IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-0661,p-ISSN: 2278-8727, Volume 19, Issue 5, Ser. 3 (Sept. – October. 2023), PP 15-32
www.iosrjournals.org

A two warehouse EOQ inventory model considering partial backlogging and time-varying holding costs for deteriorating items in a financial trade credit environment

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In this paper, Author's extent the two warehouses inventory models for non-instantaneous deteriorating items by considering shortages under progressive trade credit policy. In this paper we derived some profit functions for several realistic cases sub cases and scenarios based on the non-instantaneous deterioration and the trade credit period along with the time varying holding cost. The functions formulated as non-linear constrained optimization problem along with the solution procedure. Impact of shortages are observed and to illustrate the robustness of the model, a comprehensive sensitivity analysis has been performed on the optimal case, which is obtained by solving the hypothetical numerical examples with the help of proposed algorithm using Mathematica.

Keywords: *Inventory theory, non-instantaneous deterioration, Progressive tradecredit, two-ware house, Shortages, Time-varying holding cost.*

Date of Submission: 14-10-2023 Date of Acceptance: 29-10-2023

I. Introduction

The wholesalers and suppliers are in intense rivalry with one another to grow their businesses as a result of the globalisation of the market economy. As a result they provide various facilities to their retailers. One of these options is to offer tosell an extensive amount of items on credit. They generally give a specific credit period in that situation. The supplier doesn't charge interest during this time. However, beyond this time, the supplier will charge interest in accordance with certain terms and circumstances set forth in their contract with the store. This kind of inventory issues are termed as inventory problem with permissible delay in payment.

The available literature claims that Haley and Higgins (1973) were the first to explore this kind of issue. Then, Goyal (1985) created an economic order quantity (EOQ) model for when there is a legal payment delay. In actual life, there are some frequently used physical goods like wheat, paddy, or any other type of food grain, vegetables, fruits, drugs, pharmaceuticals, etc., where a portion of these goods are damaged, decayed, vaporised, or affected by some other factors and are not in a condition to satisfy the demand. As a result, the loss due to this natural phenomenon can not be ignored in the analysis of inventory system. Shah and Naik (2019) developed an inventory model for items that are losing market share by optimising the retailer's overall profit. The model includes both the customer's cash discount and the retailer's cash discount depending on the order amount. A three parameter Weibull distributed degrading item with variable demand based on price and frequency of advertising under trade credit was the subject of an inventory model developed by Shaikh et al. in 2019.

In the expanding technology industry, Kumar and Chanda (2018) created a two- warehouse inventory model for degrading goods with demand affected by innovationcriteria. Two models were created by Gautam et al. (2019), the first of which covers the integrated problem-solving technique and the second of which employs the Stackelberg policy. By concurrently optimising the number of shipments, order amount, and backordering quantity, the overall profit is increased. In addition, they have emphasised the need of defect management in order to meet environmental goals without sacrificing financial ones.

A unified inventory model was developed by Khanna et al. (2020) in which production takes place at the vendor's end to meet customer expectations at the buyer's doorstep. They believed that the manufacturing process was flawed and that it might suddenly swing from being "in control" to being "out of control," producing non-conforming products. To keep the manufacturing system running properly, the vendor performs routine preventive maintenance on it and provides a free basic repair warranty on the items given to the customer. The vendor employs rework and restoration activities in addition to preventative maintenance to reduce problems in the production system. The inventory model for non-instantaneous degrading items under progressive trade credit policy was created by Jaggi et al. (2020) after researching the effects of progressive trade credit on the inventory management system. Two warehouse inventory models for non-immediately degrading goods with progressive trade credit policy were taken into consideration by Tripathy et al. in 2021. Theyhave picked a variety of financial scenarios and are working to create a cost functionfor each one based on the trade credit duration in the current economic climate.

In business world, stock-out situation plays an important role. Due to some unavailable circumstances, stock-out situation may occur in any business. According tothe literature of inventory control theory, most of the inventory models were developed under the assumption "shortages are allowed and completely backlogged". In practice, this assumption is not realistic. Generally, customers are not interested to wait for a long time to purchase goods from a particular shop. Only a fraction of the customers will wait to purchase the good from a particular shop due to goodbehaviours, genuine price and quality of the goods and also the locality of the shop. As a result, shortages are considered as partially backlogged with a rate dependent onthe length of waiting time up to the arrival of fresh lot.

Therefore, this paper extends the permissible delay in payments based two-warehouse on the inventory model by considering partially backlogged shortages. According to the condition of trade credit, several cases, sub cases and situations have been considered and the corresponding constrained optimization problems have been formulated and discussed the solution procedure. For solving those problems, solution procedure has been introduced. Then, to illustrate the model and its validity, two numerical examples have been considered and solved. Finally, to the study theeffect of changes of ordering cost, demand and own warehouse capacity on the optimal policies of the model, sensitivity analyses have been performed considering first example.

II. Assumptions

The following assumptions and notations have been used in the entire paper.

(i) Replenishment rate is infinite and lead-time is constant.

(ii) The inventory planning horizon is infinite and the inventory system involves onlyone item.

(iii) The entire lot size is delivered in one batch.

(iv) The goods of RW are transported from RW to OW in continuous release pattern. The time lag between selling from OW and filling up the space by new units from RW is negligible.

(v) The demand rate is known and constant.

(vi) Deterioration is considered only after the inventory stored in the warehouse. There is neither repair nor replacement of the deteriorated units during the inventory cycle.

(vii) Shortages, if any, are allowed and partially backlogged. During the stock-out period,the backlogging rate is dependent on the length of the waiting time up to the arrivalof fresh lot. Considering this situation, the rate is defined as $\left[1+\delta\left(T-t\right)\right]^{-1}, \delta > 0$.

III. Notations

$Q_r(t), Q_o(t)$	: Instantaneous stock levels in RW and OW at time t, respectively.
S	: at the beginning of the cycle it is the maximum stock level
R	: maximum shortage level
W	: Storage capacity of OW
$\alpha, \beta (\alpha > \beta)$	: Rates of deterioration in OW and RW respectively. $(0 < \alpha, \beta << 1)$
c_t	: Transportation cost per unit for transferring the items from RW to OW
A	: Replenishment cost (ordering cost) for replenishing the items

δ	: Parameter for Backlogging
c	: Purchase cost per unit quantity of item
$p\,(p > c)$	: Selling price per unit of item
D	: Constant demand rate
$H, F(F > H)$	:Time varying Holding cost per unit per unit time at OW and RW respectively where $H = a + bt$, and $F = a_1 + b_1 t$,
π	: Shortage cost per unit per unit time
π_1	: Lost sale cost per unit per unit time
T	: Cycle length
t_1	: Time point when the stock level of OW reaches to zero
t_w	: Time point when the stock level of RW reaches to zero
M	: Credit period offered by supplier
I_e	: Interest earned by the retailer
I_p	: Interest payable to the supplier

IV. Mathematical Model

At first, a store buys $(S + R)$ units of the product. Following the completion of the backlogged amounts, the level of on-hand inventory is S units, out of which W units are kept in OW and the remaining $(S - R)$ units are kept in RW. Due to client demand and the deteriorating impact of the goods, the inventory $(S - W)$ decline throughout the course of the interval $0 \le t \le t_w$ and disappear at $t = t_w$.

In OW, the stock level decreases during $0 \le t \le t_w$ due to deterioration only and during $t_w < t \le t_1$ due to both demand and deterioration. At time $t = t_1$, the inventory level in OW reaches to zero. There after the partially backlogged shortages are allowed to occur during the time interval $t_1 < t \le T$ with a rate $\left[1 + \delta\left(T - t\right)\right]^{-1} D,\ \delta > 0$.

At time $t = T$, the maximum shortage level is R. The entire inventory cycle will be repeated after the cycle length T. The pictorial representation of the system is show in Fig. – 1.

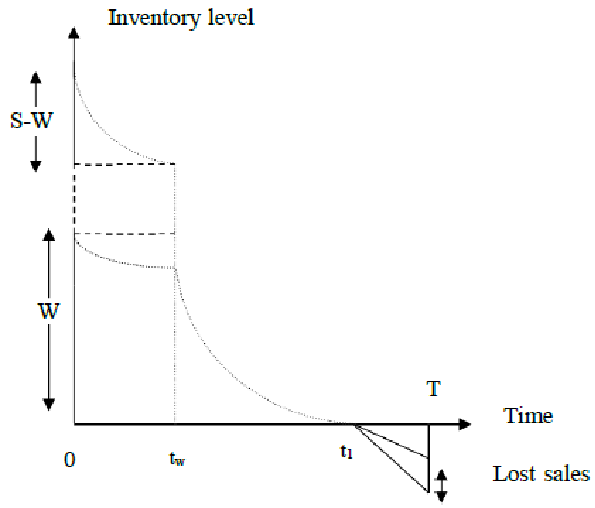

Fig. 1: Inventory situations in OW and RW

The stock depletion in RW during the time interval $0 \le t \le t_w$ is mainly due to meet up the demand and partly due to deterioration effect of items. Hence the inventory level $Q(t)$ in RW satisfies the following differential equation:

$$\frac{dQ_r(t)}{dt} + \beta Q_r(t) = -D, \quad 0 \le t \le t_w \tag{1}$$

subject to condition that $Q_r(t) = 0$ at $t = t_w$. (2)

Again, $Q_r(t) = S - W$ at $t = 0$. (3)

Using (2), the solution of (1) is given by

$$Q_r(t) = D\left[\exp\{\beta(t_w - t)\} - 1\right] / \beta, \qquad 0 \le t \le t_w \tag{4}$$

Now, from (3) and (4), we have

$$S = W + D\{\exp(\beta t_w) - 1\} / \beta \tag{5}$$

$$\frac{dQ_o(t)}{dt} + \alpha Q_o(t) = 0, \qquad 0 \le t \le t_w \tag{6}$$

$$\frac{dQ_o(t)}{dt} + \alpha Q_o(t) = -D, \qquad t_w < t \le t_1 \tag{7}$$

$$\frac{dQ_o(t)}{dt} = -D / \{1 + \delta(T - t)\}, \quad t_1 < t \le T \tag{8}$$

subject to the conditions

$$Q_o(0) = W \text{ at } t = 0, \tag{9}$$

$$Q_o(t) = 0 \text{ at } t = t_1 \tag{10}$$

$$\text{and } Q_o(t) = -R \text{ at } t = T. \tag{11}$$

Also, $Q_o(t)$ is continuous at $t = t_w$, and $t = t_1$.

Using the conditions (9) – (11), the solutions of the differential equations (6) – (8) are given by

$$Q_o(t) = W\exp(-\alpha t), \qquad 0 \le t \le t_w \tag{12}$$

$$Q_o(t) = D\left[\exp\{\alpha(t_1 - t)\} - 1\right]/\alpha, \quad t_w < t \le t_1 \tag{13}$$

$$Q_o(t) = -R + D\log(1+\delta(T-t))/\delta, \qquad t_1 < t \le T \tag{14}$$

Now using the continuity condition of $Q_o(t)$ at time $t = t_w$, we have

$$W\exp(-\alpha t_w) = D\left[\exp\{\alpha(t_1 - t_w)\} - 1\right]/\alpha \tag{15}$$

which implies

$$t_1 = t_w + \log\left[1 + \frac{\alpha W\exp(-\alpha t_w)}{D}\right]/\alpha \tag{16}$$

Again, from the continuity of $Q_o(t)$ at $t = t_1$, we have

$$R = D\log\{1+\delta(T-t_1)\}/\delta \tag{17}$$

Now the holding cost HC over the entire cycle is given by

$$HC = F\int_0^{t_w} Q_r(t)dt$$

$$+H\left[\int_0^{t_w} Q_o(t)dt + \int_{t_w}^{t_1} Q_o(t)dt\right]$$

$$\Rightarrow HC = (a_1 + b_1 t)\int_0^{t_w} Q_r(t)dt$$

$$+(a+bt)\left[\int_0^{t_w} Q_o(t)dt + \int_{t_w}^{t_1} Q_o(t)dt\right]$$

$$= (a_1 + b_1 t)\int_0^{t_w} D\left[\exp(\beta(t_w - t)) - 1\right] / \beta dt +$$

$$(a+bt)\left[W \exp(-\alpha t)dt + \int_{t_w}^{t_1} D\left[\left[\exp(\beta(t_w - t)) - 1\right] / \alpha\right]dt\right]$$

$$= (a_1 + b_1 t)\frac{D}{\beta}\left[-\frac{1}{\beta} + \frac{\exp(\beta t_w)}{\beta} - t_w\right]$$

$$+ (a+bt)\left[\frac{W}{\alpha}(1 - \exp(\alpha t_w)) + \frac{D}{\alpha}\left[-\alpha - t_1 + t_w + (\exp \alpha(t_1 - t_w)) / \alpha\right]\right] \quad (18)$$

$$= \frac{(a_1 + b_1 t)D}{\beta^2}\left[\exp(\beta t_w) - \beta t_w - 1\right]$$

$$+ (a+bt)\left[\frac{W}{\alpha}\{1 - \exp(-\alpha t_w)\}\right.$$

$$\left. + \left(\frac{D}{\alpha^2}\right)\{\exp(\alpha(t_1 - t_w)) - \alpha(t_1 - t_w) - 1\}\right]$$

Again, the total shortage cost SC over the entire cycle is given by

$$SC = \pi \int_{t_1}^{T} \left[-Q_o(t)\right] dt$$

$$= \pi\left[R(T - t_1) - \frac{D}{\delta^2}\{(1 + \delta(T - t_1))\log|1 + \delta(T - t_1)| - \delta(T - t_1)\}\right] \quad (19)$$

The lost sale cost LS over the cycle is given by

$$LS = \pi_L \int_{t_1}^{T} D\left\{1 - \frac{1}{1 + \delta(T - t)}\right\} dt$$

$$= \frac{\pi_1 D}{\delta}\left[\delta(T - t_1) - \log|1 + \delta(T - t_1)|\right] \quad (20)$$

The transportation cost TC for transferring the items from RW to OW is given by

$$TC = c_t D t_w$$

As M be the trade credit period offered by the supplier, there may arise four cases as follows:

Case 1: $0 < M \le t_w$

Case 2: $t_w < M \le t_1$

Case 3: $t_1 < M \le T$

Case 4: $T < M$

Case 1: $0 < M \le t_w$

In this instance, the entire amount owed to the supplier at this moment is c(S + R), while the amount of revenue received (due to sales and interest gained) is represented by

$$U_1 = DMp\left(1+\frac{1}{2}I_e M\right) + pR\left(1+I_e M\right) \tag{21}$$

Again, two sub cases may arise:

Case 1.1: $U_1 \ge c\left(S+R\right)$

Case 1.2: $U_1 < c\left(S+R\right)$

Case 1.1: $U_1 \ge c\left(S+R\right)$

In this subcase, the retailer must only pay the supplier c $(S + R)$ at time $t = M$. He will thus get interest for the time period $[M, T]$ from the surplus amount $U_1 - c\left(S+R\right)$. After time $t = M$, the store will begin to constantly accumulate sales revenue and earn interest on that money.

Therefore, the interest earned by the retailer is given by,

$$Z_1^{(1)}\left(t_w, T\right) = \frac{X_1}{T} \tag{22}$$

where X_1 = < Excess amount after paying the amount to the supplier > + < interest earned for the excess amount during $\left[M, T\right]$ > + < total selling price during $\left[M, t_1\right]$ > + < interest earned during $\left[M, t_1\right]$ > + < interest earned during $\left[t_1, T\right]$ > – < ordering cost > – < holding cost > – < shortage cost > – < lost sale cost> - < transportation cost >

$$i.e., X_1 = \left\{U_1 - c\left(S+R\right)\right\}\left\{1 + I_e\left(T-M\right)\right\}$$

$$+ Dp\left(t_1 - M\right)\left\{1+\frac{1}{2}I_e\left(t_1+M\right)\right\}+$$

$$I_e\left[D(t_1-M)p\left\{1+I_e\frac{\left(t_1+M\right)}{2}\right\}\left(T-t_1\right)\right] \tag{23}$$

$$-A-HC-SC-LS-TC$$

Hence the corresponding optimization problem is given by

Problem-1: Maximize $Z_1^{(1)}\left(t_w, T\right) = \dfrac{X}{T}$

subject to $0 < M \le t_w < t_1 < T$

Case 1.2: $U_1 < c(S+R)$

In this sub case, two scenarios may appear:

Case 1.2.1: When partial payment is made at $t = M$ and the rest amount is to be paid after $t = M$.

Case1.2.2: When full payment is to be made after $t = M$ due to non willingness of partial payment.

Case 1.2.1: When partial payment is made at $t = M$ and the rest amount is to be paid after $t = M$.

In this scenario, let the rest amount $c(S+R) - U_1$ will be paid at time $t = B(B > M)$. As a result, the retailer has to pay the interest of amount $c(S+R) - U_1$ for the period $[M, B]$.

Hence, at $t = B$, the total payable amount will be |

$$\{c(S+R) - U_1\} + \{c(S+R) - U_1\} I_p (B - M)$$
$$= \{c(S+R) - U_1\}\{1 + I_p (B - M)\}$$

Again, up to $t = B$, the interest earned is $\frac{1}{2} DpI_e (B^2 - M^2)$

and the total amount available to the retailer = the sum of selling amount during $[M, B]$ + interest earned =

$$Dp(B - M) + \frac{1}{2} DpI_e (B^2 - M^2)$$
$$= Dp(B - M)\left\{1 + \frac{1}{2} I_e (B + M)\right\}$$

Therefore, the amount payable to the supplier = total amount available to the retailer at time $t = B$

i.e.,

$$\{c(S+R) - U_1\}\{1 + I_p (B - M)\} = Dp(B - M)\left\{1 + \frac{1}{2} I_e (B + M)\right\} \tag{24}$$

Now the total selling price for the period $[B, t_1]$ is $Dp(t_1 - B)$ and the interest earned during that period is $\frac{1}{2} DpI_e (t_1^2 - B^2)$.

Again the interest earned during the period $[t_1, T]$ is

$$Dp(t_1 - B)\left\{1 + \frac{1}{2} I_e (t_1 + B)\right\} I_e (T - t_1)$$

Therefore, the average profit for the cycle is given by

$$Z_1^{(2.1)}\left(t_w,T\right)=\frac{X_2}{T} \tag{25}$$

where X_2 = < total selling price during $\left[B,t_1\right]$> + < interest earned during $\left[B,t_1\right]$ > + < interest earned during $\left[t_1,T\right]$> - < ordering cost > – < holding cost > – < shortage cost> – < lost sale cost> - < transportation cost >

i.e.,

$$X_2 = Dp\left(t_1-B\right)\left\{1+\frac{1}{2}I_e\left(t_1+B\right)\right\} \tag{26}$$
$$-\left\{1+I_e\left(T-t_1\right)\right\}-A-HC-SC-LS-TC$$

Problem-2: Maximize $Z_1^{(2.1)}\left(t_w,T\right)=\dfrac{X_2}{T}$

subject to $0<M\le t_w<t_1<T$

Casel.2.2: When full payment is to be made after $t=M$ due to non willingness of partial payment.

In this scenario, full payment is to be made after $t=M$ when possible. Let this time be *B*. In that case, retailer has to pay the interest for the period [*M*, *B*].

Hence the total amount paid to the supplier is $c\left(S+R\right)\left\{1+I_p\left(B-M\right)\right\}$

Up to $t=B$, the total amount earned by the retailer is

$$DBp+\frac{1}{2}DpI_eB^2=DBp\left(1+\frac{1}{2}I_eB\right)$$

According to the condition, the amount payable to the supplier = total amount available to the retailer at time $t=B$

i.e.

$$c\left(S+R\right)\left\{1+I_p\left(B-M\right)\right\} = DBp\left(1+\frac{1}{2}I_eB\right) \tag{27}$$

Now the total selling price during $\left[B,t_1\right]$ is $Dp\left(t_1-B\right)$ and the interest earned during $\left[B,t_1\right]$ is $\frac{1}{2}DpI_e\left(t_1^2-B^2\right)$.

Again, the interest earned during $\left[t_1,T\right]$ is,

$$\left\{Dp\left(t_1-B\right)+\frac{1}{2}DpI_e\left(t_1^2-B^2\right)\right\}I_e\left(T-t_1\right).$$

Therefore, the average profit for the cycle is given by

$$Z_1^{(1.2)}(t_w,T)=\frac{X_3}{T} \tag{28}$$

where X_3 = < total selling price during $[B,t_1]$ > + < interest earned during $[B,t_1]$ >

+< interest earned during $[t_1,T]$ > – < ordering cost > – < holding cost > – < shortage cost> – < lost sale cost> - < transportation cost >

$$\begin{aligned} i.e., X_3 &= Dp(t_1-B)+\frac{1}{2}DpI_e(t_1^2-B^2) \\ &+\left\{Dp(t_1-B)+\frac{1}{2}DpI_e(t_1^2-B^2)\right\}I_e(T-t_1) \\ &-A-HC-SC-LS-TC \end{aligned} \tag{29}$$

Problem-3: Maximize $Z_1^{(2.2)}(t_w,T)=\frac{X_3}{T}$

subject to $0<M\le t_w<t_1<T$

Case 2: $t_w<M\le t_1$

In this case, the interest earned during $[0,M]=\frac{1}{2}DpI_eM^2$

Hence the total revenue up to $t=M$ is given by

$$U_2=DMp\left(1+\frac{1}{2}I_eM\right)+pR(1+I_eM) \tag{30}$$

Again, two sub cases may arise:

Case 2.1: $U_2\ge c(S+R)$

Case 2.2: $U_2<c(S+R)$

Case 2.1: $U_2\ge c(S+R)$

In this sub case, retailer has to pay only $c(S+R)$ amount to the supplier at time $t=M$. So from the excess amount $U_2-c(S+R)$, he will earn the interest for the time period $[M,T]$. After time $t=M$, the retailer will start to accumulate the revenues continuously on the sales and earn interest on that revenue.

Therefore, the interest earned by the retailer is given by

I_2 = < Interest earned during $[M,t_1]$ > + < Interest earned during $[t_1,T]$ >

$$=\frac{1}{2}pI_eD\left(t_1^2-M^2\right)$$
$$+\left(pD\left(t_1-M\right)+\frac{1}{2}pI_eD\left(t_1^2-M^2\right)I_e\left(T-t_1\right)\right)$$

Therefore, the average profit for the cycle is given by

$$Z_2^{(1)}\left(t_w,T\right)=\frac{X_4}{T} \tag{31}$$

where X_4 = < Excess amount after paying the amount to the supplier > + < interest earned for the excess amount during $\left[M,T\right]$ > + < total selling price during $\left[M,t_1\right]$ > + < interest earned during $\left[M,t_1\right]$ > + < interest earned during $\left[t_1,T\right]$ > - < ordering cost > – < holding cost > – < shortage cost > – < lost sale cost> - < transportation cost >

i.e.,

$$\begin{aligned}X_4&=U_2-c\left(S+R\right)\left\{1+I_e\left(T-M\right)\right\}\\&+Dp\left(t_1-M\right)\left\{1+\frac{1}{2}I_e\left(t_1+M\right)\right\}\left\{1+I_e\left(T-t_1\right)\right\}\\&-A-HC-SC-LS-TC\end{aligned} \tag{32}$$

Problem-4: Maximize $Z_2^{(1)}\left(t_w,T\right)=\frac{X}{T}$

subject to $0<t_w\le M<t_1<T$

Case 2.2: $U_2<c\left(S+R\right)$

In this sub case, two scenarios may appear:

Case 2.2.1: When partial payment is made at $t=M$ and the rest amount is to be paid after $t=M$.

Case 2.2.2: When full payment is to be made after $t=M$ due to non willingness of partial payment.

Case 2.2.1: When partial payment is made at $t=M$ and the rest amount is to be paid after $t=M$.

In this scenario, let the rest amount $c\left(S+R\right)-U_2$ will be paid at time $t=B$. In this case, the interest of amount $c\left(S+R\right)-U_2$ for the period $\left[M,B\right]$ is to be paid.

Now, the interest payable to the supplier at $t=B$ is

$$\left\{c\left(S+R\right)-U_2\right\}I_p\left(B-M\right).$$

Hence, at $t=B$, the total payable amount will be

$$\left\{c\left(S+R\right)-U_2\right\}+\left\{c\left(S+R\right)-U_2\right\}I_p\left(B-M\right)$$
$$=\left\{c\left(S+R\right)-U_2\right\}\left\{1+I_p\left(B-M\right)\right\}$$

Again, up to $t=B$, the interest earned is $\frac{1}{2}DpI_e\left(B^2-M^2\right)$ and the total amount available to the retailer is the sum of selling amount during $\left[M,B\right]$ + interest earned

i.e., $$Dp\left(B-M\right)+\frac{1}{2}DpI_e\left(B^2-M^2\right)$$
$$=Dp\left(B-M\right)\left\{1+\frac{1}{2}I_e\left(B+M\right)\right\}$$

Now, according to condition, the amount payable to the supplier = total amount available to the retailer at time $t=B$

$$i.e.,\ \left\{c\left(S+R\right)-U_2\right\}\left\{1+I_p\left(B-M\right)\right\}$$
$$=D\left(B-M\right)p+\frac{1}{2}DpI_e\left(B^2-M^2\right) \tag{33}$$

Again, the total selling price for the period $\left[B,t_1\right]$ is $Dp\left(t_1-B\right)$ and the interest earned during that period is $\frac{1}{2}DpI_e\left(t_1-B\right)^2$.

Again the interest earned during the period $\left[t_1,T\right]$ is

$Dp\left(t_1-B\right)\left\{1+\frac{1}{2}I_e\left(t_1-B\right)\right\}I_e\left(T-t_1\right)$.

Therefore, the average profit for the cycle is given by

$$Z_2^{(2.1)}\left(t_w,T\right)=\frac{X_5}{T} \tag{34}$$

where X_5 X = < total selling price during $\left[B,t_1\right]$> + < interest earned during $\left[B,t_1\right]$ > + < interest earned during $\left[t_1,T\right]$> - < ordering cost > – < holding cost > – < shortage cost> – < lost sale cost> - < transportation cost >

$$i.e.,\ X=Dp\left(t_1-B\right)\left\{1+\frac{1}{2}I_e\left(t_1-B\right)\right\}\left\{1+I_e\left(T-t_1\right)\right\}$$
$$-A-HC-SC-LS-TC \tag{35}$$

Problem-5: Maximize $Z_2^{(2.1)}\left(t_w,T\right)=\frac{X_5}{T}$

subject to $0<t_w\le M<t_1<T$

Case 2.2.2: When full payment is to be made after $t = M$ due to non willingness of partial payment.

In this scenario, full payment is to be made after $t = M$ when possible. Let the full payment be made at time $t = B$. In that case, retailer has to pay the interest for the period $[M, B]$.

Here the total amount paid to the supplier is $c(S+R)\{1+I_p(B-M)\}$.

Now, up to $t = B$, the total amount earned by the retailer is

$$DBp + \frac{1}{2}DpI_eB^2 = DBp\left(1+\frac{1}{2}I_eB\right)$$

Hence according to condition, the amount payable to the supplier = total amount available to the retailer at time $t = B$

i.e., $c(S+R)\{1+I_p(B-M)\}$

$$= DBp\left(1+\frac{1}{2}I_eB\right) \tag{36}$$

Therefore, the average profit for the cycle is given by

$$Z_2^{(2.2)}(t_w, T) = \frac{X_6}{T} \tag{37}$$

where X_6 = < total selling price during $[B, t_1]$ > + < interest earned during $[B, t_1]$ > + < interest earned during $[t_1, T]$ > - < ordering cost > – < holding cost > – < shortage cost> – < lost sale cost> - < transportation cost >

i.e.,

$$\begin{aligned} X_6 &= Dp(t_1 - B) + DpI_e(t_1^2 - B^2)/2 \\ &+ \{Dp(t_1 - B) + DpI_e(t_1^2 - B^2)/2\}I_e(T - t_1) \\ &- A - HC - SC - LS - TC \end{aligned} \tag{38}$$

Problem-6: Maximize $Z_2^{(2.2)}(t_w, T) = \frac{X_6}{T}$

subject to $0 < t_w \le M < t_1 < T$

Case 3: $t_1 < M \le T$

In this case, the total selling price is Dpt_1 and the interest earned during $[0, t_1]$ is

$$\frac{1}{2}DpI_et_1^2$$

Now, the interest earned for the time period $[t_1, M]$ is

$$Dpt_1\left(1+\frac{1}{2}I_e t_1\right)I_e\left(M-t_1\right)$$

Again, the interest earned for the amount pR during $\left[0,M\right]$ is $pR+pRI_eM$

Hence the total revenue is given by

$$U_3 = Dpt_1\left(1+\frac{I_e t_1}{2}\right)\left\{1+I_e\left(M-t_1\right)\right\}$$
$$+pR\left(1+I_eM\right)$$

In this case, U_3 must be greater than $c\left(S+R\right)$. So, up to $t=M$, the excess amount would be $U_3-c\left(S+R\right)$.

Hence the average profit for the cycle is given by

$$Z_3\left(t_w,T\right)=\frac{X_7}{T}$$

where X_7 = < Excess amount > + < interest earned for that excess amount during $\left[M,T\right]$ > - < ordering cost > – < holding cost > – < shortage cost> – < lost sale cost> - < transportation cost >

i.e.,

$$X = U_3 - c\left(S+R\right)\left\{1+I_e\left(T-M\right)\right\}$$
$$-A-HC-SC-LS-TC \tag{39}$$

Problem-7: Maximize $Z_3\left(t_w,T\right)=\dfrac{X_7}{T}$

subject to $0<t_w<t_1<M\leq T$

Case 4: $M>T$

In this case, the total selling price would be $pR+Dt_1p$.

Hence the total revenue earned by the retailer is given by

$$U_4 = pR\left(1+I_eM\right)+\ Dpt_1+\frac{1}{2}DpI_et_1^2$$
$$+\ \left(Dpt_1+\frac{1}{2}DpI_et_1^2\right)I_e\left(M-t_1\right)$$

Therefore, the average profit for the cycle is given by

$$Z_4\left(t_w,T\right)=\frac{X_8}{T}$$

where X_8 = < interest earned for that excess amount during $[M,T]$ > - < ordering cost > – < holding cost > – < shortage cost> – < lost sale cost> - < transportation cost >

i.e.,

$$\begin{aligned} X_8 &= pR(1+I_eM) + Dpt_1 + DpI_et_1^2/2 \\ &+ \left(Dpt_1 + DpI_et_1^2/2\right) I_e (M-t_1) \\ &- A - HC - SC - LS - TC \end{aligned} \tag{40}$$

Problem-8: Maximize $Z_4(t_w,T) = \dfrac{X_8}{T}$

subject to $0 < t_w < t_1 < T < M$

V. Solution Procedure

In a similar way, the problems of other cases can be solved and stored the solutions. Let the optimal average profits of these cases (i.e., Case 2, Case 3 and Case 4) are denoted by $Z^{(2)}$, $Z^{(3)}$ and $Z^{(4)}$. The corresponding solutions of these cases be $t_w^{(2)}$, $t_1^{(2)}$, $T^{(2)}$, $S^{(2)}$, $R^{(2)}$, $t_w^{(3)}$, $t_1^{(3)}$, $T^{(3)}$, $S^{(3)}$, $R^{(3)}$, $t_w^{(4)}$, $t_1^{(4)}$, $T^{(4)}$, $S^{(4)}$, $R^{(4)}$, respectively.

The optimal solution of the inventory system can be found by comparing the average profits of all the cases. Hence the optimal average profit of the system is given by

$$Z^* = \max\{Z^{(1)}, Z^{(2)}, Z^{(3)}, Z^{(4)}\}$$

The corresponding values of t_w, t_1, T, S, R are denoted by t_w; t_1; T ; S and R which will be the optimal solution of the problem. However, the optimality condition can not tested theoretically due to high nonlinearity of the problem. For this reason, this condition is tested numerically.

VI. Numerical Analysis

To illustrate the developed model of the inventory system, two numerical examples have been considered. The values of the model parameters considered in these numerical examples are not collected from any real life case study, but these values considered here are realistic. Both these examples have been solved to find optimal values of t_w, t_1, T, S, R along with the optimal profit of the system.

Table 1: Values of parameters of different examples

Parameters	A ($)	D	H ($)	F ($)	C ($)	P ($)	c_t ($)	I_e ($)	I_p ($)	π ($)	π_1 ($)	δ	M	α	β	W
Example 1	1500	2000	1	3	10	15	0.1	0.12	0.15	11.0	12.0	0.5	0.25	0.10	0.06	100
Example 2	300	500	1	2	15	22	0.2	0.12	0.15	16	17	1.5	0.30	0.08	0.05	150

Table 2 : Results of Example 1 for different cases, subcases and scenarios

Case	Subcase	Scenario	t_w	t_1	T	S	R	Avg. Profit	Remarks
1	1.1	-	0.4149	0.4628	0.6340	940.30	328.52	6277.44	
	1.2	1.2.1	0.4834	0.5309	0.6958	1081.00	316.77	6316.51	
		1.2.2	0.5090	0.5564	0.5564	1133.67	0.00	5417.76	
2	2.1	-	0.2500	0.2986	0.4657	603.77	320.89	5817.32	
	2.2	2.2.1	-	-	-	-	-	-	Infeasible solution
		2.2.2	0.2500	0.2986	0.2986	603.77	0.00	4327.81	
3	-	-	0.2011	0.2500	0.4203	504.67	326.82	5436.71	
4	-	-	0.1448	0.1939	0.2500	501.40	110.60	3834.69	

Table 3 : Results of Example 2 for different cases, subcases and scenarios

Case	Subcase	Scenario	t_w	t_1	T	S	R	Avg. Profit	Remarks
1	1.1	-	0.3000	0.5895	0.9639	301.13	148.56	1946.41	
	1.2	1.2.1	0.3000	0.5895	0.6535	331.67	30.54	2789.23	
		1.2.2	0.3000	0.5895	0.5895	301.13	0.00	2689.56	
2	2.1	-	0.2045	0.4962	0.5564	252.76	28.84	2794.68	
	2.2	2.2.1	0.2269	0.5181	0.5784	264.10	28.87	2798.27	
		2.2.2	0.2058	0.4975	0.4975	253.43	0.00	2706.59	
3	-	-	0.0036	0.3000	0.3579	151.89	27.79	2658.80	
4*	-	-	0.0000 (-)	0.2964 (0.2738)	0.3000 (0.3000)	150.00 (138.40)	1.77 (12.85)	2539.77 (2563.49)	

*The bracket results are due to single warehouse case

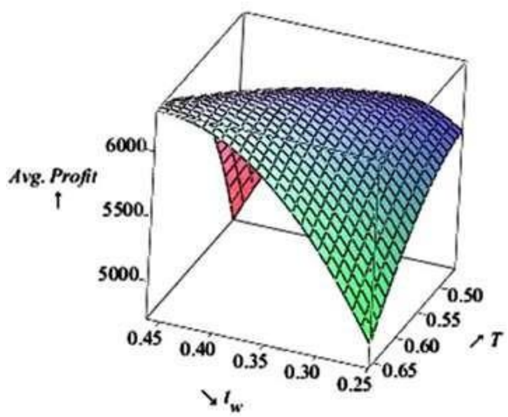

Fig. 6a. Average profit versus tw and T for Case 1.1

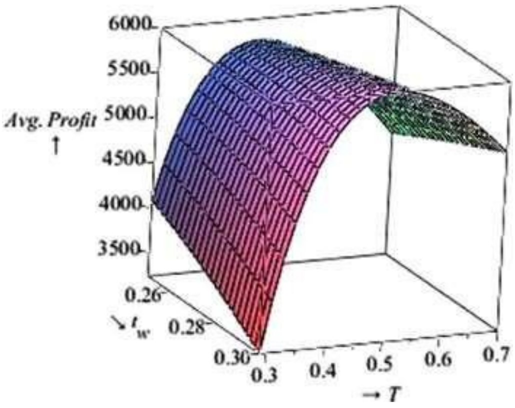

Figure. 6b. Average profit versus tw and T for Case 2.1.

VII. Sensitivity Analysis

Considering Example 1 mentioned in earlier section, we have performed the sensitivity analyses to study the effect of changes of the parameters A (Ordering cost), D (Demand) and W (owned warehouse capacity) on the optimal policies. The results of these analysis are given in Table 4.

Table 4 : Effects of changes in the system parameters of the model

W	A	D	t_w	t_1	T	S	R	profit	Remarks
100	1500	2000	0.4834	0.5309	0.6958	1081.00	316.77	6316.51	Case1.2.1
		3500	0.3815	0.4090	0.5326	1450.66	419.79	12790.62	Case1.1
		5000	0.3231	0.3424	0.4459	1731.21	504.33	19530.47	Case1.1.
100	2000	2000	0.5589	0.6061	0.7990	1236.71	368.43	5647.55	Case1.21
		3500	0.4230	0.4504	0.5955	1599.49	490.46	11930.30	Case 1.1
		5000	0.3758	0.3951	0.5150	2000.52	582.35	18489.76	Case1.1
	2500	2000	0.6249	0.6718	0.8901	1373.74	414.44	5055.54	Case1.2.1
		3500	0.4917	0.5188	0.6810	1846.49	545.61	11145.16	Case1.2.1
		5000	0.4221	0.4413	0.5759	2237.69	651.52	17573.12	Case1.1
250	1500	2000	0.4149	0.5341	0.6938	1090.18	307.38	6483.59	Case1.2.1
		3500	0.3416	0.4104	0.5309	1457.99	409.43	12974.12	Case1.1
		5000	0.2500 (=M)	0.2986	0.4001	1509.42	494.89	19744.89	Case2.1
	2000	2000	0.4905	0.6088	0.7966	1245.62	358.95	5812.67	Case1.2.1
		3500	0.4012	0.4696	0.6103	1671.25	475.93	12098.87	Case1.2.1
		5000	0.3476	0.3958	0.5134	2006.16	571.52	18679.67	Case 1.1
	2500	2000	0.5567	0.6743	0.8873	1382.29	404.88	5218.84	Case1.2.1
		3500	0.4518	0.5198	0.6788	1852.87	535.36	11323.14	Case1.2.1
		5000	0.3940	0.4420	0.5743	2243.58	640.76	17760.31	Case 1.1
300	1500	2000	0.3926	0.5358	0.6940	1094.48	304.64	6534.63	Case1.2.1
		3500	0.3286	0.4112	0.5307	1461.45	406.26	13031.73	Case 1.1
		5000	0.2500 (=M)	0.3083	0.4095	1559.42	493.20	19823.81	Case 2.1
	2000	2000	0.4682	0.6103	0.7965	1249.66	356.12	5863.77	Case1.2.1

		3500	0.3881	0.4702	0.6099	1674.14	472.76	12156.01	Case1.2.1
		5000	0.3383	0.3961	0.5130	2008.79	568.12	18740.41	Case 1.1
	2500	2000	0.5344	0.6756	0.8870	1386.14	402.00	5269.82	Case1.2.1
		3500	0.4387	0.5204	0.6783	1855.72	532.17	11379.78	Case1.2.1
		5000	0.3848	0.4423	0.5740	2246.20	637.36	17820.44	Case 1.1

VIII. Conclusion

In this paper, an attempt has been made to develop a two-warehouse deteriorating inventory model considering trade credit financing. For this purpose, a new approach for trade credit policy has been introduced in the formulation of the model. In this formulation, it is also assumed that the preservation facility in RW is better than in OW and the stocks of RW are transferred with a continuous release pattern.

This model can be applied in many practical situations. Due to introduction of open market policy, the business competition becomes very high to occupy maximum possible market. For this reason, in order to attract more customers, a retailer is forced to provide a better purchasing – environment to the customers such as well decorated show-room with modern light and electronic arrangements and enough freespace for choosing items, etc. Again, due to the expanding market situation, there is a crisis of space in the market-places especially in the super market, corporation market, etc. As a result, the retailer is bounded to hire a separate warehouse on rental basis at a distance place for storing of excess items.

For further research, the proposed model can be extended in several ways. Firstly, one can extend this model for variable demand dependent on price, displayed stock level, time and their combination. Secondly, this model can be generalised by considering two level credit policies.

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