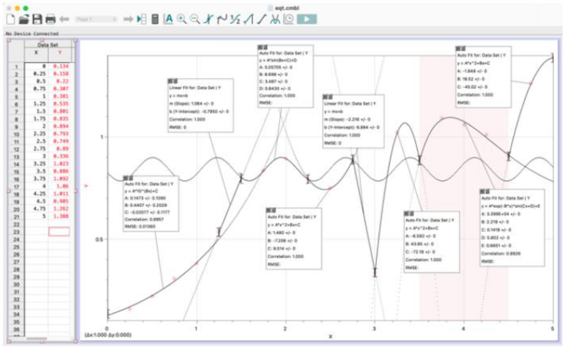
$$= [0.0417x^{3} + 0.07945x^{2} + 0.1306x]_{0}^{1.875}$$

$$+ [0.03708x^{5} - 0.6295x^{4} + 4.123x^{3} - 12.915x^{2} + 20.11x]_{1.875}^{5}$$

$$= 0.799 + 2.968 = 3.767$$

$$= 3.767$$





10kms

$$\int_{0}^{1.25} (0.1473\times10^{0.4407x} - 0.0208)dx$$

$$+ \int_{1.25}^{1.5} (1.064x - 0.795)dx$$

$$+ \int_{1.5}^{2.25} (0.05705 \sin sin\,(8.698x + 3.487) + 0.843\,)dx$$

$$+ \int_{2.25}^{2.75} \left(1.480x^{2} - 7.206x + 9.514\right)dx$$

$$+ \int_{2.75}^{3} (-\,2.216x + 6.984)dx$$

$$+ \int_{3}^{3.5} (-\,6.592 + 43.95x - 72.18)dx$$

$$+ \int_{3.5}^{4.5} \left(33990 \sin sin\,(0.1418x + 5.802)\, e^{-2.219x} + 0.6651\right)dx$$

$$+ \int_{4.5}^{5} \left(-\,1.848x^{2} + 18.52 - 45.02\right)dx$$

$$= [\frac{0.334\times10^{0.4407x}}{\ln ln\,(10)} - 0.02077]_{0}^{1.25}$$

$$+ [0.532x^{2} - 0.795x]_{1.25}^{1.5}$$

$$+ [0.843x - 0.00655897cos(8.698x + 3.487)]_{1.5}^{2.25}$$

$$+ [\frac{1480x^{3} - 10809x^{2} + 28542x}{3000}]_{2.25}^{2.75}$$

$$+ [-\frac{554x^{2} + 3447x}{500}]_{2.75}^{3}$$

$$+ [-\,2.197x^{3} + 21.975x^{2} - 72.18x]_{3}^{3.5}$$

$$+ \left[\frac{60247275000\left(-\frac{11095 \sin sin\left(\frac{709x+29010}{5000}\right)}{709} - cos\left(\frac{709x+29010}{5000}\right)\right)e^{\frac{-2219x}{1000}}}{61800853} + \frac{6651x}{10000} \right]_{3.5}^{4.5}$$

$$+ [- 0.616x^3 + 9.26x^2 - 45.02x]_{4.5}^{5}$$

$$= 0.34496 + 0.167 + 0.63097 + 0.3899 + 0.13075 + 0.44608 + 1.0216 + 0.608$$

$$= 3.733$$

II. Conclusion

The fractal dimension of the coastline of Gujarat which I have computed to value of 1.205 is an enlightening number. This is in the range of 1 through 2, which means that the coastline is not a smooth line or a one-dimensional object but neither does it occupy a two-dimensional space. The value provides a measurable indication of the irregularity or complexities of the coast. The greater the fractal dimension, the more twisted and complex a form it takes. Thus, my estimated value of the dimension 1.205 lends credence to the theory of fractal coastlines. This fractal dimension is a fixed magnitude of the coastline no matter what the measuring stick.

It is important to note that the area calculated around the land-mass at Surat did not vary much even though different length measuring sticks were used: 100km, 50km, 25km, 10km. I calculated that the areas were *3.7685, 3.913, 3.767, and 3.733*. Although there were some slight variations, they were not too dramatic as the changes in the perimeter. This implies that although the boundary of a fractal object can tend towards infinity, as the ruler shrinks, the space is limited. This proves the original assumption of the Coastline Paradox and a better insight of how fractal geometry works in real-life geographical objects. It further supports the notion that the seemingly non-substantial bumps and edges that the author first thought about can impose an intense impact on the length precisely but do not necessarily distort the aggregate space of the landmass.

Limitations and Future Refinements

1. Subjective Ruler Positioning - One major limitation was positioning the measuring sticks "by eye". This provided a possibility of subjectivity and error in my measurements. For future investigations to refine this, the measuring sticks can be positioned using computer-aided software in a precise and consistent manner.
2. Data Set Size - Employing just four measuring stick lengths (100km, 50km, 25km, and 10km) might not accurately reflect the fractal character of the coastline. Employing a larger data set with more gradual measuring stick lengths would result in an accurate fractal dimension measurement.
3. Random Starting Points - My selection of the coastline around Surat to calculate the area was random. Repeating this experiment over various parts of the coastline and on other continents would confirm my results and yield a broader conclusion.
4. Integral Calculations - Integration for the process, particularly of the 10km and 25km areas, was done using several equations and is therefore a potential source of error. Access to higher-level mathematical software or alternative calculation methods for area could enhance accuracy.

In total, this investigation effectively illustrated the Coastline Paradox principles and fractal geometry, revealing that a coastline's length is a function of measurement scale, whereas its fractal dimension is a tangible, fixed value.

Bibliography:

[1] *What is the total length of the Indian Coastal Line?* Quora. (n.d.). Retrieved July 5, 2022, from https://www.quora.com/What-is-the-total-length-of-the-Indian-coastal-line

[2] *The coastline paradox*. Sketchplanations. (n.d.). Retrieved July 5, 2022, from https://sketchplanations.com/the-coastline-paradox

[3] *How Long is the Indian Coastline - assets.researchsquare.com*. (n.d.). Retrieved July 6, 2022, from https://assets.researchsquare.com/files/rs-138313/v1/af8a402d-4955-48c6-94c2-35562d8 ef76f.pdf?c=1631869856

[4] Study.com | Take Online Courses. Earn College Credit. Research Schools, Degrees & Careers. (n.d.). Retrieved July 6, 2022, from https://study.com/learn/lesson/what-is-dimension-in-math-examples.html

[5] Husain, A., Reddy, J., Bisht, D., & Sajid, M. (2021, March 18). *Fractal dimension of coastline of Australia*. Nature News. Retrieved July 6, 2022, from https://www.nature.com/articles/s41598-021-85405-0

[6] Image credits: www.quora.com

[7] *What is the actual length of Gujarat coastline?* DeshGujarat. (2013, May 9). Retrieved July 6, 2022, from

https://www.deshgujarat.com/2013/05/09/what-is-the-actual-length-of-gujarat-coastline/
[8] *Fractals and the Fractal Dimension*. Fractals & the Fractal Dimension. (n.d.). Retrieved July 6, 2022, from https://www.vanderbilt.edu/AnS/psychology/cogsci/chaos/workshop/Fractals.html
[9] YouTube. (n.d.). *Hausdorff Measure Through Example*. YouTube. Retrieved July 6, 2022, from https://www.youtube.com/watch?v=YbuzXemwwlY
[10] *Fractal Dimension - Koch Snowflake*. Lindenmayer fractals - fractal dimension - koch snowflake. (n.d.). Retrieved July 6, 2022, from https://personal.math.ubc.ca/~cass/courses/m308-03b/projects-03b/skinner/ex-dimens ion-koch_snowflake.html
[11] *2D and 3D shapes - TOPPR-guides*. (n.d.). Retrieved July 6, 2022, from https://www.toppr.com/guides/maths/visualising-solid-shapes/2d-and-3d-figures/
[12] *Fractals and self-similarity Indiana University mathematics Journal*. (n.d.). Retrieved July 6, 2022, from https://maths-people.anu.edu.au/~john/Assets/Research%20Papers/fractals_self-similari ty.pdf