IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 17, Issue 1 Ser. III (Jan. – Feb. 2021), PP 34-38
www.iosrjournals.org

On some topological properties on a certain hyperspace topology

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Abstract: *In this paper, some topological properties, especially those that associated with compactness properties as in [8] were studied on hyperspaces $\mathcal{A}(X)$ and $\mathcal{A}_c(X)$ of X and compared with the original space X*

Keywords: *$\mathcal{A}$ - closed set, Z – closed set, Lindelöf space, Extremely disconnected space, Paracompact space, hereditary disconnected.*

Date of Submission: 06-02-2021 Date of Acceptance: 20-02-2021

I. Introduction And Definitions

The study of hyperspace topology started in 1924 with Hausdorff [1] where he started topologizing a collection of subsets of a topological space by defining a metric space X. Later, Vietoris [11] then introduced a new topology on the collection of all nonempty closed subsets of a topological space X, which is known as Vietoris or finite topology. After this, Micheal in his work [2] dealt with various types of subsets for the construction of topology. Fell [9] also introduced a compact Hausdorff topology for the space of all closed subsets of a topological space. Then, much work has been done on hyperspace of topological spaces [10]. In [3] the authors have introduced a new topology τ on the collection $\mathcal{A}(X)$ of all nonempty $\mathcal{A}$ - closed subsets of a topological space X. In this paper, we will first discuss on the new hyperspace topology τ on the collection $\mathcal{A}(X)$ of all nonempty $\mathcal{A}$ - closed subsets of a topological space X. In this work, we provides necessary and sufficient conditions for a topological space X and its hyperspace to be paracompact, Lindelöf, extremely disconnected, first countable, hereditary disconnected and also shows the behaviour of X and its hyperspace under constructions.

Definition 1.1: [3] Let X be a topological space A point $x \in X$ is called $\mathcal{A}$ - compact point of a subspace $Y \subseteq X$ if for every neighbourhood O_x, we have $Cl(O_x) \cap Y \neq \emptyset$. The set of all $\mathcal{A}$ – compact points of a set Y is known as $\mathcal{A}$ – closure of Y and is denoted by $Cl_{\mathcal{A}}(Y)$.

Definition 1.2: [3] A set $Y \subseteq X$ is said to be $\mathcal{A}$ – closed if $Cl_{\mathcal{A}}(Y) = Y$ and set Y is $\mathcal{A}$ – open if $X \setminus Y$ is $\mathcal{A}$ – closed.

Definition 1.3: [3] A topological space X is said to be $\mathcal{Z}$ – closed if for any open cover $\{U_i : i \in \Lambda\}$ of $X \exists$ a finite subsets $\Lambda_o \subseteq \Lambda$ such that $X = \cup\{Cl(U_i): i \in \Lambda\}$.

Definition 1.4: [4] A topological space X is said to be Lindelöf if every open cover has a countable subcover.

The following examples of Lindelöf spaces areare well konwn:

(i) Countabele and cocountable spaces are Lindelöf

(ii) The Sorgenfrey line is Lindelöf

Definition 1.5: [7] A topological space X is said to be paracompact, if every open cover of X has a locally finite open refinement.

Proposition: [7] Every paracompactHuasdorff space is normal.

The following examples of Paracompact spaces areare also well known:

(i) Every compact space is paracompact.

(ii) Lindelöf τ_3 – spaces are paracompact.

(iii) Metrizable spaces are paracompact.

Definition 1.6: [7] A topological space X is said to be first countable space if there is a countable neighbourhood base or local base at each of its points, for example, every metric space is first countable.

Definition 1.7: [7] A topological space X is first countable if it satisfies the first axioms of countability in $\mathcal{ZFC}$ below.

(i) If $(\forall x \in X)(\exists B(x))$, $|B(x)| \leq \chi_o$ and $B(x)$ is a local base at x

(ii) If $\left(\exists (B(x))_{x \in X}\right)(\forall x \in X)$, $|B(x)| \leq \chi_0$ and $B(x)$ is a local base at x and

(iii) If $(\exists\{B(n, x): n \in \mathbb{N}, x \in X\})(\forall x)\{B(n, x): n \in \mathbb{N}\}$ is a local base at x

In the definition (i), (ii) and (iii) above, one can take only the open neighbourhoods without changing the logic value of them. This fact is pointed out because it will be seen other situation where that is not the case.

Definition 1.8: [4] A topological space X is said to be locally compact if every point $x \in X$ has a compact Neighbourhood of X. Examples include:
(i) Every compact space is locally compact
(ii) Every discrete space is locally compact
(iii) Every indiscrete space is locally compact
(iv) Complex space and space of integers are locally compact.
Definition 1.9: [6] A topological space X is said to be extremely disconnected if the closure of every open set in X is open, for example, every discrete and indiscrete space is extremely disconnected.
Definition 1.10: [5] On the hyperspace $\mathcal{A}(X)$, we define a topology as follows:
For every $U \subseteq X$, let $U^+ = \{K \in \mathcal{A}(X): K \subseteq U\}$ and $U^- = \{K \in \mathcal{A}(X): K \cap U \neq \emptyset\}$. If we consider $N_{\mathcal{A}} = \{U^-: U\ is\ open\ in\ X\} \cup \{U^+: U\ is\ \mathcal{A} - open\ in\ X\ and\ the\ complement\ of\ U\ is\ a\ Z - set\}$. Then $N_{\mathcal{A}}$ form a subbase for some topology on $\mathcal{A}(X)$ which is written as τ.
In the topology τ above, every basic open set is written in the form $S_1^- \cap S_2^- \cap \dots \cap S_n^- \cap S_0^+$, where $S_i \subseteq S_0$ for $1 \leq i \leq n$, and $S_1, S_2, \dots, S_n$ are open sets, S_0 is a $\mathcal{A} - open$ set with $X \setminus S_0$ a Z - set.
Definition 1.11: [5] A Hausdorff space $(\tau_2)\, X$ is said to be a locally Z - space if every point of X has a neighbourhood that is a Z -set.
Definition 1.12: [5] A family $\mathcal{F} \subseteq Z(X)$ is said to be cofinal in $Z(X)$ if for any $Z \in Z(X), \exists\, Z^* \in \mathcal{F}$ such that $Z \subseteq Z^*$.
Definition 1.13: [3] A topological space X is called hemi Z – closed, if there exist a countable subfamily $\mathcal{F} \subseteq \mathcal{Z}(X)$ that is cofinal in $Z(X)$.
Definition 1.14: [5] Let X be a τ_1 topological space. Let $\mathcal{A}_n(X)$ be the set of all nonempty closed subsets of X which has cordinality χ_0 such that $\chi_0 \leq n$. We have the following as the hyperspace:
$\mathcal{A}_n(X) = \{A \in \mathcal{A}_n(X): |A| \leq n\}$. $\mathcal{A}_c(X) = \{A \in \mathcal{A}(X): A\ is\ compact\ in\ X\}$.
Remark: From **definition 1.14** above, we have, for any topological space X. $\mathcal{A}_n(X) \subset \mathcal{A}_c(X) \subset \mathcal{A}(X)$.

II. Results

Here, we discuss some topological properties like countability, locally compactness, Lindelöf, Extremely disconnectedness for any of hyperspaces defined in **section 1** above and study how some of these properties for these hyperspaces are related to the properties of the topological space X.
Theorem 2.1: Let X be a Hausdorff topological space and $\mathcal{A}(X)$ be an hyperspace of X. Then the following statements are equivalents.
(i) If $\mathcal{A}(X)$ is first countable, then X is first countable.
(ii) From **(i)**, if X is first countable and each proper $\mathcal{A}$ – open subset of X is a locally Z – closed. Then X is a locally Z - space and
(iii) $\mathcal{A}(X)$is extremely disconnected iff X is discrete.
Proof: (i) Suppose $\exists\, x \in X$, since $\mathcal{A}(X)$ is first countable, $\exists$ a local base $B = \{a_i^- \cap A_i^+: i \in N\}$ at set $\{x\}$, and a_i is a finite family of open subsets of topological space X, also, A_i is $\mathcal{A}$ – open in X and the complement $X \setminus A_i$ of A_i is a Z - set in a topological space X. Let us have
$\mathcal{W} = \{U_i \cap (\cap a_i): i \in N\}$ (1)
Next is to prove that $\mathcal{W}$ in **(1)** above is local base at $x \in X$ for every $Y \in \mathcal{W}$, $x \in Y$ and Y is open in X. Let O_x be a neighbourhood of $x \in X$ and also open in X, so, $\{x\} \in O^-$. Then there exists$i \in N$ such that $\{x\} \in a^- \cap A^+ \subseteq O^-$. Certainly, $x \in A_i \cap (\cap a_i) \subseteq O$. This shows that a topological space X is first countable. (ii) Since X is first countable as shown in **(1)** above. Let $x \in X$. Suppose $X = \{x\}$. Then there is nothing to prove anymore. But if we let $X \neq \{x\}$, there is some point $a \in X \setminus \{x\}$. If we let $A = X \setminus \{a\}$. Then A will be a proper $\mathcal{A}$ – open subset of X and by hypothesis, there is divergence sequence $\{Z: n \in N\}$ of Z - sets of A such that all Z - set of A are in some Z. Since X is first countable as shown above, there is a convergent sequence $\{K_n: n \in N\}$ which serves as a neighbourhood base at $x \in X$and $K_i \subseteq A$. Next, we shall prove that there is $n \in N$ such that $K_n \subseteq Z_n$. If otherwise, for all $n \in N$, we take $x_n \in K_n \setminus Z_n$, so that the sequence $\{x_n: n \in N\}$ has x as its limit point. Therefore, there is $y \in N$ such that $\{x_n: n \in N \cup \{x\} \subseteq Z_y \Rightarrow x_k \in Z_k\}$which is a contradiction. Then, (ii) is proved. Before we provide a proof for (iii), we recall **Lemma 2.0** below:
Lemma 2.0: A topological space X is discrete if, and only if the hyperspace $\mathcal{A}(X)$ is discrete.
Proof: Note that from **Lemma 2.0** above, the discreteness of$X$ implies that$\mathcal{A}(X)$ is discrete. By contradiction: suppose X is not discrete and there is a non isolated point $x$in X. If we let K be the set of all collection $\mathcal{F}$ such that every $m \in \mathcal{F}$ are painwise disjoint nonempty open subsets of X, further if $Y \in \mathcal{F}$, then x must not be a member of $Cl_X(Y)$. And, by considering the Kuratowski – Zorn Lemma, we have a C - maximal element T in K provided that X is τ_2 - space, $Cl(\cup T) = X$. Letting $Q = U\{Y^+ \cap \mathcal{A}(X): Y \in T\}$and, assume $\mathcal{F}$ as the filter of open neighbourhood $O_x$of x, then for each $V \in \mathcal{F}, \exists\, Y_V, A_V \in T$ such that $Y_V \neq A_V$ and $V \cap Y_V = V \cap A_V = \emptyset$.

Now, if we $P = U\{\langle Y_V, A_V\rangle \cap \mathcal{A}(X): V \in \mathcal{F}\}$, then Q and P are pairwise disjoint nonempty open subsets of $\mathcal{A}(X)$ such that $\{x\} \in Cl_{\mathcal{A}(X)}(Q) \cap Cl_{\mathcal{A}(X)}(P)$, which ends the proof.

Lemma 2.1: A topological space X is Lindelöf iff$\mathcal{A}_n(A)$ is Lindelöf.

Lemma 2.2: Let X be a Hausdorff space. Then X is extremely disconnected iff for every $A, B \in X\ Cl(A) \cap Cl(B) = \emptyset$.

Lemma 2.3: Suppose topological space X is locally compact. Then X is paracompact iff $\mathcal{A}_n(X)$ is paracompact.

Lemma 2.4: Suppose the family of open set $\phi = \{Q_i^K: K \in A_i, i \in N\}$ is a refinement of the family $\{F_1, F_2, \dots, F_n\}$ in open subsets of a topological space X. Then family $\phi = \{0\langle Q_1^K, Q_2^K, \dots, Q_n^{K_n}\rangle: k_i \in A_i, i \in N\}$ is a refinement of the family $\{0\langle Q_1, Q_2, \dots, Q_n\rangle\}$ in the hyperspace $\mathcal{A}(X)$.

Theorem 2.2: If X is a topological space. $X$is Lindelöf space iff the hyperspace $\mathcal{A}_c(X)$ is Lindelöf space.

Proof: Suppose X is Lindelöf, we define an arbitrary open cover of hyperspace $\mathcal{A}_c(X)$ as $\phi = \{0\langle Q_1^K, Q_2^K, \dots, Q_n^K\rangle: k \in Y\}$ as shown in **Lemma 2.4** above. If $\phi = \{Q_1^K, Q_2^K, \dots, Q_n^K: k \in Y\}$ is a refinement of ϕ; we shall [prove by contradiction that the system $\{\phi_i^K: i \in N, i$ is finite and $k \in Y\}$ is a cover of the space X. Now, if there is an element $x \in \mathcal{A}_n(X) \setminus \cup\{0\langle Q_1^K, Q_2^K, \dots, Q_n^{K_n}\rangle: k \in Y\}$. Then the singleton set $\{x_i\} \in \mathcal{A}_c(X) \cup \{0\langle Q_1^K, Q_2^K, \dots, Q_n^K\rangle: k \in Y\}$. This follows that a singleton set $\{x\}$ is a member of $\mathcal{A}_c(X)$, where $\{x\}$ is not a member of $\cup\{0\langle Q_1^K, Q_2^K, \dots, Q_n^K\rangle: k \in Y\}$.By this, we arrived at a contradiction. Then the system $\{Q_i^K: i \in N,$ is fininte and $k \in Y\}$ will certainly be a cover of the topological space X. Since X is Lindelöf, there exists a countable cover Q_2 such that $\phi_2 = \{Q_i^t: i \in N, i\ is\ finite\ and\ t \in T \subset Y, |T| \leq \chi_0\}$ such that ϕ_2 covers the space X. By considering all the finite open subsets of the family ϕ_2 above, where $0\langle Q_1^t, Q_2^t, \dots, Q_n^s\rangle, t \in T, |S| \leq \chi_0$, we shall show that the system $\{0\langle Q_1^t, Q_2^t, \dots, Q_n^s\rangle, t \in T\}$ is a cover of the hyperspace $\mathcal{A}_c(X)$ by assuming that there is a compact set A such that $A \in \mathcal{A}_c(X) \setminus \cup\{0\langle Q_1^t, Q_2^t, \dots, Q_K^t\rangle: t \in T, K \in N\}$. Provided that the system ϕ_2 is a cover of X, there exists$Q_1^t, Q_2^t, \dots, Q_K^t$ such that A is a member of $0\langle Q_1^t, Q_2^t, \dots, Q_K^t\rangle$, and by this, we arrived at contradiction. So, $\phi_3 = \{0\langle Q_1^t, Q_2^t, \dots, Q_K^t\rangle: t \in T, n \in N, |T| \leq \chi_0\}$ will be a countable cover of the hyperspace $\mathcal{A}_c(X)$. Therefore the hyperspace $\mathcal{A}_c(X)$ is Lindelöf.

Conversely, If $\mathcal{A}_c(X)$ is Lindelöf as shown above, the space X is closed in $\mathcal{A}(X)$, and X is also closed in $\mathcal{A}_c(X)$. This shows that the topological space X is Lindelöf.

Theorem 2.3: Suppose X is a topological space. It is said to be extremely disconnected if, and only its hyperspace $\mathcal{A}(X)$ is extremely disconnected.

Proof: As in **theorem 2.2** above, we prove this as follow:

Suppose X is extremely disconnected, then for every open set $A \subset X, Cl(A) \subset X$ and open in X. We shall prove that the hyperspace $\mathcal{A}(X)$ is extremely disconnected. By definition of extremely disconnectedness, if $0\langle Q_1, Q_2, \dots, Q_n\rangle$ be an arbitrary open set in $\mathcal{A}(X)$, we need to show that $Cl(0\langle Q_1, Q_2, \dots, Q_n\rangle) = 0\langle Cl(Q_1), Cl(Q_2), \dots, Cl(Q_n)\rangle$ is open in $\mathcal{A}(X)$. Now, suppose we take an arbitrary element $K \in 0\langle Cl(Q_1), Cl(Q_2), \dots, Cl(Q_n)\rangle$ such that $K \subset \cup_{i=1}^n Cl(Q_i)$ and both K and $Cl(Q_i)_{i=1,2,\dots,n}$ are disjoint. Since X is extremely disconnected, for every point $x$in $K \cap Cl(Q_i)$, there exists a neighbourhood $O_x \subset Cl(Q_i), \forall i = 1, 2, \dots, n$. Let us define $\phi_i = \cup\{O_x: x \in K \cap Cl(Q_i), O_x \subset Cl(Q_i), \forall\ i = 1, 2, \dots, n\}$. Then we see that $K \subset \cup_{i=1}^n \phi_i$and $K \cap \phi_i \neq \emptyset\ \forall\ i = 1, 2, \dots, n$. Hence, $K \in 0\langle \phi_1, \phi_2, \dots, \phi_n\rangle$ and $0\langle \phi_1, \phi_2, \dots, \phi_n\rangle$ is a neighbourhood of the point $a \in K$. Next, we shall prove that $0\langle \phi_1, \phi_2, \dots, \phi_n\rangle$ containedin $0\langle Cl(\phi_1), Cl(\phi_2), \dots, Cl(\phi_n)\rangle$. Take an arbitrary set $M \in 0\langle \phi_1, \phi_2, \dots, \phi_n\rangle$.Since $M \subset \cup_{i=1}^n \phi_i$and $M \cap \phi_i \neq \emptyset, i = 1,2, \dots, n$, then from the fact that $\phi_i \subset Cl(Q_i)$ for $i = 1, 2,, \dots, n$, we have $M \subset \cup_{i=1}^n Cl(Q_i)$,where both M and $Cl(Q_i)$ are disjoint, for $i = 1,2, \dots, n$. Therefore, $M \in 0\langle Cl(Q_1), Cl(Q_2), \dots, Cl(Q_n)\rangle$ and every point of $K \in 0\langle Cl(Q_1), Cl(Q_2), \dots, Cl(Q_n)\rangle$ is an interior. Therefore the hyperspace $\mathcal{A}(X)$ is extremely disconnected.

Conversely, let the hyperspace $\mathcal{A}(X)$ be extremely disconnected as shown above, and $Q \subset X$ and open. We shall prove that $Cl(Q)$ is open in X, from definition of extremely disconnectedness. Since we know that $0\langle Q\rangle$ is open in the hyperspace $\mathcal{A}(X)$. Since hyperspace $\mathcal{A}(X)$ is extremely disconnected as shown above, then $Cl(0\langle Q\rangle) = 0\langle Cl(Q)\rangle$ is open in $\mathcal{A}(X)$. Next, we need to show that $Q \subset X$ is open in X. If $x \in Cl(Q)$, then $\{x\} \in 0\langle Cl(Q)\rangle$. Since the set $0\langle Cl(Q)\rangle$ is open, there exists a neighbourhood $0\langle A\rangle$ such that $x \in 0\langle A\rangle$and $0\langle A\rangle \subset 0\langle Cl(Q)\rangle$, hence, $A \subset Cl(Q)$. Therefore, $Cl(Q) \subset X$ and open. Which shows that X is extremely disconnected. This ends the proof.

Theorem 2.4: Suppose X is a discrete topological space. Then X is paracompact iff $\mathcal{A}_c(X)$ is paracompact.

Proof: If X is a discrete space, we shall prove that the hyperspace $\mathcal{A}_c(X)$ is paracompact. Let $\phi = \{0\langle Q_1^K, Q_2^K, \dots, Q_n^K\rangle: K \in A\}$ of $\mathcal{A}_c(X)$ be an arbitrary open cover and $\phi_1 = \{Q_i^K: K \subset A, i = 1,2, \dots, n\}$ is a refinement of the cover ϕ. Certainly, ϕ_1 is also an open cover of the topological space X. Provided that the topological space X is paracompact, there is a locally finite open cover $\sigma = \{K^t: t \in B\}$ which serves as refinement of ϕ_1, by **Lemma 2.4** above. If we take every final combinations of elements of the cover σ and define $\sigma_1 = \{0\langle K_1^t, K_2^t, \dots, K_n^t\rangle: t \in B\}$, certainly, σ_1 is cover of the hyperspace $\mathcal{A}_c(X)$ and σ_1 will be a refinement of the cover ϕ_1 due to **Lemma 2.4** above. Now, we shall show that the system $\boldsymbol{\sigma_1}$is locally finite by letting $E \in \mathcal{A}_n(X)$ an arbitrary element, then E is compact and contained in the topological space X. Provided that the cover $\sigma = \{K^t : t \in B\}$ is locally finite, every point $x \in E$ has a neighbourhood O_x such that $\{t \in B : O_x \cap K^t \neq \emptyset\}$ is finite. Again, provided that E is compact, there exist$O_{x_1}, O_{x_2}, \dots, O_{x_m}$ such that $E \subset U_{i=1}^{m} O_{x_i}$ and $\{t \in B : O_{x_i} \cap K^t = \emptyset\}$ is finite, for all $i = 1,2,\dots,m$. Then, the set $0\langle O_{x_1}, O_{x_2}, \dots, O_{x_m}\rangle$ is a neighbourhood of the compact set $E \in \mathcal{A}_c(X)$ and the set$\{t \in B : 0\langle O_{x_1}, O_{x_2}, \dots, O_{x_m}\rangle \cap 0\langle K_1^t, K_2^t, \dots, K_m^t\rangle \neq \emptyset\}$ is finite. Which shows that the hyperspace $\mathcal{A}_c(X)$ is paracompact.

Conversely, suppose $\mathcal{A}_c(X)$ is paracompact as shown above. Since$\phi = \{Q^q : q \in A\}$ is an arbitrary open cover of the space X, taking all finite combinations of the cover ϕwe cand put $\phi_1 = \{0\langle Q_1^q, Q_2^q, \dots, Q_n^q\rangle : q \in A, Q_i^q \in \phi$, *i=1,2,...,n*. Certainly, ϕ is an open cover of the hyperspace *AcX*. Since*AcX* is paracompact, there exists a finite open cover $\sigma = \{\{0\langle K_1^t, K_2^t, \dots, K_r^t\rangle : t \in B\}\}$ which is refinement of the cover ϕ_1. By taking the trace σ_1 of a family in the topological space X, we shall show that $\sigma_1 = \{K_i^t : t \in B, i = 1,2,\dots,n\}$ is locally finite open cover of X. Now, if $x \in X$ and is arbitrary, then $\{x\} \in \mathcal{A}_c(X)$. Since$\mathcal{A}_c(X)$ is paracompact and σ is locally finite open cover of topological space X, there exists a neighbourhood $0\langle N\rangle_{\{x\}}$ such that $\{t \in B : 0\langle N\rangle \cap 0\langle K_1^t, K_2^t, \dots, K_r^t\rangle \neq \emptyset\}$ is finite. This means $x \in N$ and $N \cap K_i^t \neq \emptyset$ for $i = 1,2,\dots,r$ and also for finite, where$t \in B$. Therefore σ_1 is locally finite, which shows that X is paracompact. This ends the proof.

Theorem 2.5: Suppose X is first countable topological space, Z - closed and Urysohn space. The hyperspace $\mathcal{A}(X)$ is first countable if, and onlyevery $U \in \mathcal{A}(X)$ is separable and every proper $\mathcal{A}$ - open subset of X is hemi Z - closed.

Proof: Suppose $\mathcal{A}(X)$ is first countable, then due to the fact that if X is Hausdorff and the tightness of $\mathcal{A}(X)$ is first countable, then any member $Y \in \mathcal{A}(X)$ is separable and also every proper $\mathcal{A}$ - open subset of X is hemi$Z$ – closed.

Conversely, suppose every $\mathcal{A}$ - closed. Assume $Y \in \mathcal{A}(X)$. Then by hypothesis, Y is separable and hence there exists a countable subset $K \subset Y$ such that $Cl(K) = Y$. Since X is first countable as shown in **theorem 2.1(i)** above, there exists a countable local base $B_f$at F, $\forall f \in K$. Let us define $B = \cup\{B_f : f \in K\}$ and μ be the set of all finite subsets of B. Provided that the complement of Y, $X \setminus Y$ is hemi Z – closed, $\exists$ a countable family $\mathcal{F} \subseteq Z(X \setminus Y)$ that is cofinal in $Z(X \setminus Y)$. Assume $Q = \{A^- \cap W^+ : A \in \sigma, X \setminus W \in F\}$. Then Q is countable. Next, we shall show that Q is a local base at $Y \in \mathcal{A}(X)$. Assume $A_1^- \cap A_2^- \cap \dots \cap A_n^- \cap A_o^+$ is a basic neighbourhood of $Y \in \mathcal{A}(X)$, where each $A_i \subset X, for\ i = 1,2,\dots,n$ is open in X and A_o is $\mathcal{A}$ - open subset of X such that $X \setminus A_0$ is Z - set. So, Y and A_i are disjoint and $Y \subseteq A_0$. Then $\exists\ Z \in \mathcal{F}$ such that $X \setminus A_0 \subseteq Z$ and $\{H_1^-, H_2^-, \dots, H_n^-\} \in \sigma$ such that $x_i \in H_i \subseteq A_i$for $x_i \in A_i \cap K, i = 1,2,\dots,n$. Then $Y \in H_i \cap H_2 \cap \dots \cap H_n \cap (X \setminus$ *Z+⊆⋂i=1nAi−⋂A0+*. Therefore the hyperspace *AX* is first countable.

Theorem 2.6: A topological space X is hereditary disconnected if, and only if its hyperspace $\mathcal{A}(X)$ is hereditary disconnected.

Proof: We prove this by contradiction. Assume X is not hereditary disconnected. Then there exists $x \in X$ such that the component of $X, C(x) \neq \{x\}$. So, there is a point $m \in C(x)$ such that $m \neq x$. From the fact that if $A_1, A_2, \dots, A_n$ are connected subsets of a topological space X, the subsets $\{A_1, A_2, \dots, A_n\}$ is a connected subset of the hyperspace $\mathcal{A}(X)$. The component $C(x)$ of x is connected implies that$\{C(x)\} = O\langle C(x)\rangle$ is connected. Thus $\{m\}, \{x\} \in O\langle C(x)\rangle$and $O\langle C(x)\rangle \subset c(\{x\})$. Therefore,if$c(\{x\}) \neq (\{x\})$, then, the hyperspace $\mathcal{A}(X)$ is not hereditary disconnected.

Conversely, suppose that X is hereditary disconnected. Assume $P, Q \in \mathcal{A}(X)$ such that there is an element $x \in Q \notin P$. Then, for every $m \in P$, there exists a clopen set B_m such that $m \in B_m$ and $x \notin B_m$. Therefore, the set $\{B_m : m \in P\}$ is an open cover of set P. So, there is a finite subcover $\{B_{m_k}\}_{k=1}^{n}$ of compact set P. Then, $\bigcup_{k=1}^{n} B_{m_k}$ is clopen subset of topological space X. Thus $o\langle \bigcup_{k=1}^{n} B_{m_k}\rangle$ is clopen in the hyperspace $\mathcal{A}(X)$ of X such that $P \in \mathcal{A}(X)$and $Q \notin \mathcal{A}(X)$. Therefore, the hyperspace $\mathcal{A}(X)$ is hereditary disconnected.

III. Conclusion

This paper shows the necessary and sufficient conditions for both a topological space X and its hyperspace to be: (i) Lindelöf space (ii) Extremely disconnected (iii) Paracopactness (iv) First countable space (v) Hereditary disconnected and also shows the behaviours of X and its hyperspace under constructions.

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