Group Divisible Design \((4, n, n + 1, 4; \lambda_1, \lambda_2)\), for \(n \geq 4\)

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**Abstract:** This work is about Group Divisible Designs (GDDs) of block size four on three groups of different sizes \(n_1 = 4, n_2 = n\) and \(n_3 = n + 1\) where \(n \geq 4\). We first establish necessary conditions for the existence of the GDD using relationships between the parameters of the GDD and then prove that these conditions are sufficient for several families of GDDs.

**Key words:** Group Divisible Designs, Blocks, Parameters

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**I. Introduction**

A group divisible design, GDD \((n, m, k; \lambda_1, \lambda_2)\), is a collection of \(k\) -element subsets, called blocks, of an \(nm\), \(V\) -set where the elements of \(V\) are partitioned into \(m\) groups of size \(n\) each; each point of \(V\) appearing in \(r\) of the \(b\) blocks; each pair of symbols from the same group appearing in exactly \(\lambda_1\) blocks and each pair of symbols from different groups occurring in \(\lambda_2\) blocks. Group divisible designs have been studied for their usefulness in statistics, designs of experiments and their important applications in scheduling, group testing and construction of other types of combinatorial designs such as pairwise balanced designs, packings and frames [1]. A GDD is called uniform if all its groups have the same size, otherwise it is called non uniform.

When GDDs are used to construct other combinatorial designs, the groups are preferred to have different sizes, that is, non-uniform GDDs are used to fit in various situations that may arise [1]. Unfortunately, comparing with uniform GDDs, much less is known on the construction of non-uniform ones. One major reason is that no appropriate algebraic or geometric structures have been found for the construction. Therefore, the construction of non-uniform GDDs is still a challenging problem.

Non-uniform GDDs of block size three have been studied in [2, 3, 4, 5, 6, 7] while those of block size four have been studied only for \(n_1 = 1\) or 2 in [8] and \(n_1 = 3\) in [9]. It is therefore still reasonable to focus on group divisible designs of block size four, solving the problem when the design has three groups of different sizes \(n_1 = 4, n_2 = n\) and \(n_3 = n + 1\) where \(n \geq 4\).

**Definition 1.1.** [10] A balanced incomplete block design, \((V, B)\) is a finite non empty multiset, \(B\) of \(b\) nonempty subsets (called blocks) of size \(k\) of a \(V\) -set such that each element (point) in \(V\) appears in exactly \(r\) of the blocks, every pair of distinct elements of \(V\) occurs in \(\lambda\) blocks and \(2 \leq k < v\).

**Lemma 1.1.** [11] In a \((v, b, r, k, \lambda)\)-BIBD, the parameters satisfy the necessary conditions;

(i) \(v r = bk\)

(ii) \(\lambda (v - 1) = r (k - 1)\)

**Theorem 1.2.** [12] Let \(v\) be a positive integer. Then the necessary and sufficient conditions for existence of a \((v, 4, \lambda)\)-BIBD are as follows;

(i) \(\lambda \equiv 1, 5 \) (mod 6), then \(v = 1, 4 \) (mod 12)

(ii) \(\lambda \equiv 2, 4 \) (mod 6), then \(v = 1 \) (mod 3)

(iii) \(\lambda \equiv 3 \) (mod 6), then \(v = 0, 1 \) (mod 4)

(iv) \(\lambda \equiv 0 \) (mod 6), then \(v \geq 4\)

**Definition 1.2.** [5] A group divisible design, GDD \((V = n_1 + n_2 + \cdots + n_m, k; \lambda_1, \lambda_2)\) is a collection of \(k\) -element subsets (called blocks) of a \(V\) -set of symbols where the \(V\) -set is partitioned into \(m\) groups of sizes \(n_1, n_2, \ldots, n_m\) ; each pair of symbols from the same group appearing in exactly \(\lambda_1\) blocks and
each pair of symbols from different groups occurring in $\lambda_2$ blocks.

**Example 1.3.** A GDD $(3, 3, 4; 2, 2)$ with groups $G_1 = \{1, 2, 3\}$, $G_2 = \{4, 5, 6\}$ and $G_3 = \{7, 8, 9, 10\}$ has $r_1 = r_2 = r_3 = 6$ and $b = 15$. Its blocks are:

\[
\begin{array}{cccccccc|cccc}
1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 5 \\
2 & 2 & 3 & 4 & 6 & 3 & 3 & 4 & 5 & 6 & 5 & 5 & 6 \\
4 & 7 & 5 & 8 & 7 & 8 & 8 & 5 & 7 & 6 & 8 & 7 & 9 & 7 \\
6 & 9 & 6 & 10 & 9 & 9 & 9 & 10 & 10 & 10 & 8 & 10 & 10 & 8 \\
\end{array}
\]

**II. GDD $(4, n, n + 1, 4; \lambda_1, \lambda_2)$**

A group divisible design GDD $(4, n, n+1, 4; \lambda_1, \lambda_2)$ is a GDD with three groups of different size $n_1 = 4$, $n_2 = n$ and $n_3 = n + 1$ where first associate pairs occur in $\lambda_1$ blocks, and second associate pairs occur in $\lambda_2$ blocks. We establish necessary conditions and prove that they are sufficient for the existence of this GDD.

Difficulties arise in the construction of GDDs when the number of groups is smaller than the block size [13]. A case in point is when the block size is four and the number of groups is three. Earlier results on such constructions include those of Clatworthy [14] who in 1973 listed only 11 GDDs with three groups and block size four up to $r = 10$.

Henson and Sarvate in [15] generalized one of these designs, namely; GDD $(8, 3, 4; 2, 1)$ and proved that the necessary conditions are sufficient for the existence of the GDD. They consequently constructed a new family of group divisible designs $(6s + 2, 3, 4; 2, 1)$ using mutually orthogonal latin squares (MOLS) of order $3s+1$ and $(3s+1, 4, 2)$-BIBDs for all positive integers, $s$.

The problem of existence of group divisible designs of block size 4 on three groups can be answered using results in [16] where the authors show that the necessary conditions are sufficient for the existence of odd and even GDD $(n, 3, 4; \lambda_1, \lambda_2)$ for any $n$, and for all mixed designs except for the minimal index case for group size 5t where $(t \geq 1)$. These GDDs were also studied by Zhu and Ge in [13] who completed the undetermined families of mixed GDDs using two constructions based on idempotent self-orthogonal latin squares and skew room squares.

In all these studies, the authors considered GDDs of block size four on three groups. However, they did not take care of the cases where the three groups are of different sizes.

The existence problem of GDDs when the three groups have different sizes has been studied recently. Hurd and Sarvate in [3] found all ordered pairs $(n, \lambda)$ of positive integers such that a GDD $(v = 1 + 1 + n, 3; 1, \lambda)$ exists and all ordered triples $(n, \lambda_1, \lambda_2)$ of positive integers with $\lambda_1 < \lambda_2$ such that a GDD $(v = 1 + 2 + n, 3; \lambda_1, \lambda_2)$ exists. They later in [6] found all ordered triples $(n, \lambda_1, \lambda_2)$ of positive integers, with $\lambda_1 > \lambda_2$, such that a GDD $(v = 1 + 2 + n, 3; \lambda_1, \lambda_2)$ exists and completely solved the problem when $k = 3$ on three groups of sizes $(n, 2, 1)$ for any two indices, $\lambda_1$ and $\lambda_2$ with $\lambda_1 > \lambda_2$ for any $n$. They show that the necessary conditions are sufficient for the existence of GDD $(n, 2, 1; \lambda_1, \lambda_2)$ for which the indices are both even or both odd and that if $n$ is odd, the indices must be both even. The same authors in [7] found all ordered pairs $(n, \lambda)$ of positive integers such that a GDD $(v = 1 + 1 + n, 3; \lambda_1)$ exists.

Chaiyasena, et al in [2] published a paper in the direction of solving the problem of determining the existence of a GDD $(v = n_1 + n_2 + n_3, 3; \lambda_1, \lambda_2)$ for small values of $n_1, n_2, n_3$. In particular, for each $n = \{2, 3, 4, 5, 6\}$ they found all ordered pairs $(\lambda_1, \lambda_2)$ of positive integers such that a GDD $(v = 1 + 2 + n, 3; \lambda_1, \lambda_2)$ exists.

Lapchinda, et al in [4] found all ordered triples $(n, \lambda_1, \lambda_2)$ of positive integers, with $\lambda_1 < \lambda_2$, such that a GDD $(v = 1 + n + n, 3; \lambda_1, \lambda_2)$ exists and later in [5] considered the problem of determining all ordered triples $(n, \lambda_1, \lambda_2)$ of positive integers, with $\lambda_1 \leq \lambda_2$, such that a GDD $(v = 1 + n + n, 3; \lambda_1, \lambda_2)$ exists by using graph decomposition. They prove that a GDD $(v = 1 + n + n, 3; \lambda_1, \lambda_2)$ exists if and only if a $(2n + 1, 3, \lambda)$-BIBD exists. Whereas the authors in these studies worked on GDDs on three groups of different sizes, they only gave solutions for GDDs of block size three and did not consider the case of block size four.

GDDs of block size four on three groups of different sizes have been studied in very few papers such as in [8] where necessary conditions are proved to be sufficient for the existence of GDD$(1, n, n + 1, 4; \lambda_1, \lambda_2)$. For $n \geq 4$.

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4; λ1, λ2) whenever λ1 ≥ λ2 and GDD(2, n, n + 1, n; λ1, λ2) while in [9], the authors proved that the necessary conditions are sufficient for the existence of GDD(3, n, n + 1, 4; λ1, λ2). Therefore, the next step is to study a GDD (n1, n2, n3, 4; λ1, λ2) where n1 = 4, n2 = n and n3 = n + 1 with n ≥ 4.

III. Results on existence of GDD (4; n, n + 1, 4; λ1, λ2)

Modern design theory includes many existence as well as non-existence results. Thus, one fundamental question one may ask about GDD (4; n, n + 1, 4; λ1, λ2) under study is: “Does such a GDD exist?” In this section we establish results for the existence of the GDD.

Parameters of GDD (4; n, n + 1, 4; λ1, λ2): For GDDs with block size four and three groups of different sizes, 4, n and n + 1, the replication numbers, r1 = 1, 2, 3 are; from r1 = \( \frac{3λ_1 + (2n + 1)λ_2}{3} \), r2 = \( \frac{(n-1)λ_1 + (n+5)λ_2}{3} \) and r3 = \( \frac{2λ_1 + (n+4)λ_2}{3} \). The GDD has \( (n^2 + 6)λ_1 \) first associate pairs and \( (n^2 + 9n + 4)λ_2 \) second associate pairs.

The Necessary Conditions for GDD (4; n, n + 1, 4; λ1, λ2): One of the main items while doing combinatorial design constructions is to find necessary conditions for a particular design. In this subsection, we establish the necessary conditions for a GDD (4; n, n + 1, 4; λ1, λ2) and prove that they are sufficient.

**Theorem 3.1.** If a GDD (4; n, n + 1, 4; λ1, λ2) exists, then:

i) \( 3λ_1 + (2n + 1)λ_2 \equiv 0 \pmod{3} \) and \( (2n + 1)λ_2 \equiv 0 \pmod{3} \).

ii) \( (n - 1)λ_1 + (n + 5)λ_2 \equiv 0 \pmod{3} \).

iii) \( nλ_1 + (n + 4)λ_2 \equiv 0 \pmod{3} \).

iv) \( (n^2 + 6)λ_1 + (n^2 + 9n + 4)λ_2 \equiv 0 \pmod{6} \).

**Proof.** By counting the replication numbers \( r_i \) for elements of the \( i^{th} \) group,

i) The replication number for elements in \( G_1 \) is obtained from \( r_1 = \frac{3λ_1 + (2n + 1)λ_2}{3} \). Since \( r_1 \) is an integer, then \( 3λ_1 + (2n + 1)λ_2 \equiv 0 \pmod{3} \). Again since \( 3λ_1 \), it remains that \( 3 \) | \( (2n + 1)λ_2 \), that is \( (2n + 1)λ_2 \equiv 0 \pmod{3} \).

ii) The replication number for elements in \( G_2 \) is obtained from \( r_2 = \frac{(n-1)λ_1 + (n+5)λ_2}{3} \). Since \( r_2 \) is an integer, \( (n - 1)λ_1 + (n + 5)λ_2 \equiv 0 \pmod{3} \).

iii) The replication number for elements in \( G_3 \) is obtained from \( r_3 = \frac{nλ_1 + (n+4)λ_2}{3} \). Since \( r_3 \) is an integer, then \( nλ_1 + (n + 4)λ_2 \equiv 0 \pmod{3} \).

iv) The number of blocks in the GDD is obtained from \( b = \frac{(n^2 + 6)λ_1 + (n^2 + 9n + 4)λ_2}{6} \). Therefore \( (n^2 + 6)λ_1 + (n^2 + 9n + 4)λ_2 \equiv 0 \pmod{6} \).

These necessary conditions on \( b \) and \( r_1 \) determine possibilities for the parameter \( n \), and the indices \( λ_1 \) and \( λ_2 \) which are summarized in Table 1 below where “None” means that the design does not exist for any value of \( n \).

**Table 1: The restrictions on \( n \) for GDD (4; n, n + 1, 4; λ1, λ2)**

<table>
<thead>
<tr>
<th>( \lambda_1/\lambda_2 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>None</td>
<td>None</td>
<td>None</td>
<td>Any n</td>
<td>None</td>
<td>None</td>
<td>Any n</td>
</tr>
<tr>
<td>1</td>
<td>None</td>
<td>( n = 4 ) (mod 6)</td>
<td>None</td>
<td>None</td>
<td>None</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>2</td>
<td>None</td>
<td>None</td>
<td>( n = 1 ) (mod 3)</td>
<td>None</td>
<td>None</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>3</td>
<td>Even n</td>
<td>None</td>
<td>None</td>
<td>Even n</td>
<td>None</td>
<td>None</td>
<td>Even n</td>
</tr>
</tbody>
</table>

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### Group Divisible Design (4, n, n + 1, 4; λ₁, λ₂), for n ≥ 4

<table>
<thead>
<tr>
<th>n</th>
<th>GDD</th>
<th>h = 1 (mod 3)</th>
<th>GDD</th>
<th>h = 1 (mod 3)</th>
<th>GDD</th>
<th>h = 4 (mod 6)</th>
<th>GDD</th>
<th>h = 4 (mod 6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>None</td>
<td>None</td>
<td>None</td>
<td>None</td>
<td>None</td>
<td>None</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>5</td>
<td>None</td>
<td>None</td>
<td>None</td>
<td>None</td>
<td>None</td>
<td>None</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>6</td>
<td>Any n</td>
<td>None</td>
<td>Any n</td>
<td>None</td>
<td>Any n</td>
<td>None</td>
<td>Any n</td>
<td>None</td>
</tr>
</tbody>
</table>

Observations from this table show that there are other necessary conditions as shown below.

1. **GDD (4, n, n + 1, 4; λ₁, λ₂) when λ₁, λ₂ ≡ 0 (mod 6).**

   **Example 3.1.** A GDD (4, 5, 6, 4; 12, 6) exists with \( r₁ = 34, r₂ = 36, r₃ = 38 \) and \( b = 136 \). The groups of the GDD are \( G₁ = \{1, 2, 3, 4\}, G₂ = \{5, 6, 7, 8, 9\} \) and \( G₃ = \{a, b, c, d, e, f\} \). The design can be constructed taking the 36 blocks of a BIBD (9, 4, 6) on \( G₁ \cup G₂ \) add 45 blocks of a BIBD (10, 4, 6) on \( G₁ \cup G₃ \) together with the 55 blocks of a BIBD (11, 4, 6) on \( G₂ \cup G₃ \).

   **Theorem 3.2.** A GDD (4, n, n + 1, 4; 2λ, λ) exists if a BIBD (4 + n, 4, λ), a BIBD (5 + n, 4, λ) and a BIBD (2n + 1, 4, λ) exist. It is well known that a BIBD (n, 4, 6) exists for \( n ≥ 4 \) and thus a GDD (4, n, n+1, 4; 12, 6) will always exist. Hence a GDD (4, n, n+1, 4; 12s, 6s) always exists for all positive integers, \( s \).

   In general, a GDD (4, 5t, 5t+1, 4; 12t, 6t) exists with \( r₁ = 20t² + 14t, r₂ = 30t² + 6t, r₃ = 30t² + 8t \) and \( b = 25t³ + 95t² + 16t \) where \( t \) is a positive integer.

2. **GDD (4, n, n + 1, 4; λ₁, λ₂) when λ₁ = λ₂.**

   A GDD (4, n, n + 1, 4; 0, λ₂) does not exist. This is because there are only three groups and the block size is four. So, each block must contain at least a pair from the same group \( (λ₁ ≥ 1) \) to complete the block size. A GDD (4, n, n + 1, 4; λ₁, 0) exists as a \((2n + 5, 4, λ₁)-\)BIBD for particular values of \( n \) and \( λ₁ \). So, a GDD (4, n, n + 1, 4; 0, 0) does not exist.

   **Theorem 3.3.** A GDD (4, n, n + 1, 4; λ, λ) exists if and only if a \((2n + 5, 4, λ)-\)BIBD exists.

   **Example 3.2.** A GDD (4,10,11,4;1,1) exists as a (25,4,1)-BIBD.

   The GDD exists with \( G₁ = \{1, 2, 3, 4\}, G₂ = \{5, 6, 7, 8, 9, 10, 11, 12, 13, 14\} \) and \( G₃ = \{a, b, c, d, e, f, g, h, i, j, k\} \). The respective replication numbers are \( r₁ = r₂ = r₃ = 8 \) and the number of blocks is \( b = 50 \). The blocks of the GDD are those of the BIBD. They are:

```
1 1 1 1 1 1 1 1 2 2 2 2 2 2 3 3 3
2 4 5 6 7 8 9 10 4 5 6 7 8 9 11 4 5 6
3 b 12 13 14 c e 11 14 C 10 f 13 b 12 11 a d
k j g h a d f i i h e g a d j f j i
```

```
3 3 3 3 4 4 4 4 4 5 5 5 5 6 6 6 6
7 8 9 10 5 7 8 10 13 7 8 11 14 7 9 12 a
b e 13 12 6 12 9 a c 9 10 13 d 8 11 14 b
c g 14 h k e h d g I f b e j g c f
```

```
7 7 8 8 9 9 10 11 12 13 14 a b c d
10 11 11 12 10 12 14 a 13 e f g e f g
13 d 14 b c a b c d i h h i i j
k h k i j k g e f j j i k k k
```
Group Divisible Design $(4, n, n + 1, 4; \lambda_1, \lambda_2)$, for $n \geq 4$

To generalize, $(4, 10t, 10t+1, 4; t, t)$ exists for all positive integers, $t$ except for $t \equiv 2 \pmod{3}$.

3. GDD $(4, n, n + 1, 4; \lambda_1, \lambda_2)$ when $\lambda_1 < \lambda_2$.

**Theorem 3.4.** A GDD $(4, n, n + 1, 4; \lambda_1 < \lambda_2, \lambda_2)$ exists only when:

i) $\lambda_1 \equiv 0 \pmod{3}$ and $\lambda_2 \equiv 0 \pmod{6}$.

ii) $\lambda_1 \equiv 2 \pmod{3}$ and $\lambda_2 \equiv 2 \pmod{6}$.

**Example 3.3.** A GDD $(4, 4, 5, 4; 3, 6)$ exists with $r_1 = r_2 = 21$, $r_3 = 20$ and 67 blocks.

In general, GDD $(4, 4t, 4t+1, 4; 3t, 6t)$ exists for all positive integers, $t$.

**Example 3.4.** A GDD $(4, 4, 5, 4; 2t, 8t)$ exists with $r_1 = r_2 = 26$, $r_3 = 24$ and $b = 93$ blocks.

In fact, a GDD $(4, 4t, 4t+1, 4; t, t)$ exists for all positive integral values of $t$ except $t = 2 \pmod{3}$.

4. GDD $(4, n, n + 1, 4; \lambda_1, \lambda_2)$ when $\lambda_1 > \lambda_2$.

**Theorem 3.5.** A GDD $(4, n, n + 1, 4; \lambda_1 > \lambda_2, \lambda_2)$ exists when $\lambda_1, \lambda_2 = 0, 1, 2 \pmod{3}$.

**Example 3.5.** A GDD $(4, 5, 6, 4; 6, 3)$ exists with $r_1 = 17$, $r_2 = 18$, $r_3 = 19$ and 68 blocks.

**Example 3.6.** A GDD $(4, 4, 5, 4; 4, 1)$ exists. The groups of the GDD are: $G_1 = \{1, 2, 3, 4\}$, $G_2 = \{5, 6, 7, 8\}$ and $G_3 = \{a, b, c, d, e\}$ with $r_1 = r_2 = 7$, $r_3 = 8$ and 24 blocks.

A GDD $(4, 4t, 4t+1, 4; 4t, t)$ exists for all positive integers, $t$ except when $t = 2 \pmod{3}$.

**Example 3.7.** A GDD $(4, 7, 8, 4; 8, 2)$ exists with $r_1 = 18$, $r_2 = 24$, $r_3 = 26$ and 112 blocks.

In fact, a GDD $(4, 7t, 7t+1, 4; 8t, 2t)$ exists for all positive integral values of $t$ except $t = 2 \pmod{3}$.

**Theorem 3.6.** Let a GDD $(4, n, n + 1, 4; \lambda_1, \lambda_2)$ be a design.

i) If $n$ is even, there is no restriction on the parity of $\lambda_1$ and $\lambda_2$.

ii) If $n$ is odd, $\lambda_1$ must be even.

**Proof.** The number of blocks, $b$, is given by $b = \frac{(n^2 + 6)\lambda_1 + (n^2 + 9n + 4)\lambda_2}{6}$.

i) Let $n = 4t$ where $t \geq 1$ is an integer. Then $b = \frac{(16t^2 + 6)\lambda_1 + (16t^2 + 36t + 4)\lambda_2}{6}$ which implies that $\lambda_1$ and $\lambda_2$ can be of any parity.

ii) Let $n = 4t+1$ where $t \geq 1$ is an integer. Then $b = \frac{(16t^2 + 28t + 7)\lambda_1 + (16t^2 + 44t + 14)\lambda_2}{6}$ which implies that $\lambda_1$ must be even.

**Example 3.8.** A GDD $(4,4,5,4;1,1)$ exists with $n$-even, and both $\lambda_1$ and $\lambda_2$ odd. The groups of the GDD are: $G_1 = \{1, 2, 3, 4\}$, $G_2 = \{5, 6, 7, 8\}$ and $G_3 = \{a, b, c, d, e\}$ with $r_1 = r_2 = r_3 = 4$ and $b = 13$. The blocks are:

$$
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 6 \\
2 & 3 & 5 & 7 & 3 & 6 & 8 & 4 & 7 & 5 & 8 & a & b \\
4 & a & 6 & c & 5 & 7 & d & 6 & 8 & 7 & A & b & c \\
b & e & 8 & d & C & a & e & d & B & e & C & d & e
\end{array}
$$

In general, a GDD $(4,4t,4t+1,4;t,t)$ exists for all positive integers, $t$ except $t \equiv 2 \pmod{3}$.

**Example 3.9.** A GDD $(4, 6, 7, 4, 6, 3)$ with $r_1 = 19$, $r_2 = 21$, $r_3 = 22$ and $b = 74$, a GDD $(4, 4, 5, 4; 12, 6)$ with $r_1 = 30$, $r_2 = 26$, $r_3 = 32$ and $b = 116$, and a GDD $(4, 4, 5, 4, 3, 6)$ of Example 3.3 all exist with $n$ even and no restriction on the parity of $\lambda_1$ and $\lambda_2$.

In fact for all positive integral values of $t$, a GDD $(4, 6t, 6t+1, 4; 6t, 3t)$ with $r_1 = 12t^2 + 7t$, $r_2 = 18t^2$.
+ 3t, \( r_3 = 18t^2 + 4t \) and \( b = 39t^3 + 27t^2 + 8t \), and a GDD \((4, 4t, 4t + 1, 4; 12t, 6t)\) with \( r_1 = 16t^2 + 14t, r_2 = 20t^2 + 6t, r_3 = 24t^2 + 8t \) and \( b = 64t^3 + 36t^2 + 16t \) all exist.

**Example 3.10.** A GDD \((4, 5, 6, 4; 6, 3)\) of Example 3.5 exists with \( n \) odd and \( \lambda_1 \) only even.

**Theorem 3.7.** If a GDD \((4, n, n + 1, 4; \lambda_1, \lambda_2)\) exists, then \( b \geq \max (2r_1 - \lambda_1, 2r_1 - \lambda_2) \).

**Proof.** Consider any first associate pair, say \( a \) and \( b \) from any group \( G_i \) where \( i = 1, 2, 3 \). This pair must appear in the design coming together \( \lambda_1 \) times. There are \( r_i \) blocks containing each of \( a \) and \( b \). There are \( r_i - \lambda_1 \) blocks containing only one of them. Thus, the number of blocks must be at least \( 2r_i - \lambda_1 \) to accommodate the pair \( r_i \) times. Similarly, consider any second associate pair, say \( a \) and \( c \) from two different groups \( G_i \). As both of them come together \( \lambda_2 \) times, the number of blocks must be at least \( 2r_i - \lambda_2 \) to accommodate the pair \( r_i \) times in the design.

**Example 3.11.** Consider a GDD \((4, 5, 6, 4; 6, 3)\) of Example 3.5. In this design, \( r_1 = 17, r_2 = 18, r_3 = 19 \) and \( b = 68 \).

\[
\begin{align*}
\text{Max} \ [2r_1 - \lambda_1, 2r_1 - \lambda_2] &= \max \ [2(17) - 6 = 28, 2(17) - 3 = 31] = 31 < 68 \\
\text{Max} \ [2r_2 - \lambda_1, 2r_2 - \lambda_2] &= \max \ [2(18) - 6 = 30, 2(18) - 3 = 33] = 33 < 68 \\
\text{Max} \ [2r_3 - \lambda_1, 2r_3 - \lambda_2] &= \max \ [2(19) - 6 = 32, 2(19) - 3 = 35] = 35 < 68
\end{align*}
\]

**Theorem 3.8.** If a GDD \((4, n, n + 1, 4; \lambda_1, \lambda_2)\) exists, then \( \lambda_1 \geq \frac{(n^2 + 9n + 4)\lambda_2}{5n^2 + 30} \).

**Proof.** The design has three groups of size \( n_1 \geq 4 \), the block size. Therefore, each block must have at least one first associate pair. This means that the total number of first associate pairs is at least equal to the number of blocks. Since there are \((n^2 + 6)\lambda_1\) first associate pairs and \(\frac{(n^2 + 6)\lambda_1 + (n^2 + 9n + 4)\lambda_2}{6}\) blocks,

\[
\frac{(n^2 + 6)\lambda_1}{5} \geq \frac{(n^2 + 6)\lambda_1 + (n^2 + 9n + 4)\lambda_2}{6} \geq \frac{5(n^2 + 9n + 4)\lambda_2}{6} \geq \frac{5(n^2 + 9n + 4)\lambda_2}{5n^2 + 30}.
\]

**Corollary 1.** If a GDD \((4, n, n + 1, 4; \lambda_1, \lambda_2)\) exists, then \( b \leq (n^2 + 6) \lambda_1 \).

**Proof.** The design has \( b \) blocks and \((n^2 + 6)\lambda_1\) first associate pairs. The total number of blocks cannot exceed the total number of first associate pairs. Thus \( b \leq (n^2 + 6) \lambda_1 \).

**Example 3.14.** A GDD \((4, 5, 6, 4; 12, 6)\) of Example 3.1 exists with 372 first associates and 136 blocks.

**References**


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