IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 19, Issue 3 Ser. 2 (May. – June. 2023), PP 11-15
www.iosrjournals.org

The Metric Properticès Of The Nilpotent Cayley Graph Of The Ring $(Z_n, \oplus, \odot)$

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ABSTRACT
The authors have studied a new class of arithmetic Cayley graphs, namely, the nilpotent Cayley graphs $G(Z_n, N)$ associated with the set N of nilpotent elements in the residue class ring $(Z_n, \oplus, \odot), n \geq 1$, an integer. The metric properties, such as the eccentricity of a vertex, the radius, the diameter, the girth and the circumference of the nilpotent Cayley graph $G(Z_n, N)$ associated with the residue class ring $(Z_n, \oplus, \odot)$ are determined in this paper.

Key Words: *Nilpotent Cayley Graph, Eccentricity, Radius, Diameter, Girth, Circumference*

Date of Submission: 25-05-2023 Date of Acceptance: 05-06-2023

I. INTRODUCTION

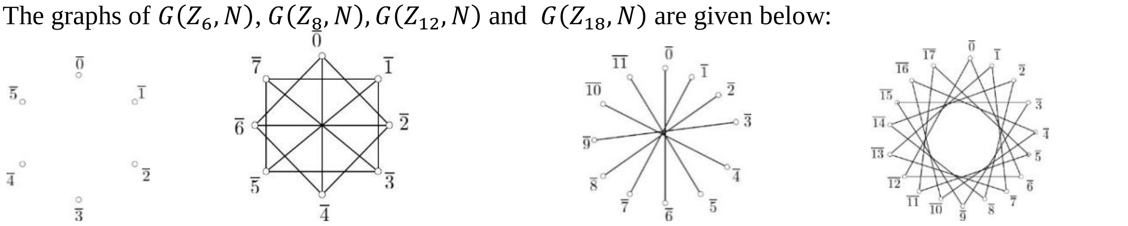
The concept of a Cayley graph was introduced to study, whether given a group $(X,.)$, there is a graph Γ, whose automorphism group is isomorphic to the group $(X,.)$ [11]. Extensive studies have been carried out on the Cayley graphs by many graph theorists [3, 4, 6, 12]. Given a group $(X,.)$ and a symmetric subset S of X, (a subset S of a group $(X,.)$ is called a symmetric subset, if $s^{-1} \in S$ for every $s \in S$) is the graph $G(X,S)$, whose vertex set V is X and the edge set $E = \{(x,y)/\text{either } xy^{-1} \in S, \text{ or } yx^{-1} \in S\}$. If S does not contain e, the identity element of the group $(X,.)$, then $G(X,S)$ is a simple undirected graph. Further $G(X,S)$ is $|S|$-regular and contains $\frac{|X||S|}{2}$ edges [12]. Madhavi [12] introduced Cayley graphs associated with the arithmetical functions, namely, the Euler totient function $\varphi(n)$, the set of quadratic residues modulo a prime p and the divisor function $d(n)$, $n \geq 1$, an integer and obtained various properties of these graphs.

In recent times Chen [7], Nikmehr and Khojasteh [20] and Dhiren Kumar Basnet *et al.*, [8] have studied the nilpotent graphs associated with a finite commutative ring R and the $n \times n$ matrix ring $M_n(R)$. In [18, 19], the authors have introduced a new class of arithmetic Cayley graphs, namely, the nilpotent Cayley graphs associated with the set of nilpotent elements in the residue class ring $(Z_n, \oplus, \odot), n \geq 1$, an integer. An element $\bar{a} \neq \bar{0}$, in the ring $(Z_n, \oplus, \odot)$ is called a nilpotent element, if there exists a positive integer l such that $(\bar{a})^l = \bar{0}$. It is an easy verification that the set N of all nilpotent elements in the ring $(Z_n, \oplus, \odot)$ is a symmetric subset of the group $(Z_n, \oplus)$. The Cayley graph $G(\mathrm{Z}_n, N)$ associated with the group $(Z_n, \oplus)$ and its symmetric subset N, is the graph whose vertex set V is Z_n and the edge set $E = \{(x,y)/x, y \in Z_n$ and either $x - y \in N$, or, $y - x \in N\}$ and it is called the **nilpotent Cayley graph of the ring $(Z_n, \oplus, \odot)$**.

In [19], it is proved that, if $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$, are primes, $\alpha_i \geq 1$ and $1 \leq i \leq r$, are integers and $m = p_1 p_2 p_3 \ldots p_r$, then

i. the graph $G(Z_n, N)$ is $(\prod_{i=1}^{r} p_i^{\alpha_i - 1} - 1) -$ regular and contains $\frac{n}{2}(\prod_{i=1}^{r} p_i^{\alpha_i - 1} - 1)$ edges,
ii. the graph $G(Z_m, N)$ contains only vertices and
iii. the graph $G(\mathrm{Z}_n, N)$ is a union of m disjoint connected components of $G(\mathrm{Z}_n, N)$, each of which is a complete subgraph of $G(\mathrm{Z}_n, N)$.

The graphs of $G(Z_6, N)$, $G(Z_8, N)$, $G(Z_{12}, N)$ and $G(Z_{18}, N)$ are given below:



$G(Z_6, N)$ $G(Z_8, N)$ $G(Z_{12}, N)$ $G(Z_{18}, N)$

Figure 1.1

The terminology and notations that are used in this paper can be found in [5] for graph theory, [10] for algebra and [1] for number theory.

II. THE ECCENTRICITY OF A VERTEX, THE RADIUS AND THE DIAMETER OF THE NILPOTENT GRAPH $G(Z_n, N)$

For any two vertices u, v in a graph G, the **distance** $d(u, v)$ is defined as the length of the shortest path, if any, joining u and v. If there is no path joining the vertices u and v, then it is defined that $d(u, v) = \infty$. The **eccentricity** $e(v)$ of any vertex in G is defined as $e(v) = \max\{d(v, u) : u \in V\}$. The **radius** $r(G)$ and the **diameter** $d(G)$ of G are respectively defined by $r(G) = \min\{e(v) : v \in V\}$ and $d(G) = \max\{e(v) : v \in V\}$.

Example 2.1: Consider the graph G, whose vertex set V and the edge set E are given by $V = \{a, b, c, d, e, f\}$ and $E = \{(a, b), (a, d), (a, e), (b, c), (c, d), (c, f)\}$. The diagram of the the graph G is given below.

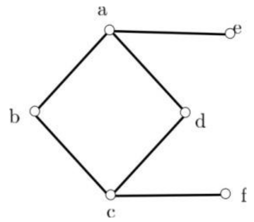


Figure 2.1

The following table gives the values of the distances $d(u, v)$, for all $u, v \in V$, the eccentricities $e(v)$, for all $v \in V$, the diameter $d(G)$ and the radius $r(G)$ of the graph G.

$d(u,v)$	a	b	c	d	e	f
a	0	1	2	1	1	3
b	1	0	1	2	2	2
c	2	1	0	1	3	1
d	1	2	1	0	2	2
e	1	2	3	2	0	3
f	3	2	1	2	4	0
$\boldsymbol{e(v)}$	**3**	**2**	**3**	**2**	**4**	**3**
$\boldsymbol{d(G)}$	$\boldsymbol{max\{3,2,2,2,4,3,4\} = 4}$					
$\boldsymbol{r(G)}$	$\boldsymbol{min\{3,2,2,2,4,3,4\} = 2}$					

Table 2.1

Let us now find the eccentricity of a vertex in the nilpotent Cayley graph $G(Z_n, N)$.

Theorem 2.2: If $n = \prod_{i=1}^{r} {p_i}^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$, are primes and $\alpha_i \geq 1$, $1 \leq i \leq r$ are integers such that $\alpha_i > 1$, for at least one i and if $m = p_1 p_2 \ldots p_r$, then for any vertex v in $G(Z_n, N)$,

i. the eccentricity $e(v)$ is ∞,

ii. the radius $r(G(Z_n, N)) = \infty$ and the diameter $d(G(Z_n, N)) = \infty$.

Proof :

i. Let v be any vertex of $G(Z_n, N)$**.** Then by the Theorem 3.5 of [19]**,** the nilpotent Cayley graph $G(Z_n, N)$ is decomposed into m disjoint complete components $\langle C_0 \rangle, \langle C_1 \rangle, \langle C_2 \rangle, \ldots, \langle C_{m-1} \rangle$, where

$$C_k = \{\overline{k}, \overline{m+k}, \overline{2m+k}, \overline{3m+k}, \ldots, \overline{\imath m+k}, \overline{\jmath m+k}, \ldots, \overline{(n-m)+k}\}.$$

That is, $Z_n = \bigcup_{k=1}^{m-1} C_k$. So $v \in C_i$ for some $i, 0 \leq i \leq m-1$. Let u be any vertex of the graph $G(Z_n, N)$, such that $u \neq v$.

If $u \in C_i$, then the vertices v and u belong to the same component C_i of the graph $G(Z_n, N)$**.** Since each C_i is a complete subgraph of the graph $G(Z_n, N)$, the vertices v and u are adjacent, so that, $d(v, u) = 1$.

If $u \notin C_i$, then $u \in C_j$, for some $j \neq i$, $0 \leq j \leq m-1$. Since $\langle C_i \rangle$ and $\langle C_j \rangle$ are edge disjoint components of the graph $G(Z_n, N)$, there is no edge between the vertices v and u, so that $d(v, u) = \infty$. So,

$e(v) = \max\{1, \infty\} = \infty$.

ii. By part (i), the eccentricity $e(v)$ of a vertex v of the graph $G(Z_n, N)$ is ∞, for every vertex $v \in Z_n$. So, the radius $r(G) = \min\{e(v) : v \in V\} = \infty$ and the diameter $d(G) = \max\{e(v) : v \in V\} = \infty$.

■

III. ENUMERATION OF THE GIRTH AND THE CIRCUMFERENCE OF THE NILPOTENT CAYLEY GRAPH $G(Z_n, N)$

Let G be a graph G with vertex set V and edge set E. The length of the smallest cycle in G is called the **girth** of G and it is denoted by $g(G)$. The length of the largest cycle in G is called the **circumference** of G and it is denoted by $c(G)$. If the graph G has **no cycles** then the terms girth and circumference of G are **undefined**.

Lemma 3.1: If $n = 4$, the girth and the circumference of the nilpotent Cayley graph $G(Z_n, N)$ are undefined.

Proof: If $n = 4$, then the set of N of nilpotent elements of the ring $(Z_4, \oplus, \odot)$ is the singleton set $\{\bar{2}\}$, and the graph $G(Z_4, N)$ is the following bi-partite graph, which has no cycles. The girth and the circumference are undefined. ■

Figure 3.1: $G(Z_4, N)$

Lemma 3.2: If $n = p_1 p_2 \dots p_r$, where $p_1 < p_2 < \cdots < p_r$, are primes and if $n > 4$, then the graph $G(Z_n, N)$ has no cycles, so that the girth and the circumference are undefined.

Proof : If $n = p_1 p_2 \dots p_r$, where $p_1 < p_2 < \cdots < p_r$, are primes, then the set N of nilpotent elements of the ring $(Z_n, \oplus, \odot)$ is empty, so that the edge set of $G(Z_n, N)$ is empty and the graph contains only vertices and but no edges. So the graph $G(Z_n, N)$ has no cycles, so that the girth and the circumference of the graph $G(Z_n, N)$ are undefined.

Lemma 3.3 : If $n = 2^2 p_2 p_3 \dots p_r$, where $2 < p_2 < p_3 < \cdots < p_r$ are primes and if $n > 4$, then the graph $G(Z_n, N)$ has no cycles and the girth and the circumference are undefined.

Proof : Let $n = 2^2 p_2 p_3 \dots p_r$, where $2 < p_2 < p_3 < \cdots < p_r$, are primes. By the Theorem 2.2.12 of [18], the graph $G(Z_n, N)$ is the following bipartite graph, with the bipartition (U, V), where $U = \{\bar{0}, \bar{1}, \dots, \overline{m-1}\}$ and $V = \{\bar{m}, \overline{m+1}, \dots, \overline{n-1}\}$.

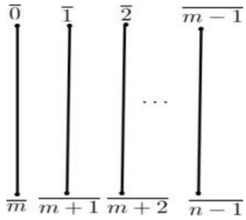


Figure 3.2: $G(Z_n, N)$

Clearly $G(Z_n, N)$ has no cycles, and hence the girth and the circumference of $G(Z_n, N)$ are undefined. ■

In the following theorem, the girth and the circumference of the graph $G(Z_n, N)$, when $n \neq 2^2 p_2 p_3 \dots p_r$, where $p_{2<} p_{3<} \dots < p_r$ are primes are found.

Theorem 3.4 : If $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$, are primes, $\alpha_i \geq 1, 1 \leq i \leq r$ are integers, such that $\alpha_i > 1$ for at least one i and $m = p_1 p_2 p_3 \dots p_r$, then the girth $g(G(Z_n, N))$ is 3 and the circumference $c(G(Z_n, N))$ is $\frac{n}{m}$.

Proof : Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$, are primes, $\alpha_i \geq 1, 1 \leq i \leq r$ are integers, such that $\alpha_i > 1$ for at least one i, and let $m = p_1 p_2 \dots p_r$. Consider the vertices $\bar{0}$, $\bar{m}$, $\overline{2m}$ in $G(Z_n, N)$. Since $\bar{m} - \bar{0} = \bar{m} \in N$, $\overline{2m} - \bar{m} = \bar{m} \in N$, and $\overline{2m} - \bar{0} = \overline{2m} \in N$,
it follows that $(\bar{0}, \bar{m}, \overline{2m}, \bar{0})$ is a cycle of length 3. Clearly, this is a cycle of smallest length, so that $g\big(G(Z_n, N)\big) = 3$.

By the Theorem 3.5 [19], the nilpotent Cayley graph $G(Z_n, N)$ is decomposed into m disjoint components $\langle C_0 \rangle, \langle C_1 \rangle, \dots, \langle C_k \rangle, \dots, \langle C_{m-1} \rangle$, where,

$$C_k = \left\{\bar{k}, \overline{m+k}, \overline{2m+k}, \overline{3m+k}, \dots, \overline{\imath m+k}, \dots, \overline{\jmath m+k}, \dots, \overline{\left(\tfrac{n}{m}-1\right)m+k}\right\}.$$

For $0 \leq k \leq m-1$, clearly each

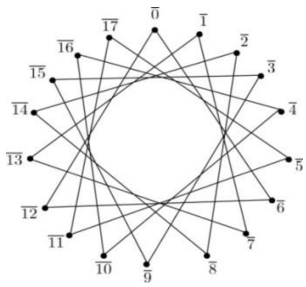
$$C_k = (\bar{k}, \overline{m+k}, \overline{2m+k}, \overline{3m+k}, \dots, \overline{\imath m+k}, \dots, \overline{\jmath m+k}, \dots, \overline{\left(\tfrac{n}{m}-1\right)m+k}, \bar{k}),$$

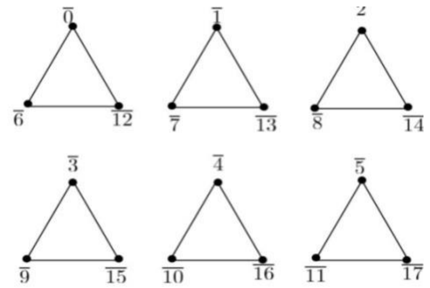
is a cycle of length $\frac{n}{m}$ and it is a cycle of maximum length in $G(Z_n, N)$. Hence the circumference $c\big(G(Z_n, N)\big)$ of $G(Z_n, N)$ is $\frac{n}{m}$. ∎

The following corollary is immediate from the Theorem 3.4.

Corollary 3.5 : If $n = p^r$, where p is a prime and $r > 2$, an integer, then $c\big(G(Z_n, N)\big) = p^{r-1}$.

Example 3.6 : Consider the graph $G(Z_{18}, N)$ together with its components given below.



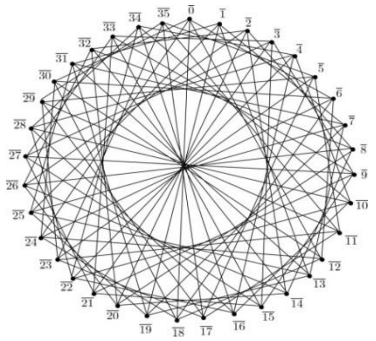


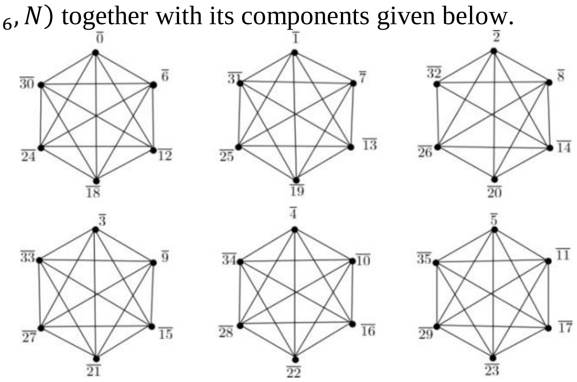
The graph $G(Z_{18}, N)$ **The components in the graph $G(Z_{18}, N)$**

Figure 3.3

Each of the six components of the graph is a triangle, which is a cycle of minimum as well as maximum length 3 in $G(Z_{18}, N)$. So $g\big(G(Z_{18}, N)\big) = c\big(G(Z_{18}, N)\big) = 3$.

Example 3.7: Consider the graph $G(Z_{36}, N)$ together with its components given below.





The graph $G(Z_{36}, N)$ **The components of the graph $G(Z_{36}, N)$**

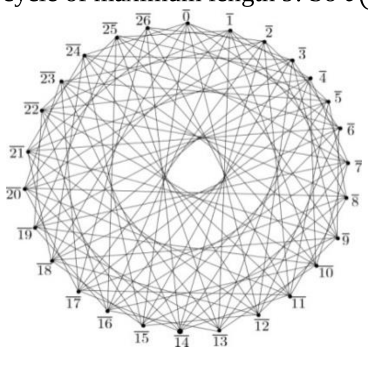
Figure 3.4

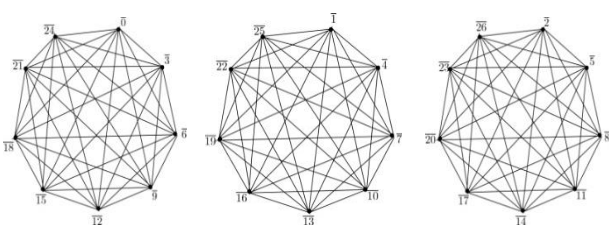
The graph $G(Z_{36}, N)$ has triangles. For example $(\overline{0}, \overline{6}, \overline{12}, \overline{0})$ is a triangle and it has length 3, so that $g\big(G(Z_{36}, N)\big) = 3$.

Further $G(Z_{36}, N)$ has six components $\langle\{\overline{0}, \overline{6}, \overline{12}, \overline{18}, \overline{24}, \overline{30}\}\rangle$, $\langle\{\overline{1}, \overline{7}, \overline{13}, \overline{19}, \overline{25}, \overline{31}\}\rangle$, $\langle\{\overline{2}, \overline{8}, \overline{14}, \overline{20}, \overline{26}, \overline{32}\}\rangle$, $\langle\{\overline{3}, \overline{9}, \overline{15}, \overline{21}, \overline{27}, \overline{33}\}\rangle$, $\langle\{\overline{4}, \overline{10}, \overline{16}, \overline{22}, \overline{28}, \overline{34}\}\rangle$, $\langle\{\overline{5}, \overline{11}, \overline{17}, \overline{23}, \overline{29}, \overline{35}\}\rangle$ and each of the six components of the graph $G(Z_{36}, N)$ gives a cycle of the length 6, which is of maximum length. So $c\big(G(Z_{36}, N)\big) = 6$.

Example 3.8: Consider the graph $G(Z_{27}, N)$. Here $27 = 3^3$. This graph has the triangle $(\overline{0}, \overline{3}, \overline{6}, \overline{0})$, so that $g\big(G(Z_{27}, N)\big) = 3$.

Further $G(Z_{27}, N)$ has the three components $\langle\{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}, \overline{24}, \}\rangle$, $\langle\{\overline{1}, \overline{4}, \overline{7}, \overline{10}, \overline{13}, \overline{16}, \overline{19}, \overline{22}, \overline{25}, \}\rangle$ and $\langle\{\overline{2}, \overline{5}, \overline{8}, \overline{11}, \overline{14}, \overline{17}, \overline{20}, \overline{23}, \overline{26}, \}\rangle$ and each of the three components gives a cycle of maximum length 9. So $c\big(G(Z_{27}, N)\big) = 9$.





The graph $G(Z_{27}, N)$ **The components of the graph $G(Z_{27}, N)$**

Figure 3.5

ACKNOWLEDGMENT

The authors express their thanks to Prof. L. Nagamuni Reddy for his suggestions during the preparation of this paper.

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