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# Certain Integral Formulae Involving The Generalized Wright Hypergeometric Function

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***Abstract:***
*The present paper, we aim to establishing certain integral formulae involving the wright function. The obtained results are in the form of hypergeometric and wright function, which are made with the help of Hadamard product. We have derived some other interesting formulae as special cases of our main results.*

***Keywords:*** *hypergeometric function, generalized wright hypergeometric function, Lavoie-Trottier, MacRobert and Edward integrals.*

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## I. Introduction

For $x \in \mathbb{C}$, $A_j, B \in \mathbb{C}$ and $a_j, b_j \in \mathbb{R}$ the definition of generalized Wright hypergeometric function ${}_p\Psi_q$ is defined [3] as below:

$$ {}_p\Psi_q[x] \equiv {}_p\Psi_q\left[\begin{matrix}(a_i, A_i)_{1,p} \\ (b_j, B_j)_{1,q}\end{matrix}\middle| \; x\right] = \sum_{\kappa=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + A_j\kappa)}{\prod_{j=1}^{q} \Gamma(b_j + B_j\kappa)} \frac{x^\kappa}{\kappa!} \tag{1.1} $$

$$ {}_p\Psi_q\left[\begin{matrix}(a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q)\end{matrix}\middle| \; x\right] = \sum_{\kappa=0}^{\infty} \frac{\Gamma(a_1 + A_1\kappa), \dots, \Gamma(a_p + A_p\kappa)}{\Gamma(b_1 + B_1\kappa), \dots, \Gamma(b_q + B_q\kappa)} \frac{x^\kappa}{\kappa!} \tag{1.2} $$

where $A_i, B_j \neq 0;\ i = 1, \dots, p;\ j = 1, \dots, q$ and for all values of the $x$ under the condition:

$$ 1 + \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > 0 \tag{1.3} $$

For specific value of parameters $A_1 = A_2 = \cdots = A_p = 1$ and $B_1 = B_2 = \cdots = B_q = 1$, the Wright function ${}_p\Psi_q$ reduce into generalized hypergeometric function such that

$$ {}_p\Psi_q\left[\begin{matrix}(a_1, 1), \dots, (a_p, 1) \\ (b_1, 1), \dots, (b_q, 1)\end{matrix}\middle| \; x\right] = \frac{\prod_{j=1}^{p} \Gamma(a_j)}{\prod_{j=1}^{q} \Gamma(b_j)} \, {}_pF_q\left[\begin{matrix}a_1, \dots, a_p \\ b_1, \dots, b_q\end{matrix}\middle| \; x\right] \tag{1.4} $$

where ${}_pF_q$ is the generalized hypergeometric series defined [3,4] by

$$ {}_pF_q\left[\begin{matrix}a_1, \dots, a_p \\ b_1, \dots, b_q\end{matrix}\middle| \; x\right] = \sum_{\kappa=0}^{\infty} \frac{(a_1)_\kappa \dots (a_p)_\kappa}{(b_1)_\kappa \dots (b_p)_\kappa} \frac{x^\kappa}{\kappa!} \tag{1.5} $$

The Pohhammer symbol is defined [2] as follows

$$ (\alpha)_n = \Gamma(\alpha + n)/\Gamma(\alpha)\,; \quad n \in \mathbb{N}, \quad \alpha \in \mathbb{C}/\mathbb{Z}_0 \tag{1.6} $$

## II. Preliminaries and Definitions

For our present investigation we recall the following interesting and useful results of Lavoie-Trottier [7], MacRobert [10] and Edward [6]:

$$ \int_0^1 \xi^{\mu-1}(1-\xi)^{2\epsilon-1}\left(1-\frac{\xi}{3}\right)^{2\mu-1}\left(1-\frac{\xi}{4}\right)^{\epsilon-1} d\xi = \left(\frac{4}{9}\right)^{\mu} \frac{\Gamma(\mu)\,\Gamma(\epsilon)}{\Gamma(\mu+\epsilon)} \tag{2.1} $$

where $Re(\mu) > Re(\epsilon) > 0$.

$$ \int_0^1 \xi^{\mu-1}(1-\xi)^{\epsilon-1}\,[C\xi + D(1-\xi)]^{-\mu-\epsilon} d\xi = \frac{1}{C^\mu D^\epsilon} \frac{\Gamma(\mu)\,\Gamma(\epsilon)}{\Gamma(\mu+\epsilon)} \tag{2.2} $$

where $Re(\mu) > 0,\ Re(\epsilon) > 0$ and $C, D$ are non zero constant with the expression $[C\xi + D(1-\xi)]$, where $0 \leq \xi \leq 1$.

$$\int_0^1 \int_0^1 \xi^{\epsilon} (1-\xi)^{\mu-1} (1-\zeta)^{\epsilon-1} (1-\xi\,\zeta)^{1-\epsilon-\mu} d\xi\, d\zeta = \frac{\Gamma(\mu)\,\Gamma(\epsilon)}{\Gamma(\mu+\epsilon)} \tag{2.3}$$

where $Re(\mu) > 0$ and $Re(\epsilon) > 0$.

Let $f(x) = \sum_{\kappa=0}^{\infty} C_\kappa\, x^\kappa$ and $g(x) = \sum_{\kappa=0}^{\infty} D_\kappa\, x^\kappa$ are two analytic functions with their radii of convergence $R_f$ and $R_g$ respectively, then their Hadamard product [8,9] is given by the following power series

$$f * g(x) = g * f(x) = \sum_{\kappa=0}^{\infty} C_\kappa\, D_\kappa\, x^\kappa; \quad (\,|x| < R\,) \tag{2.4}$$

where $R_c \geq R_f \cdot R_g$ is the radius of convergence of the composite series.

## III. Main Results

**Theorem 3.1.** Let $\xi > 0, \upsilon, \epsilon \in \mathbb{C}$ be such that $Re(\upsilon) > 0, Re(\epsilon) > 0$ and the conditions (1.3) is satisfied, then for the generalized wright hypergeometric function ${}_p\Psi_q$, the following integral formula holds true

$$\int_0^1 \xi^{\upsilon-1} (1-\xi)^{2\epsilon-1} \left(1-\frac{\xi}{3}\right)^{2\upsilon-1} \left(1-\frac{\xi}{4}\right)^{\epsilon-1} {}_p\Psi_q[X]\ d\xi = \left(\frac{4}{9}\right)^{\upsilon} \frac{\Gamma(\upsilon)\,\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)}$$
$$\times\ {}_p\Psi_q \left[\begin{matrix}(a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q)\end{matrix}\middle|\ \theta\right] * {}_2F_1\left[\begin{matrix}\epsilon, 1 \\ \upsilon+\epsilon\end{matrix}\middle|\ \theta\right] \tag{3.1}$$

where $X = (1-\xi)^2 \left(1-\frac{\xi}{4}\right) \theta$

**Proof.** First we refer to the left hand side of equation (3.1) as the sign $I_1$ then making the use of equation (1.1) in equation (3.1), we have

$$I_1 \equiv \int_0^1 \xi^{\upsilon-1} (1-\xi)^{2\epsilon-1} \left(1-\frac{\xi}{3}\right)^{2\upsilon-1} \left(1-\frac{\xi}{4}\right)^{\epsilon-1} \sum_{\kappa=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + A_j\kappa)}{\prod_{j=1}^{q} \Gamma(b_j + B_j\kappa)} \frac{(1-\xi)^{2\kappa} \left(1-\frac{\xi}{4}\right)^{\kappa} \theta^\kappa}{\kappa!} d\xi$$

After interchanging the order of integration and summation under the theorem's condition

$$I_1 \equiv \sum_{\kappa=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + A_j\kappa)}{\prod_{j=1}^{q} \Gamma(b_j + B_j\kappa)} \frac{\theta^\kappa}{\kappa!} \int_0^1 \xi^{\upsilon-1} (1-\xi)^{2\epsilon+2\kappa-1} \left(1-\frac{\xi}{3}\right)^{2\upsilon-1} \left(1-\frac{\xi}{4}\right)^{\epsilon+\kappa-1} d\xi$$

By using equation (2.1) and after some simplification, we get

$$I_1 \equiv \left(\frac{4}{9}\right)^{\upsilon} \sum_{\kappa=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + A_j\kappa)}{\prod_{j=1}^{q} \Gamma(b_j + B_j\kappa)} \frac{\theta^\kappa}{\kappa!} \frac{\Gamma(\upsilon)\,\Gamma(\epsilon+\kappa)}{\Gamma(\upsilon+\epsilon+\kappa)}$$

$$I_1 \equiv \frac{\Gamma(\upsilon)\,\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)} \left(\frac{4}{9}\right)^{\upsilon} \sum_{\kappa=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + A_j\kappa)}{\prod_{j=1}^{q} \Gamma(b_j + B_j\kappa)} \frac{\theta^\kappa}{\kappa!} \frac{(\epsilon)_\kappa\,(1)_\kappa}{(\upsilon+\epsilon)_\kappa\,\kappa!}$$

Now apply Hadamard product *i.e.* $\sum_{\kappa=0}^{\infty} C_\kappa\, y^\kappa * \sum_{\kappa=0}^{\infty} D_\kappa\, y^\kappa = \sum_{\kappa=0}^{\infty} C_\kappa\, D_\kappa\, y^\kappa$

$$I_1 \equiv \frac{\Gamma(\upsilon)\,\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)} \left(\frac{4}{9}\right)^{\upsilon} {}_p\Psi_q \left[\begin{matrix}(a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q)\end{matrix}\middle|\ \theta\right] * {}_2F_1\left[\begin{matrix}\epsilon, 1 \\ \upsilon+\epsilon\end{matrix}\middle|\ \theta\right]$$

**Theorem 3.2.** Let $\xi > 0, \upsilon, \epsilon \in \mathbb{C}$ be such that $Re(\upsilon) > 0, Re(\epsilon) > 0$ and the conditions (1.3) is satisfied, then for the generalized wright hypergeometric function ${}_p\Psi_q$, the following integral formula holds true

$$\int_0^1 \xi^{\upsilon-1} (1-\xi)^{2\epsilon-1} \left(1-\frac{\xi}{3}\right)^{2\upsilon-1} \left(1-\frac{\xi}{4}\right)^{\epsilon-1} {}_p\Psi_q[Y]\ d\xi = \left(\frac{4}{9}\right)^{\upsilon} \frac{\Gamma(\upsilon)\,\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)}$$
$$\times\ {}_p\Psi_q \left[\begin{matrix}(a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q)\end{matrix}\middle|\ \theta\right] * {}_2F_1\left[\begin{matrix}\upsilon, 1 \\ \upsilon+\epsilon\end{matrix}\middle|\ \theta\right] \tag{3.2}$$

where $Y = \xi \left(1-\frac{\xi}{3}\right)^2 \theta$

**Proof.** First we refer to the left hand side of equation (3.2) as the sign $I_2$ then making the use of equation (1.1) in equation (3.2), we have

$$I_2 \equiv \int_0^1 \xi^{\upsilon-1} (1-\xi)^{2\epsilon-1} \left(1-\frac{\xi}{3}\right)^{2\upsilon-1} \left(1-\frac{\xi}{4}\right)^{\epsilon-1} \sum_{\kappa=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + A_j\kappa)}{\prod_{j=1}^{q} \Gamma(b_j + B_j\kappa)} \frac{\xi^\kappa \left(1-\frac{\xi}{3}\right)^{2\kappa} \theta^\kappa}{\kappa!} d\xi$$

After interchanging the order of integration and summation under the theorem's condition

$$I_2 \equiv \sum_{\kappa=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma\left(a_j + A_j\kappa\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + B_j\kappa\right)} \frac{\theta^{\kappa}}{\kappa!} \int_0^1 \xi^{\upsilon+\kappa-1}(1-\xi)^{2\epsilon-1} \left(1-\frac{\xi}{3}\right)^{2\upsilon+2\kappa-1} \left(1-\frac{\xi}{4}\right)^{\epsilon-1} d\xi$$

By using equation (2.1) and after some simplification, we get

$$I_2 \equiv \left(\frac{4}{9}\right)^{\upsilon} \sum_{\kappa=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma\left(a_j + A_j\kappa\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + B_j\kappa\right)} \frac{\theta^{\kappa}}{\kappa!} \frac{\Gamma(\upsilon+\kappa)\,\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon+\kappa)}$$

$$I_2 \equiv \frac{\Gamma(\upsilon)\,\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)} \left(\frac{4}{9}\right)^{\upsilon} \sum_{\kappa=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma\left(a_j + A_j\kappa\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + B_j\kappa\right)} \frac{\theta^{\kappa}}{\kappa!} \frac{(\upsilon)_\kappa\,(1)_\kappa}{(\upsilon+\epsilon)_\kappa\,\kappa!}$$

Now apply Hadamard product *i.e.* $\sum_{\kappa=0}^{\infty} C_\kappa\, y^\kappa * \sum_{\kappa=0}^{\infty} D_\kappa\, y^\kappa = \sum_{\kappa=0}^{\infty} C_\kappa\, D_\kappa\, y^\kappa$

$$I_2 \equiv \frac{\Gamma(\upsilon)\,\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)} \left(\frac{4}{9}\right)^{\upsilon} {}_p\Psi_q \left[\begin{matrix}(a_1,A_1),\dots,(a_p,A_p)\\ (b_1,B_1),\dots,(b_q,B_q)\end{matrix}\middle|\ \theta\right] * {}_2F_1\left[\begin{matrix}\upsilon,1\\ \upsilon+\epsilon\end{matrix}\middle|\ \theta\right]$$

**Theorem 3.3.** Let $\xi > 0, \upsilon, \epsilon \in \mathbb{C}$ be such that $Re(\upsilon) > 0, Re(\epsilon) > 0$ and the conditions (1.3) is satisfied, then for the generalized wright hypergeometric function ${}_p\Psi_q$, the following integral formula holds true

$$\int_0^1 \xi^{\upsilon-1}(1-\xi)^{\epsilon-1}\,[C\xi + D(1-\xi)]^{-\upsilon-\epsilon}\ {}_p\Psi_q[Z]\ d\xi = \frac{1}{C^{\upsilon}D^{\epsilon}} \frac{\Gamma(\upsilon)\,\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)} \times {}_p\Psi_q \left[\begin{matrix}(a_1,A_1),\dots,(a_p,A_p)\\ (b_1,B_1),\dots,(b_q,B_q)\end{matrix}\middle|\ \frac{\theta}{4CD}\right] * {}_3F_2\left[\begin{matrix}\upsilon,\epsilon,1\\ \frac{\upsilon+\epsilon}{2},\frac{\upsilon+\epsilon+1}{2}\end{matrix}\middle|\ \frac{\theta}{4CD}\right] \quad (3.3)$$

where $Z = \frac{\xi\,(1-\xi)}{[C\xi + D(1-\xi)]^2}\,\theta$

**Proof.** First we refer to the left hand side of equation (3.3) as the sign $I_3$ then making the use of equation (1.1) in equation (3.3), we have

$$I_3 \equiv \int_0^1 \xi^{\upsilon-1}(1-\xi)^{\epsilon-1}\,[C\xi + D(1-\xi)]^{-\upsilon-\epsilon} \sum_{\kappa=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma\left(a_j + A_j\kappa\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + B_j\kappa\right)} \frac{\xi^{\kappa}\,(1-\xi)^{\kappa}\,\theta^{\kappa}}{[C\xi + D(1-\xi)]^{2\kappa}\,\kappa!} d\xi$$

After interchanging the order of integration and summation under the theorem's condition

$$I_3 \equiv \sum_{\kappa=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma\left(a_j + A_j\kappa\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + B_j\kappa\right)} \frac{\theta^{\kappa}}{\kappa!} \int_0^1 \xi^{\upsilon+\kappa-1}(1-\xi)^{\epsilon+\kappa-1}\,[C\xi + D(1-\xi)]^{-\upsilon-\epsilon-2\kappa}\, d\xi$$

By using equation (2.2) and after some simplification, we get

$$I_3 \equiv \frac{1}{C^{\upsilon}D^{\epsilon}} \sum_{\kappa=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma\left(a_j + A_j\kappa\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + B_j\kappa\right)} \frac{\theta^{\kappa}}{\kappa!} \frac{1}{C^{\kappa}D^{\kappa}} \frac{\Gamma(\upsilon+\kappa)\,\Gamma(\epsilon+\kappa)}{\Gamma(\upsilon+\epsilon+2\kappa)}$$

$$I_3 \equiv \frac{1}{C^{\upsilon}D^{\epsilon}} \frac{\Gamma(\upsilon)\,\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)} \sum_{\kappa=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma\left(a_j + A_j\kappa\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + B_j\kappa\right)} \frac{\left(\frac{\theta}{C\,D}\right)^{\kappa}}{\kappa!} \frac{(\upsilon)_\kappa\,(\epsilon)_\kappa\,(1)_\kappa}{2^{2\kappa}\left(\frac{\upsilon+\epsilon}{2}\right)_\kappa \left(\frac{\upsilon+\epsilon+1}{2}\right)_\kappa \kappa!}$$

Now apply Hadamard product *i.e.* $\sum_{\kappa=0}^{\infty} C_\kappa\, y^\kappa * \sum_{\kappa=0}^{\infty} D_\kappa\, y^\kappa = \sum_{\kappa=0}^{\infty} C_\kappa\, D_\kappa\, y^\kappa$

$$I_3 \equiv \frac{1}{C^{\upsilon}D^{\epsilon}} \frac{\Gamma(\upsilon)\,\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)}\ {}_p\Psi_q \left[\begin{matrix}(a_1,A_1),\dots,(a_p,A_p)\\ (b_1,B_1),\dots,(b_q,B_q)\end{matrix}\middle|\ \frac{\theta}{4CD}\right] * {}_3F_2\left[\begin{matrix}\upsilon,\epsilon,1\\ \frac{\upsilon+\epsilon}{2},\frac{\upsilon+\epsilon+1}{2}\end{matrix}\middle|\ \frac{\theta}{4CD}\right]$$

**Theorem 3.4.** Let $\xi > 0, \upsilon, \epsilon \in \mathbb{C}$ be such that $Re(\upsilon) > 0, Re(\epsilon) > 0$ and the conditions (1.3) is satisfied, then for the generalized wright hypergeometric function ${}_p\Psi_q$, the following integral formula holds true

$$\int_0^1 \int_0^1 \xi^{\epsilon}\,(1-\xi)^{\upsilon-1}(1-\zeta)^{\epsilon-1}(1-\xi\,\zeta)^{1-\epsilon-\upsilon}\ {}_p\Psi_q[W]\ d\xi\, d\zeta = \frac{\Gamma(\upsilon)\,\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)} \times {}_p\Psi_q \left[\begin{matrix}(a_1,A_1),\dots,(a_p,A_p)\\ (b_1,B_1),\dots,(b_q,B_q)\end{matrix}\middle|\ \frac{\theta}{4}\right] * {}_3F_2\left[\begin{matrix}\epsilon,\upsilon,1\\ \frac{\upsilon+\epsilon}{2},\frac{\upsilon+\epsilon+1}{2}\end{matrix}\middle|\ \frac{\theta}{4}\right] \quad (3.4)$$

where $W = \frac{\xi\,(1-\xi)\,(1-\zeta)}{(1-\xi\zeta)^2}\,\theta$

**Proof.** First we refer to the left hand side of equation (3.4) as the sign $I_4$ then making the use of equation (1.1) in equation (3.4), we have

$$I_4 \equiv \int_0^1 \int_0^1 \frac{\xi^{\epsilon}(1-\xi)^{\upsilon-1}(1-\zeta)^{\epsilon-1}}{(1-\xi\,\zeta)^{\epsilon+\upsilon-1}} \sum_{\kappa=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma\left(a_j + A_j\kappa\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + B_j\kappa\right)} \frac{\xi^{\kappa}(1-\xi)^{\kappa}(1-\zeta)^{\kappa}\theta^{\kappa}}{(1-\xi\,\zeta)^{2\kappa}\ \kappa!}\, d\xi\, d\zeta$$

After interchanging the order of integration and summation under the theorem's condition

$$I_4 \equiv \sum_{\kappa=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + A_j\kappa)}{\prod_{j=1}^{q} \Gamma(b_j + B_j\kappa)} \frac{\theta^\kappa}{\kappa!} \int_0^1 \int_0^1 \frac{\xi^{\epsilon+\kappa}(1-\xi)^{\upsilon+\kappa-1}(1-\zeta)^{\epsilon+\kappa-1}}{(1-\xi\,\zeta)^{\epsilon+\upsilon+2\kappa-1}} \, d\xi\, d\zeta$$

By using equation (2.3) and after some simplification, we get

$$I_4 \equiv \sum_{\kappa=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + A_j\kappa)}{\prod_{j=1}^{q} \Gamma(b_j + B_j\kappa)} \frac{\theta^\kappa}{\kappa!} \frac{\Gamma(\epsilon+\kappa)\,\Gamma(\upsilon+\kappa)}{\Gamma(\epsilon+\upsilon+2\kappa)}$$

$$I_4 \equiv \frac{\Gamma(\upsilon)\,\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)} \sum_{\kappa=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + A_j\kappa)}{\prod_{j=1}^{q} \Gamma(b_j + B_j\kappa)} \frac{\theta^\kappa}{\kappa!} \frac{(\epsilon)_\kappa\,(\upsilon)_\kappa\,(1)_\kappa}{2^{2\kappa}\left(\frac{\upsilon+\epsilon}{2}\right)_\kappa \left(\frac{\upsilon+\epsilon+1}{2}\right)_\kappa \kappa!}$$

Now apply Hadamard product *i.e.* $\sum_{\kappa=0}^{\infty} C_\kappa\, y^\kappa * \sum_{\kappa=0}^{\infty} D_\kappa\, y^\kappa = \sum_{\kappa=0}^{\infty} C_\kappa\, D_\kappa\, y^\kappa$

$$I_4 \equiv \frac{\Gamma(\upsilon)\,\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)} \; {}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\middle|\; \frac{\theta}{4}\right] * {}_3F_2\left[\begin{matrix}\epsilon,\upsilon,1\\ \frac{\upsilon+\epsilon}{2},\frac{\upsilon+\epsilon+1}{2}\end{matrix}\middle|\; \frac{\theta}{4}\right]$$

## IV. Special Cases:

$(\boldsymbol{i})$. On taking $\upsilon = \epsilon = 1$ in $Theorem$ 3.1, we get

$$\int_0^1 (1-\xi)\left(1-\frac{\xi}{3}\right)\; {}_p\Psi_q[X]\; d\xi = \left(\frac{4}{9}\right) \times {}_{p+1}\Psi_{q+1}\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p),(1,1)\\(b_1,B_1),\ldots,(b_q,B_q),(2,1)\end{matrix}\middle|\; \theta\right] \tag{4.1}$$

where $X = (1-\xi)^2\left(1-\frac{\xi}{4}\right)\theta$

$(\boldsymbol{ii})$. On taking $\upsilon = \epsilon = 1$ in $Theorem$ 3.2, we get

$$\int_0^1 (1-\xi)\left(1-\frac{\xi}{3}\right)\; {}_p\Psi_q[Y]\; d\xi = \left(\frac{4}{9}\right) \times {}_{p+1}\Psi_{q+1}\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p),(1,1)\\(b_1,B_1),\ldots,(b_q,B_q),(2,1)\end{matrix}\middle|\; \theta\right] \tag{4.2}$$

where $Y = \xi\left(1-\frac{\xi}{3}\right)^2 \theta$

$(\boldsymbol{iii})$. On taking $\upsilon = 1$ in $Theorem$ 3.3, we get

$$\int_0^1 (1-\xi)^{\epsilon-1}\,[C\xi + D(1-\xi)]^{-\epsilon-1}\; {}_p\Psi_q[Z]\; d\xi = \frac{1}{C^\upsilon D^\epsilon}\; {}_{p+2}\Psi_{q+1}\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p),(\epsilon,1),(1,1)\\(b_1,B_1),\ldots,(b_q,B_q),(\epsilon+1,2)\end{matrix}\middle|\; \frac{\theta}{CD}\right] \tag{4.3}$$

where $Z = \frac{\xi\,(1-\xi)}{[C\xi+D(1-\xi)]^2}\theta$

$(\boldsymbol{iv})$. On taking $\epsilon = 1$ in $Theorem$ 3.3, we get

$$\int_0^1 \xi^{\upsilon-1}\,[C\xi + D(1-\xi)]^{-\upsilon-1}\; {}_p\Psi_q[Z]\; d\xi = {}_{p+2}\Psi_{q+1}\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p),(\upsilon,1),(1,1)\\(b_1,B_1),\ldots,(b_q,B_q),(\upsilon+1,2)\end{matrix}\middle|\; \frac{\theta}{CD}\right] \tag{4.4}$$

where $Z = \frac{\xi\,(1-\xi)}{[C\xi+D(1-\xi)]^2}\theta$

$(\boldsymbol{v})$. On taking $\upsilon = 1$ in $Theorem$ 3.4, we get

$$\int_0^1\int_0^1 \xi^{\epsilon}\,(1-\zeta)^{\epsilon-1}\,(1-\xi\,\zeta)^{-\epsilon}\; {}_p\Psi_q[W]\; d\xi\, d\zeta = {}_{p+2}\Psi_{q+1}\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p),(\epsilon,1),(1,1)\\(b_1,B_1),\ldots,(b_q,B_q),(\epsilon+1,2)\end{matrix}\middle|\; \theta\right] \tag{4.5}$$

where $W = \frac{\xi\,(1-\xi)\,(1-\zeta)}{(1-\xi\zeta)^2}\theta$

$(\boldsymbol{vi})$. On taking $\epsilon = 1$ in $Theorem$ 3.4, we get

$$\int_0^1 \int_0^1 \xi \ (1-\xi)^{\upsilon-1} \ (1-\xi\,\zeta)^{-\upsilon} \ {}_p\Psi_q[W] \ d\xi \ d\zeta$$
$$= {}_{p+2}\Psi_{q+1}\left[\begin{matrix}(a_1, A_1), \dots, (a_p, A_p), (\upsilon, 1), (1,1) \\ (b_1, B_1), \dots, (b_q, B_q), (\upsilon+1,2)\end{matrix}\middle| \ \theta\right] \quad (4.6)$$

where $W = \frac{\xi\,(1-\xi)\,(1-\zeta)}{(1-\xi\zeta)^2}\,\theta$

## References


[1]. A. Erdelye, W. Magnus, F. Oberhettinger And F. G. Tricomi (1953). Higher Transcedental Functions, Vol. I, Mcgraw-Hill, New York, London
[2]. E. D. Rainville (1960). Special Functions, Chelsea Publication Co., New York
[3]. E. M. Wright (1940). The Asymptotic Expansion Of The Generalized Hypergeometric Function Ii, Proc. London Math. Soc., Vol. 46 (2), Pp.389-408
[4]. F. Oberhettinger (1974). Tables Of Mellin Transforms, Springer, New York
[5]. H. M. Srivastava (1972). A Contour Integral Involving Fox's H-Function, Indian J. Math. Vol. 14, Pp. 1-6
[6]. J. Edward (1922). A Treatise On The Integral Calculus, Vol. Ii, Chelsea Publishing Company, New York
[7]. J. L. Lavoie And G. Trottier (1969). On The Sum Of Certain Appell's Series, Ganita, Vol. 20 Pp. 43–46
[8]. R. K. Saxena And R. K. Parmar (2017). Fractional Integration And Differention Of The Generalized Mathieu Series, Axioms, Mdpi
[9]. T. Pohlen (2009). The Hadamard Product And Universal Power Series, Ph.D Thesis, Universitat Trier, Germany
[10]. T. M. Macrobert (1961). Beta Functions Formulae And Integrals Involving E-Function, Math. Annalen., Vol. 142, Pp. 450-452