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Minimal proof of the four-color theorem

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Abstract
The proof of the four-color theorem is led from its definition to the notion of a "complete triangular network" CTN(n) to a description of its construction to see if it is possible to construct a CTN(n) enforcing the 5th color. A negative answer then allows us to claim that the four-color theorem is thereby proved.[1]
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54C99
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I. Introduction

The proof of the four-color theorem seems to be possible by a reverse procedure from the usual one: instead of constructing all possible graphs and coloring them, we use the notion of color forced to construct graphs that force the maximum number of colors, namely 5. If we fail to do this, we can conclude that it is impossible to construct a graph with a maximum number of enforced colors of 5, while enforcing 4 colors is provable, which implies that the maximum number of enforced colors is 4 and that the four-color theorem holds.

Theory/calculation
We will give the proof of the four-color theorem (T1) in the following formulation [3]:

T1 Four colors are sufficient to color any map so that no two adjacent states are colored the same color.

For the proof, we must first establish the conceptual apparatus.

A map consists of colored fields that are touching (must be linear touching, not point touching [4]) or not touching. We can convert the map into a graph, where the individual fields will be represented by nodes, and the touches will be the edges connecting the corresponding nodes, which will be assigned colors from the m-color set (m) = {x_1, x_2, x_3, x_4, ..., x_m}, where x_1, x_2, x_3, x_4, ..., x_m are the specific colors A, B, C, D, ... X. The nodes will be numbered sequentially with the natural numbers 1, 2, ..., n, where n is the number of nodes in the graph. If two nodes r, s are connected by an edge, they must be assigned different colors, i.e., if node r is assigned color x, node s is enforced to have color y.

Then we can reformulate the theorem to be proved as follows:

T2 No more than the 4th color is enforced in any map chart.

The number of enforced colors depends on the number of touches, i.e. the number of edges. If we construct a graph with a larger number of edges, it should enforce more colors.

Then we can reformulate the theorem to be proved as follows:

T3 The graph with the maximum number of edges enforces at most 4 colors.

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[1] Abbreviations: CTN – complete triangular network, SC – set of colours, C – circle, F – formation, GTN – general triangular network.

The graph with the maximum number of edges is one in which no more edges can be added. We call it a complete triangular network, CTN(n), where n is the number of nodes contained in it, $n > 2$, and the number of edges in it is $3(n - 2)$. It is made up of so-called "triangles", that is, elementary figures whose 3 vertices are 3 adjacent nodes that are connected by 3 edges (for our purposes, the edges need not be straight lines).

This definition relies on a convention introduced in the construction of CTN(n) (its usefulness will become apparent later when drawing the graph):
1. Draw $n - 1$ nodes in the form of a semicircle and number them sequentially from the left with the natural numbers 2, ..., n. Place a node numbered 1 below the semicircle.
2. We connect the nodes by edges as follows:
3. Connect node 1 to all remaining nodes (in the number $n - 1$).
4. We connect node 2 with node 3. Similarly, we continue with node 3 (connect it with node 4), and so on, until we connect node $n - 1$ with node n. (These edges will be $n - 2$.)
5. We join nodes 2, n. We get a circle C(2-n) with nodes 2, ..., n. We add edges to this circle as long as we can (including the 2–n node, there will be $n - 3$ of them). This gives the formation F(2-n). This problem has multiple solutions. First, we keep the edges starting from node 2, then follow the solutions with starting node 3, and so on, until we exhaust all possibilities. (The number of edges in the resulting CTN(n) will be equal to $(n - 1) + (n - 2) + (n - 3)$, i.e. $3(n - 2)$).

Remark. For a graph of n nodes, the convention ensures their union by a maximum number of edges of $3(n - 2)$. In any case, its purpose is not to construct all possible CTN(n), but only to construct a CTN(n) enforcing the 5th color. Thus, the counterexample illustrating the inappropriateness of the convention is not the presentation of a CTN(n) that is not constructed according to the convention, but the presentation of an "unconventional" CTN(n) that enforces the 5th color.

To determine the color assignment behavior of CTN(n), we construct graphs CTN(3) to CTN(6):

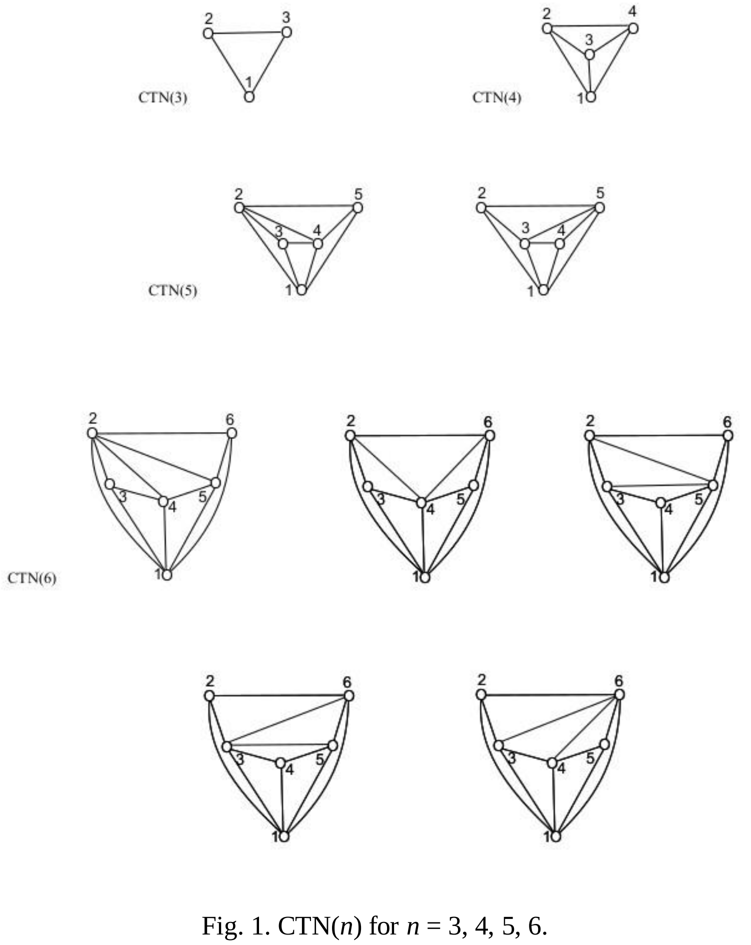

Fig. 1. CTN(n) for n = 3, 4, 5, 6.

We assign colors to the graphs according to the following convention (the meaning of which will also be illuminated in the next section):
We assign color A to node 1, color B to node 2, color C to node *n*, and we assign the remaining nodes the enforced color by assigning the remaining color from SC(3) = {B, C, D}.

The color assignment looks like this:

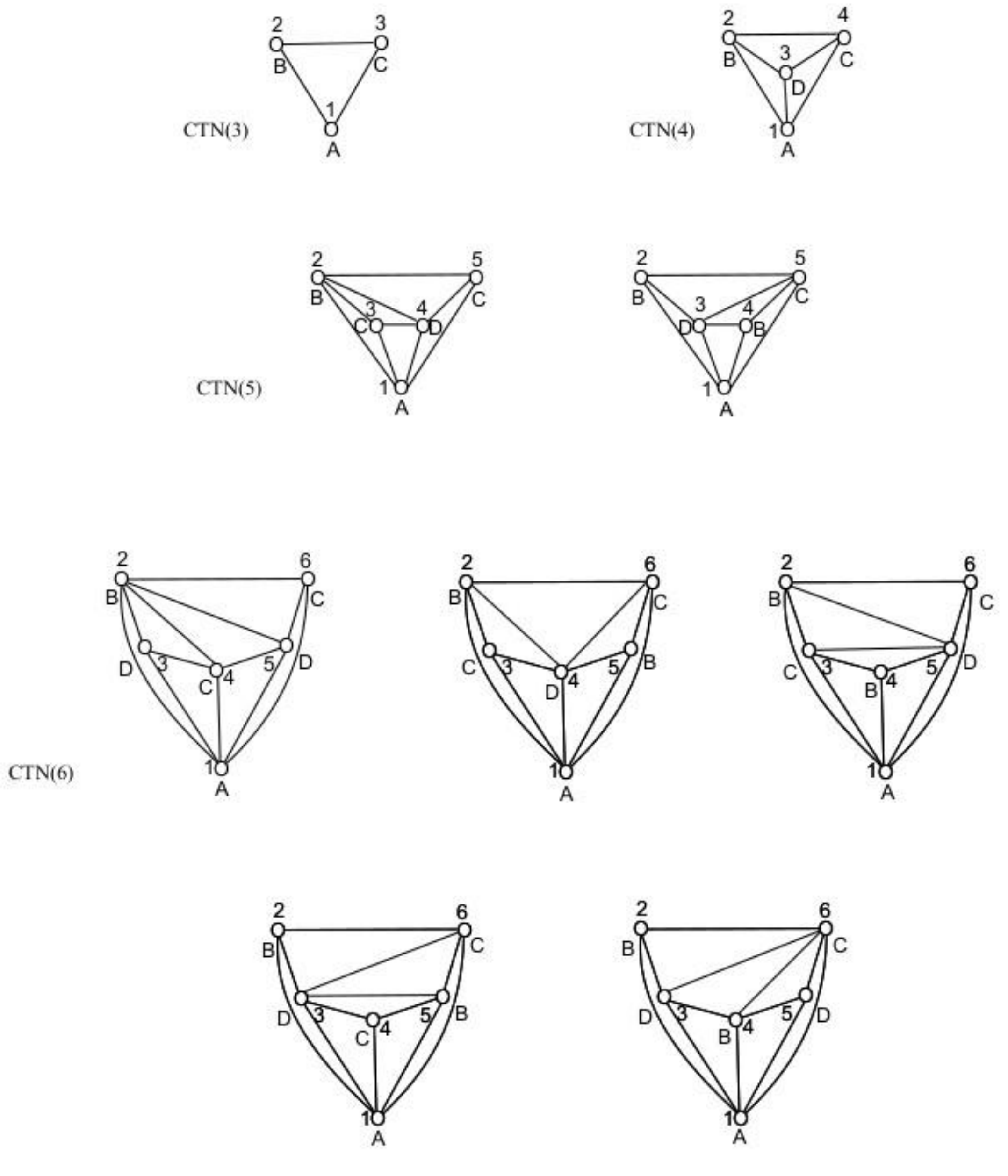

Fig. 2. CTN(*n*) for *n* = 3, 4, 5, 6 after color assignment.

Based on the Partial Proof below, it is clear that the formation F(2-*n*) consists of “triangles” that enforce three colors, i.e., the maximum number of enforced colors of the formation F(2-*n*) is MAX(F(2-*n*)) = 3. Since array 1 has a fourth color, A, MAX(CTN(*n*)) = 4.

Depending on this, we can reformulate the theorem to be proved as follows:

T4 For the maximum number of enforced colors, MAX(CTN(*n*)) = 4.

The construction of CTN(n) implies that MAX(CTN(n)) = 4.
Forcing the 5th color in this way will never occur.

Therefore, we can claim that T1 holds.

Appendix
Partial proof of the theorem
Ta The F(2-n) formations enforce at most the third color, i.e., MAX(F(2-n)) = 3 holds.

1. The behavior of a general triangular network (i.e., a triangular network that need not have a maximum number of edges) GTN(k) generated by adding all possible edges to a circle C(k) of k nodes, natural number $k > 3$

Two adjacent "triangles" in GTN(k) are said to have the same edge and 2 nodes so that in the color assignment the remaining two nodes can have the same color.
Therefore, we can proceed as follows when assigning colors:
From the set of colors SC(3) = {A, B, C}, we assign three different colors to the nodes of an arbitrary "triangle". We assign the remaining 3rd color to the adjacent "triangle" for the remaining 3rd node. Repeat this process until all nodes have been assigned colors.
The procedure of awarding colors from SC(3) guarantees that all nodes of GTN(k) are awarded a color. This proves that MAX(GTN(k)) = 3.

2. F(2-n) is GTN(k), where $k + 1 = n$. Thus it holds for F(2-n) that MAX(F(2-n)) = 3, thus proving the Ta theorem.

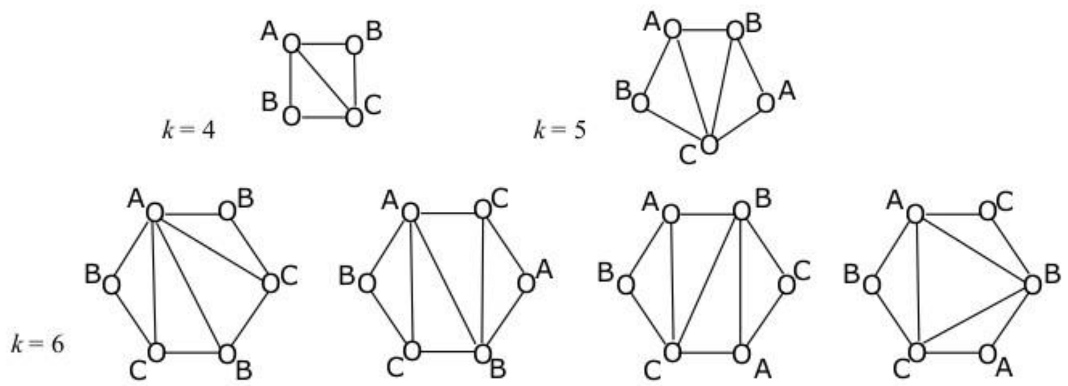

Fig. 3. GTN(k) arising from the circle C(k) for k = 4, 5, 6.
(The number of "triangles" in GTN(k) is equal to $k – 2$ (when k is 1 higher, only 1 "triangle" is added to the circle, so the way the colors are awarded does not change).)

II. Results

The previous train of thought suggests that the four-color theorem is proved again.

III. Conclusions

The change of approach has made it possible to dispense with the demanding computational technology.

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