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Mathematical Formulation of the Frontier Element Method for the Laplace and Poisson Equation

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Abstract: *This article aims to obtain mathematical formulation, as well as the discretization expression of the equations to obtain the numerical results of the solution to the Laplace and Poisson operators, it should be mentioned that the development and updating in the architecture and computational devices have allowed greater capacity to store information and speed in its processing, different software applications have been developed and used to implement the finite element method, however, as far as the fronter element method is concerned, there are very few. It should be noted that in its implementation and way of operating it decreases the number of equation systems to be solved, so it is not necessary to use too much memory in its development.*

Background*:The method arose in the late 1970s. To expand the history of the method, see the article by Alexander H.D. Chenga, Daisy T. Cheng (2005)[1], this numerical technique known as the frontier finite element method, has come to replace the finite difference and finite element methods; beginning the development of this in (1941) its historical development can be considered in the article by Wing Kam Liu, Shaofan Li & Harold S. Park (2022)[2], because they consider too many elements to bring it closer to the model to work, so the number of equations to solve is greater, which requires more time to obtain the solution of the problem.*

Materials and Methods*: These three methods are currently used to solve partial differential equations in those cases in which their analytical solution cannot be obtained due to their complexity. The finite element is a method that is based on the minimization of a functional associated with the given problem, which is nothing other than the total energy of the defined physical system.*

Results*: The implementation of the frontier method in its way of operating reduces the number of systems of equations to be solved, so it is not required to use too much memory in its development.*

Conclusion:*The frontier element method when only 4 elements are used having already the discretized formulation, several domain decomposition techniques can be proposed, which are more than the original model, one of the advantages of this numerical method that reduces the number of equations to solve, so the solution can also be found anywhere within the model.*

Key Word*:Frontier Element Method, Laplace Equation, Poisson Equation, finite element methods*

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I. Introduction

The method arose in the late 1970s. To expand the history of the method, see the article by Alexander H.D. Chenga, Daisy T. Cheng (2005)[1], this numerical technique known as the frontier finite element method, has come to replace the finite difference and finite element methods; beginning the development of this in (1941) its historical development can be considered in the article by Wing Kam Liu, Shaofan Li & Harold S. Park (2022)[2], because they consider too many elements to bring it closer to the model to work, so the number of equations to solve is greater, which requires more time to obtain the solution of the problem. These three methods are currently used to solve partial differential equations in those cases in which their analytical solution cannot be obtained due to their complexity. The finite element is a method that is based on the minimization of a functional associated with the given problem, which is nothing other than the total energy of the defined physical system. These two methods give specific results, providing a disadvantage with respect to the frontier element that can be calculated at any point in the model.

II. Mathematical approach

Suppose that a point charge is located at a point punto $P = (\xi, \eta)$, $Q = (\xi, \eta, \zeta)$ en $\Omega \subset \Re^2$ o $\Re^3$as the case may be. For Laplace's equation for $\Omega \subset \Re^2$. That is, solve the equation (1)

$$\nabla^2 \omega = 0 \text{in } \Omega \qquad (1)$$

where $\omega(x, y)$ has a singularity in $((\xi, \eta) \in \Omega$ in this way the equation becomes equation (2)

$$\nabla^2 \omega + \delta(\xi - x, \eta - y) = 0 \text{in } \Omega \qquad (2)$$

is satisfied by the function $\omega(x, y)$whose solution is the following:Since $\delta = 0$ for$\xi \neq x$, $\eta \neq y$. Let$\bar{D}_\varepsilon$ be a closed disk centered on$(\xi, \eta)$of radiusε, taking the polar coordinates $x = r\cos\theta$, $y = r sen\theta$the equation in$\bar{D}_\varepsilon$ remains

$$\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \omega}{\partial \theta^2} = -\delta(\xi - r\cos\theta, \eta - r sen\theta) \qquad (3)$$

but as $\delta = 0$ si $\xi \neq x, \eta \neq y$.

$$r^2 \frac{\partial^2 \omega}{\partial r^2} + r\frac{\partial \omega}{\partial r} + \frac{\partial^2 \omega}{\partial \theta^2} = 0 \text{in } D_\varepsilon \backslash P \qquad (4)$$

As the solution of this equation depends only on *r* and not on θ, since ω is constant with respect to θ.

$$r^2 \frac{\partial^2 \omega}{\partial r^2} + r\frac{\partial \omega}{\partial r} = 0$$
$$\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r} = 0 \qquad (5)$$

Integrating it, we have $\omega(r) = -\frac{1}{2\pi}\log r = \frac{1}{2\pi}\log\left(\frac{1}{r}\right)$recognized as the function with a logarithmic singularity in (ξ, η). In this way the solution of $\nabla^2 \omega + \delta(\xi - x, \eta - y) = 0 \text{in } \Omega$is:

$$\omega(r) = \frac{1}{2\pi}\log r \text{ con } r = \sqrt{(\xi - x)^2 + (\eta - y)^2} \qquad (6)$$

Now be the equation

$$\nabla^2 u = 0 \text{in } \Omega \qquad (7)$$

under

$$u|_{\Gamma_1} = g_1(s); \nabla u \cdot \hat{n}|_{\Gamma_2} = g_2(s) \qquad (8)$$

To solve it, the function $\omega(r)$ is used as a weight function.

$$\omega \nabla^2 u = \nabla \omega \cdot \nabla u - \nabla \cdot (\omega \nabla u)$$

$$u \nabla^2 \omega = \nabla u \cdot \nabla \omega - \nabla \cdot (u \nabla \omega)$$

as $\int_\Omega u \nabla^2 \omega d\Omega = -u(P)$and

$$\nabla u \cdot \nabla \omega = u \nabla^2 \omega + \nabla \cdot (u \nabla \omega)$$

$$\omega \nabla^2 u = u \nabla^2 \omega + \nabla \cdot (u \nabla \omega) - \nabla \cdot (\omega \nabla u)$$

$$0 = \int_\Omega \omega \nabla^2 u d\Omega = \int_\Omega u \nabla^2 \omega d\Omega + \int_\Omega \nabla \cdot (u \nabla \omega) d\Omega - \int_\Omega \nabla \cdot (\omega \nabla u) d\Omega$$

$$0 = -u(P) + \int_{\partial\Omega} u\left(\frac{\partial \omega}{\partial \hat{\eta}}\right) dS - \int_{\partial\Omega} \omega\left(\frac{\partial u}{\partial \hat{\eta}}\right) dS$$

$$u(P) + \int_{\partial\Omega} \omega \left(\frac{\partial u}{\partial \hat{\eta}}\right) dS = \int_{\partial\Omega} u \left(\frac{\partial \omega}{\partial \hat{\eta}}\right) dS \quad (9)$$

Frontier Integral Equation

Here $P = (\xi, \eta) \in \Omega$. There are three possible cases that the singularity is inside, frontier or outside $\bar{\Omega}$. The case in which $P \in \partial\Omega$ will be addressed, and the frontier integral equation will be solved. For this we take a semi-disc $\bar{D}(P, \varepsilon)$with center at P and radius ε and we extend the region Ω to $\Omega' = \Omega \cup \bar{D}(P, \varepsilon)$ in such a way that the frontier now of Ω'' is $\partial\Omega' = \Gamma_{-\varepsilon} \cup \Gamma_{\varepsilon}$. The frontier$\Gamma_\varepsilon \bar{D}(P, \varepsilon)$has length $\pi\varepsilon$, $P \in \Omega'$ therefore the IFE is valid as shown in Figure 1.

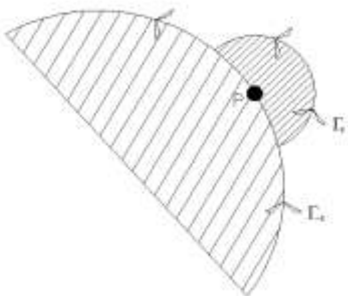

Figure 1: Semi-disc $\bar{D}(P, \varepsilon)$ with center at P and radius ε

$$u(P) + \int_{\partial\Omega'} u \left(\frac{\partial \omega}{\partial \hat{\eta}}\right) dS = \int_{\partial\Omega'} \omega \left(\frac{\partial u}{\partial \hat{\eta}}\right) dS \quad (10)$$

or

$$u(P) + \int_{\Gamma_{-\varepsilon} \cup \Gamma_\varepsilon} u \left(\frac{\partial \omega}{\partial \hat{\eta}}\right) dS = \int_{\Gamma_{-\varepsilon} \cup \Gamma_\varepsilon} \omega \left(\frac{\partial u}{\partial \hat{\eta}}\right) dS \quad (11)$$

$$u(P) + \int_{\Gamma_{-\varepsilon}} u \left(\frac{\partial \omega}{\partial \hat{\eta}}\right) dS + \int_{\Gamma_\varepsilon} u \left(\frac{\partial \omega}{\partial \hat{\eta}}\right) dS = \int_{\Gamma_{-\varepsilon}} \omega \left(\frac{\partial u}{\partial \hat{\eta}}\right) dS + \int_{\Gamma_\varepsilon} \omega \left(\frac{\partial u}{\partial \hat{\eta}}\right) dS \quad (12)$$

Let's calculate the integrals over Γ_ε

$$\int_{\Gamma_\varepsilon} u \left(\frac{\partial \omega}{\partial \hat{\eta}}\right) dS = \int_{\Gamma_\varepsilon} u \left(\frac{\partial \omega}{\partial r}\right) dS = -\frac{1}{2\pi} \int_{\Gamma_\varepsilon} u \left(\frac{1}{r}\right) dS = -\frac{1}{2\pi\varepsilon} \int_{\Gamma_\varepsilon} u dS = -\frac{1}{2\pi\varepsilon} \tilde{u} \pi\varepsilon = -\frac{u}{2}(P) \quad (13)$$

in relation to the integral equation (14)

$$\int_{\Gamma_\varepsilon} u \left(\frac{\partial \omega}{\partial \hat{\eta}}\right) dS \quad (14)$$

is obtained:

$$-\int_{\Gamma_\varepsilon} \frac{1}{2\pi} log\, r \left(\frac{\partial u}{\partial \hat{\eta}}\right) dS = -\frac{log\, \varepsilon}{2\pi} \int_{\Gamma_\varepsilon} \left(\frac{\partial u}{\partial \hat{\eta}}\right) dS = -\frac{log\, \varepsilon}{2\pi} \left(\frac{\partial \tilde{u}}{\partial \hat{\eta}}\right) \pi\varepsilon = -\frac{1}{2} \varepsilon\, log\, \varepsilon \left(\frac{\partial \tilde{u}}{\partial \hat{\eta}}\right) \quad (15)$$

making $\varepsilon \to 0$, you get:

$$\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} u \left(\frac{\partial \omega}{\partial \hat{\eta}}\right) dS = -\frac{u(P)}{2} \quad (16)$$

$$\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} \omega \left(\frac{\partial u}{\partial \hat{\eta}}\right) dS = 0 \quad (17)$$

with respect to integrals over $\Gamma_{-\varepsilon}$,

$$\lim_{\varepsilon \to 0} \int_{\Gamma_{-\varepsilon}} u \left(\frac{\partial \omega}{\partial \hat{\eta}}\right) dS = \int_{\partial\Omega} u \left(\frac{\partial \omega}{\partial \hat{\eta}}\right) dS \quad (18)$$

$$\lim_{\varepsilon \to 0} \int_{\Gamma_{-\varepsilon}} \omega \left(\frac{\partial u}{\partial \hat{\eta}}\right) dS = \int_{\partial\Omega} \omega \left(\frac{\partial u}{\partial \hat{\eta}}\right) dS \quad (19)$$

in this way the frontier integral equation is given by equation (20):

$$u(P) + \int_{\partial\Omega} u \left(\frac{\partial \omega}{\partial \hat{\eta}}\right) dS - \frac{1}{2} u(P) = \int_{\partial\Omega} \omega \left(\frac{\partial u}{\partial \hat{\eta}}\right) dS \quad (20)$$

$$\frac{u(P)}{2} + \int_{\partial\Omega} u\left(\frac{\partial\omega}{\partial\hat{\eta}}\right) dS = \int_{\partial\Omega} \omega\left(\frac{\partial u}{\partial\hat{\eta}}\right) dS \qquad (21)$$

This integral frontier equation (21) is valid as long as the singularity is at a smooth point of $\partial\Omega$; that is to say that there exists the tangent to $\partial\Omega$ in P to the smooth curve γ.

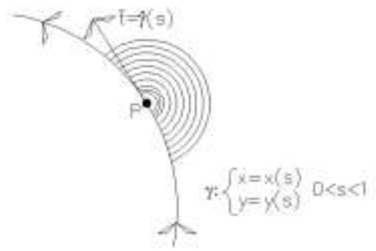

Figure 2: The tangent to $\partial\Omega$ in P to the smooth curve γ

This because as $\varepsilon \to 0$, the semidisc $\bar{D}(P, \varepsilon)$ tends to the point P and it is closing for both sides of P, before and after P. $d\gamma = \gamma(s)ds$. If $\gamma(s)$ is a parameterization of the frontier $\partial\Omega$, that it contains to $P = \gamma(s_0)$.

$$\int_{\gamma} u\left(\frac{\partial\omega}{\partial\hat{\eta}}\right) d\gamma = \int_{\gamma} u\left(\frac{\partial\omega}{\partial\hat{\eta}}\right) \dot{\gamma} ds = \int_{\gamma} u\left(\frac{\partial\omega}{\partial\hat{\eta}}\right) \hat{t}(s) ds \quad (22)$$

If the opposite occurs; that is to say that in the point $P = (\xi, \eta)$ it does not admit the tangent as it is the case of a corner or a peak.

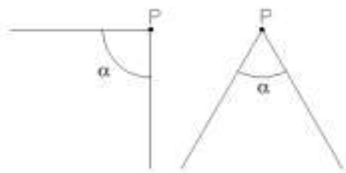

Figure 3: The cases of a corner or a peak

then the coefficient $C(P)u(P)$ must be replaced by $\frac{\alpha}{2\pi}$ the portion in radians of the angle internal to the region Ω. So, the frontier integral equation is:

$$C(P)u(P) + \int_{\Gamma} u\left(\frac{\partial\omega}{\partial\hat{\eta}}\right) dS = \int_{\Gamma} \omega\left(\frac{\partial u}{\partial\hat{\eta}}\right) dS \qquad (23)$$

$$C(P) = \begin{cases} 1 \; If \; (\xi, \eta) \in \Omega \\ \frac{1}{2} If \; (\xi, \eta) \in \Omega \text{ and tangent in } (\xi, \eta) \\ \frac{\alpha}{2\pi} \; If \; (\xi, \eta) \text{ corner or peak y } \alpha\angle \text{ internal in } \Omega \end{cases} \qquad (24)$$

Discrete Form of the Frontier Element

Given the region $\bar{\Omega} = \Omega \cup \partial\Omega$, by defining a partition$\wp(\bar{\Omega}) = \{\Delta^{(1)}, \ldots, \Delta^{(E)}\}$. $\tilde{\Omega} = \overset{e=1}{\underset{E}{U}} \Delta^{(e)} \approx \bar{\Omega}$ and it is essential to define it $\partial\tilde{\Omega} = \overset{f=1}{\underset{F}{U}} \Delta^{(f)}$; (not necessary in Ω) $F \leq E$the frontier of$\tilde{\Omega}$, $\partial\tilde{\Omega} \approx \partial\Omega$. Let $\tilde{C}$ be a polygonal curve defined in $\partial\Omega$, the points $P_{j,j+1} \in \partial\Omega$, $1 \leq j \leq N$where C_j is the edges or chords of the polygonal $1 \leq j \leq N$.

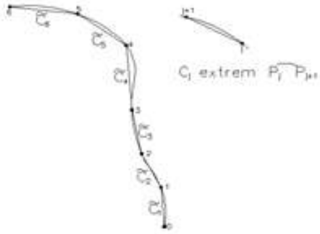

Figure 4: Curve defined in $\partial\Omega$, the points $P_{j,j+1} \in \partial\Omega$, $1 \leq j \leq N$

The chords $\tilde{C}_j$ will be called frontier elements and extreme nodes will be those that divide the portions of the border in which there are different frontier conditions, for example

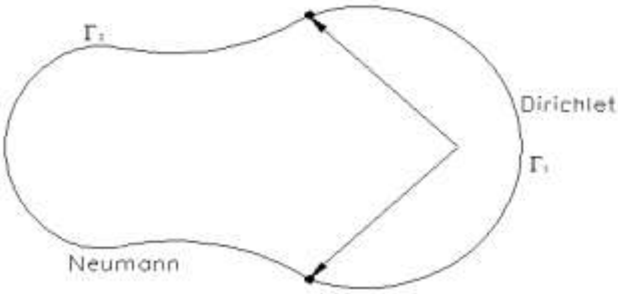

Figure 5: There are different frontier conditions

The nodes are the points found in the middle of a chord and an extreme point in relation to the points where the values of u such as $\frac{\partial u}{\partial \hat{\eta}}$are known or unknown are called *frontier nodes*, thus in this way, the chords $\tilde{C}_j$ may be constant, linear, quadratic, cubic elements, etc.

Figure 6: Nodes are the points found in the middle of a chord

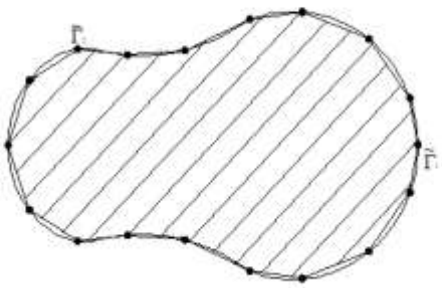

Figure 7: An extreme point in relation to the points

The constant frontier elements have a middle node, that is, if the values of u or $\frac{\partial u}{\partial \hat{\eta}}$are constant along the border, the chords $\tilde{C}_j$ that are in $\tilde{\Gamma}_1$ or in $\tilde{\Gamma}_2$can take the midpoint as the singularity $(\xi, \eta) = P_i$, in such a way that an equation can be established for each coefficient of $U(P_i)$ or in simplified form $U(i)$; $1 \leq i \leq N$, N_1 = number of chords in which the value u is known and N_2 = number of chords in which $u/\tilde{\Gamma}_1 = u_0$is

$$\left.\frac{\partial u}{\partial \hat{n}}\right|_{C_j} = q_0;\ \omega|\tilde{\Gamma}_{1,2} = u*,\ \nabla\omega \cdot \hat{\eta} = \left.\frac{\partial \omega}{\partial \hat{\eta}}\right| \tilde{\Gamma}_{1,2} = q*, \qquad (25)$$

in this way, the frontier integral equation for the singularity $(\xi,\eta) \in \partial\tilde{\Omega}$, with tangent defined in P will be:

$$\frac{U(P)}{2} + \int_{\partial\tilde{\Omega}} u\left(\frac{\partial\omega}{\partial\hat{\eta}}\right) d\tilde{S} = \int_{\partial\tilde{\Omega}} \omega\left(\frac{\partial u}{\partial\hat{\eta}}\right) d\tilde{S} \qquad (26)$$

which will be established for $\partial\tilde{\Omega} = \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2$

$$\frac{U(P)}{2} + \int_{\Gamma_1} u\left(\frac{\partial\omega}{\partial\hat{\eta}}\right) d\tilde{S} + \int_{\Gamma_2} u\left(\frac{\partial\omega}{\partial\hat{\eta}}\right) d\tilde{S} = \int_{\Gamma_1} \omega\left(\frac{\partial u}{\partial\hat{\eta}}\right) d\tilde{S} + \int_{\Gamma_2} \omega\left(\frac{\partial u}{\partial\hat{\eta}}\right) d\tilde{S} \qquad (27)$$

If N_1 represents the number of chords $\tilde{C}_j$ in which we have the frontier condition of the type $\tilde{\Gamma}_1$and N_2 that of the type $\tilde{\Gamma}_2$, $u/\tilde{\Gamma}_1 = u_0$; $u/\tilde{\Gamma}_2 = u$; $\nabla u \cdot \hat{\eta}|\tilde{\Gamma}_1 = q$; $\nabla u \cdot \hat{\eta}|\tilde{\Gamma}_2 = q_0$; $\omega|\tilde{\Gamma}_{1,2} = u*$, $\nabla\omega \cdot \hat{\eta}|\tilde{\Gamma}_{1,2} = q*$. We have

$$\int_{\Gamma_1} u\left(\frac{\partial\omega}{\partial\hat{\eta}}\right) d\tilde{S} = \sum_{j=1}^{N_1} \int_{\tilde{C}_j} u_0 q * d\tilde{S} \qquad (28)$$

$$\int_{\Gamma_2} u\left(\frac{\partial\omega}{\partial\hat{\eta}}\right) d\tilde{S} = \sum_{j=1}^{N_2} \int_{\tilde{C}_j} uq * d\tilde{S} \qquad (29)$$

$$\int_{\Gamma_1} \omega\left(\frac{\partial u}{\partial\hat{\eta}}\right) d\tilde{S} = \sum_{j=1}^{N_1} \int_{\tilde{C}_j} u * qd\tilde{S} \qquad (30)$$

$$\int_{\Gamma_2} \omega\left(\frac{\partial u}{\partial\hat{\eta}}\right) d\tilde{S} = \sum_{j=1}^{N_2} \int_{\tilde{C}_j} u * q_0 d\tilde{S} \qquad (31)$$

then the frontier integral equation for each *i* taken at each midnode of each chord $\tilde{C}_j$ will be

$$\frac{U(P)}{2} + \sum_{j=1}^{N_1} \int_{\tilde{C}_j} u_0 q * d\tilde{S} + \sum_{j=1}^{N_2} \int_{\tilde{C}_j} uq * d\tilde{S} = \sum_{j=1}^{N_1} \int_{\tilde{C}_j} u * qd\tilde{S} + \sum_{j=1}^{N_2} \int_{\tilde{C}_j} u * q_0 d\tilde{S} \qquad (32)$$

$1 \le i \le N$; $1 \le j \le N$ will represent a linear system of equations with *N* unknowns in the values of *u* and $\partial\frac{\partial u}{\partial\hat{\eta}}$ in those portions in which their values are unknown, for example $u/\tilde{\Gamma}_1 = u_0$are known, but u$u/\tilde{\Gamma}_2 = u$are unknown, the same for$\frac{\partial u}{\partial\hat{\eta}}\Big|\,\tilde{\Gamma}_2 = q_0$known, but for$\frac{\partial u}{\partial\hat{\eta}}\Big|\,\tilde{\Gamma}_1 = q$are unknown.

This system can be abbreviated as follows.

$$\frac{U(P)}{2} + \sum_{j=1}^{N} u_j \int_{\tilde{C}_j} q * d\tilde{S} = \sum_{j=1}^{N} q_j \int_{\tilde{C}_j} u * d\tilde{S} \qquad (33)$$

and defining $\hat{H}_{ij} = \int_{\tilde{C}_j} q * d\tilde{S}$; $G_{ij} = \int_{\tilde{C}_j} u * d\tilde{S}$,the previous equation mounted for every *i* in each chord, $\tilde{C}_j$, $1 \le i \le N$; $1 \le j \le N$, it will be had

$$\frac{U(i)}{2} + \sum_{j=1}^{N} u_j \cdot \hat{H}_{ij} = \sum_{j=1}^{N} q_j \cdot G_{ij} \qquad (34)$$

$$\sum_{j=1}^{N} H_{ij} \cdot u_j = \sum_{j=1}^{N} G_{ij} \cdot q_j \qquad (35)$$

where

$$H_{ij} = \begin{cases} \hat{H}_{ij} \; si \; i \neq j \\ \hat{H}_{ij} + \frac{1}{2} \; si \; i = j \end{cases} \qquad (36)$$

In this way the system will be

$$HU = GQ \qquad (37)$$

$$N = N_1 + N_2$$

where N_1 values of u and N_2 values of $\frac{\partial u}{\partial \hat{\eta}}$ are known, but N_1 values of $\frac{\partial u}{\partial \hat{\eta}}$ and N_2 values of u are unknown, or unknowns to be determined. Once these values of both u and $\frac{\partial u}{\partial \hat{\eta}}$ have been determined on the entire frontier $\partial\tilde{\Omega} = \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2$. The above matrix equation can be rearranged by passing all the unknown values N_1, N_2 of both u and $\frac{\partial u}{\partial \hat{\eta}}$ in $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ respectively in such a way that we have:

$AX = F$ (38)

where now X is the vector of unknowns of u and $\frac{\partial u}{\partial \hat{\eta}}$ in $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$ respectively. Once the unknowns $u, \frac{\partial u}{\partial \hat{\eta}}$ have been calculated, we have all the values of u, $\frac{\partial u}{\partial \hat{\eta}}$ in $\partial\tilde{\Omega}$, and from here we can proceed to calculate the values of u for each interior point of the *network* $\wp(\tilde{\Omega}) *$ de $\tilde{\Omega}$, as follows.

$u(i) = \sum_{j=1}^{N} q_j \cdot G_{ij} - \sum_{j=1}^{N} u_j \cdot \hat{H}_{ij}$ (39)

for each node of the *network*. In relation to the internal flow $q_x = \frac{\partial u}{\partial x}$; $q_y = \frac{\partial u}{\partial y}$ or gradient ∇u is obtained by deriving the expression $u(i)$, $\frac{\partial u(i)}{\partial x}$, $\frac{\partial u(i)}{\partial y}$ from the equation

$$u(i) = \int_{\partial\tilde{\Omega}} \omega \left(\frac{\partial u}{\partial \hat{\eta}}\right) d\tilde{S} - \int_{\partial\tilde{\Omega}} u \left(\frac{\partial \omega}{\partial \hat{\eta}}\right) d\tilde{S} = \int_{\tilde{\Gamma}_1 \cup \tilde{\Gamma}_2} \omega \left(\frac{\partial u}{\partial \hat{\eta}}\right) d\tilde{S} - \int_{\tilde{\Gamma}_1 \cup \tilde{\Gamma}_2} u \left(\frac{\partial \omega}{\partial \hat{\eta}}\right) d\tilde{S}$$

$$u(i) = \int_{\tilde{\Gamma}_1} \omega \left(\frac{\partial u}{\partial \hat{\eta}}\right) d\tilde{S} + \int_{\tilde{\Gamma}_2} \omega \left(\frac{\partial u}{\partial \hat{\eta}}\right) d\tilde{S} - \int_{\tilde{\Gamma}_1} u \left(\frac{\partial \omega}{\partial \hat{\eta}}\right) d\tilde{S} - \int_{\tilde{\Gamma}_2} u \left(\frac{\partial \omega}{\partial \hat{\eta}}\right) d\tilde{S}$$

$u(i) = \int_{\tilde{\Gamma}_1} u * q d\tilde{S} + \int_{\tilde{\Gamma}_2} u * q_0 d\tilde{S} - \int_{\tilde{\Gamma}_1} u_0 q * d\tilde{S} - \int_{\tilde{\Gamma}_2} uq * d\tilde{S}$ (40)

$q_x(i) = \int_{\partial\tilde{\Omega}} q \frac{\partial u*}{\partial x} d\tilde{S} - \int_{\partial\tilde{\Omega}} u \frac{\partial q*}{\partial x} d\tilde{S} = \sum_{j=1}^{N} q_j \left[\int_{\tilde{C}_j} \left(\frac{\partial u*}{\partial x}\right) d\tilde{S}\right] - \sum_{j=1}^{N} u_j \left[\int_{\tilde{C}_j} \left(\frac{\partial q*}{\partial x}\right) d\tilde{S}\right]$ (41)

$q_y(i) = \int_{\partial\tilde{\Omega}} q \frac{\partial u*}{\partial y} d\tilde{S} - \int_{\partial\tilde{\Omega}} u \frac{\partial q*}{\partial y} d\tilde{S} = \sum_{j=1}^{N} q_j \left[\int_{\tilde{C}_j} \left(\frac{\partial u*}{\partial y}\right) d\tilde{S}\right] - \sum_{j=1}^{N} u_j \left[\int_{\tilde{C}_j} \left(\frac{\partial q*}{\partial y}\right) d\tilde{S}\right]$ (42)

$\frac{\partial u*}{\partial x} = \frac{\partial \omega}{\partial x} = \frac{1}{2\pi} \frac{\partial}{\partial x} \left[log\left(\frac{1}{r}\right)\right] = -\frac{1}{2\pi r} \frac{\partial r}{\partial x}$ (43)

$\frac{\partial u*}{\partial y} = \frac{\partial \omega}{\partial y} = \frac{1}{2\pi} \frac{\partial}{\partial y} \left[log\left(\frac{1}{r}\right)\right] = -\frac{1}{2\pi r} \frac{\partial r}{\partial y}$ (44)

$\frac{\partial q*}{\partial x} = \frac{\partial}{\partial x} (\nabla\omega \cdot \hat{\eta}) = \frac{\partial}{\partial x} \left[\frac{\partial \omega}{\partial r} \frac{\partial r}{\partial x} cos\,\theta_1 + \frac{\partial \omega}{\partial r} \frac{\partial r}{\partial y} cos\,\theta_2\right] = \frac{1}{2\pi} \frac{\partial}{\partial x} \left[\frac{1}{r}\left(\frac{\partial r}{\partial x}\eta_1 + \frac{\partial r}{\partial y}\eta_2\right)\right]$ (45)

$\frac{\partial q*}{\partial y} = \frac{\partial}{\partial y} (\nabla\omega \cdot \hat{\eta}) = \frac{\partial}{\partial y} \left[\frac{\partial \omega}{\partial r} \frac{\partial r}{\partial x} cos\,\theta_1 + \frac{\partial \omega}{\partial r} \frac{\partial r}{\partial y} cos\,\theta_2\right] = \frac{1}{2\pi} \frac{\partial}{\partial y} \left[\frac{1}{r}\left(\frac{\partial r}{\partial x}\eta_1 + \frac{\partial r}{\partial y}\eta_2\right)\right]$ (46)

The integrations in the expression are calculated by *Gaussian* quadrature's $q_x(i)$, $q_y(i)$

$\int_{\tilde{C}_j} \left(\frac{\partial u*}{\partial x}\right) d\tilde{S}$, $\int_{\tilde{C}_j} \left(\frac{\partial q*}{\partial x}\right) d\tilde{S}$, $\int_{\tilde{C}_j} \left(\frac{\partial u*}{\partial y}\right) d\tilde{S}$, $\int_{\tilde{C}_j} \left(\frac{\partial q*}{\partial y}\right) d\tilde{S}$ (47)

the same for $\hat{H}_{ij}$, G_{ij}, $i \neq j$, $\hat{H}_{ij} = 0$, $\bar{r} \perp \hat{\eta}$.

Now let $L_i = \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$ be the length of element i, $r = \frac{\xi L_i}{2}$ where ξ is the coordinate natural.

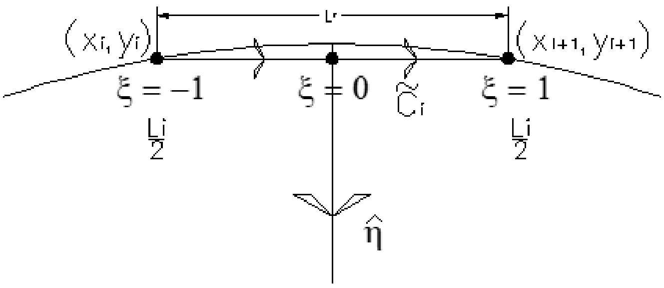

Figure 8: The length of element *i*

$G_{ii} = \frac{1}{2\pi}\int_{\tilde{C}_j} ln\left(\frac{1}{r}\right) dr = \frac{2}{2\pi}\int_0^1 ln\left(\frac{2}{\xi L_i}\right)\frac{L_i}{2} d\xi = \frac{L_i}{2\pi}\int_0^1 ln\left(\frac{2}{\xi L_i}\right) d\xi$ (48)

From the integration of
$\int ln(x)\, dx = x\, ln\, x - x$ (49)
applying it to

$ln\frac{2}{\xi L_i} = ln(2) - ln(\xi L_i)$ (50)

$$\int_0^1 ln\left(\frac{2}{\xi L_i}\right) d\xi = \int_0^1 ln(2)\, d\xi - \int_0^1 ln(\xi L_i)\frac{d\xi}{L_i} = ln(2) - \left[\frac{(\xi L_i)\, ln(\xi L_i)}{L_i} - \frac{(\xi L_i)}{L_i}\right]\Bigg|_0^1$$

$= ln(2) - [\xi\, ln(\xi L_i) - \xi]|_0^1 = ln(2) - ln(L_i) + 1 = ln\left(\frac{2}{L_i}\right) + 1$(51)

$\therefore G_{ii} = \frac{L_i}{2\pi}\int_0^1 ln\left(\frac{2}{\xi L_i}\right) d\xi = \frac{L_i}{2\pi}\left[ln\left(\frac{2}{L_i}\right) + 1\right]$ (52)

In the frontier element method, it has been seen that once the frontier integral equation is obtained for each of the strings where the singularity $u(i)$ has been assumed, a linear system of equations is established to determine the values of u and of$\frac{\partial u}{\partial \hat{\eta}}$in those chords $\tilde{C}_j$ that form the polygonal boundary $\partial\tilde{\Omega}$, $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$respectively, $\frac{\partial u}{\partial \hat{\eta}}\Big|\tilde{\Gamma}_1 = q$; $u|\tilde{\Gamma}_1 = u$ where the *Green* function $\omega(r) = \frac{1}{2\pi}log\left(\frac{1}{r}\right)$ is evaluated as $\frac{\partial \omega}{\partial \hat{\eta}}$on each chord $\tilde{C}_j$obtaining a linear equation for each node position $i \in \tilde{C}_j$ (midpoint of $\tilde{C}_j$).

$C(P)U(i) + \sum_{j=1}^{N}\int_{\tilde{C}_j} u\frac{\partial \omega}{\partial \hat{\eta}} d\tilde{S} = \sum_{j=1}^{N}\int_{\tilde{C}_j} \omega\frac{\partial u}{\partial \hat{\eta}} d\tilde{S}$ (53)

$1 \le i \le N;\ 1 \le j \le N,\ C(P) = \frac{1}{2}.$

or explicitly

$\frac{U(i)}{2} + \int_{\tilde{\Gamma}_1} u_0 q * d\tilde{S} + \int_{\tilde{\Gamma}_2} uq * d\tilde{S} = \int_{\tilde{\Gamma}_1} u * qd\tilde{S} + \int_{\tilde{\Gamma}_2} u * q_0 d\tilde{S}$ (54)

$\frac{U(i)}{2} + \sum_{j=1}^{N_1}\int_{\tilde{C}_j} u_0 q * d\tilde{S} + \sum_{j=1}^{N_2}\int_{\tilde{C}_j} uq * d\tilde{S} = \sum_{j=1}^{N_1}\int_{\tilde{C}_j} u * qd\tilde{S} + \sum_{j=1}^{N_2}\int_{\tilde{C}_j} u * q_0 d\tilde{S}$(55)

or in abbreviated form
$\frac{U(i)}{2} + \sum_{j=1}^{N_1} u_j \int_{\tilde{C}_j} q * d\tilde{S} = \sum_{j=1}^{N_1} q_j \int_{\tilde{C}_j} u * d\tilde{S}$ (56)

$1 \le i \le N$; $1 \le j \le N$, where u_j, $q_j$are partly unknown and the values of $\int_{\tilde{C}_j} q * d\tilde{S}$, as of $\int_{\tilde{C}_j} u * d\tilde{S}$are given by the values of the function $\omega(r)$, $\frac{\partial \omega}{\partial \hat{\eta}}$in each chord $\tilde{C}_j$ for each position of *i* at the midpoint of each $\tilde{C}_j$. Here the

u_j, q_j have come out of the integrand from the fact that we have only considered constant frontier elements; that is to say that u, $\frac{\partial u}{\partial \hat{\eta}}$in each chord respectively are constant, but in general it is not so. Now, as in the general case, the integrations are presented as follows

$$\int_{\tilde{C}_j} u * q d\tilde{S}; \int_{\tilde{C}_j} q * u d\tilde{S} \qquad (57)$$

where the functions $\omega/\tilde{C}_j = u *$; $\nabla\omega \cdot \hat{\eta}/\tilde{C}_j = q *$are weight functions for q and *u* respectively, we will give the *Gaussian* quadrature integration formula for the general case.

III. Results and Discussion

The derivation of the method is quite complicated, it is necessary to have knowledge of partial differential equations, frontier conditions of different types, discretization of the domain, and finally, to know how to couple the equations so that it forms a linear system of equations.

IV. Conclusion

The frontier element method when only 4 elements are used having already the discretized formulation, several domain decomposition techniques can be proposed, which are more than the original model, one of the advantages of this numerical method that reduces the number of equations to solve, so the solution can also be found anywhere within the model.

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