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# Structural Strong Domination of Graphs

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***Abstract***
*In a graph G = (V, E), a set S of vertices is said to be strong dominating set if for every $v \in V - S$, there exists a vertex $u \in S$ such that $uv \in E(G)$ and deg u >deg v. The subgraph induced by a strong dominating set is called strong dominating subgraph of G. The cardinality of a minimum strong dominating set is denoted by $y_3$ set. For a given class D of connected graphs it is an interesting problem to characterize the class SD(D) of graphs G such that each connected induced subgraph of G contains a strong dominating subgraph belonging to D. In this paper we determine SD(D) where D = {$P_1$, $P_2$, $P_3$}, D = {{$K_n$/n >1} ∪ $P_3$} and D = {connected graphs on atmost four vertices}.*

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## I. Introduction

Strong domination is an interesting field of graph theory. In a graph $G = (V, E)$, a set $S$ of vertices is said to be strong dominating set if for every $v \in V S$, there exists a vertex $u \in S$ such that $UV \in E(G)$ and *deg u >deg v*. The subgraph induced by a strong dominating set is called strong dominating sub graph of $G$. In this paper for a finite or infinite class $D$ of connected graphs, the class $SD(D)$ of those graphs in which every connected induced subgraph contains a strong dominating induced subgraph isomorphic to some $D$ in $D$ are characterized. For a given $D$ we find $SD(D)$in terms of forbidden induced subgraphs. The class $SD(D)$ for $D = \{P_1, P_2, P_3\}$, $D = \{\{K_n/n > 1\} \cup P_3\}$ and $D$ = {connected graphs on atmost four vertices}.

**Definitions and Notations:**
Throughout this sections $D$ means a nonempty class of connected graphs, $V(G)$ and $E(G)$ denote the vertex set and edge set respectively. A graph $G$ is said to be minimal non-D-strong dominated if it is connected and has no strong dominating induced subgraph in $D$, but each of its proper connected induced subgraphs does have one.
A graph minimal not in $D$, if it is not in $D$ but all of its proper induced con nectedsubgraphs belong to $D$.
A graph is hereditarily strong dominated by $D$ if each of its connected induced subgraphs is strong dominated by some graph in $D$. The class of graphs hereditarily strong dominated by $D$ is denoted by $SD(D)$.
A graph $G$ is $F$-free if it does not contain $F$ as an induced subgraph. The class of $F$-free graphs is denoted by $F$ $orb(F)$. If $F$ is a family of graphs then $G$ does not contain each graph of $F$, so that $G$ is $F$−free.
A Leaf-graph of a graph $T$, denoted $F(T)$ is the graph obtained from $T$ by putting a leaf on each of its non-cutting vertices.
Partial leaf graph of a graph $T$ is obtained from $T$ by putting leaves on some of its vertices. Here one may put leaves on cut vertices too.
Compact class of connected graphs is a class closed under connected induced subgraphs.
Star-cutset is a vertex subset $S$ of $V(G)$ such that $GS$ is disconnected and there is an $s \in S$ adjacent to all vertices of $S\{s\}$.
**Theorem 1.1.***A graph G is hereditarily strong dominated by $D_1 = \{P_1, P_2P_3\}$ iff among its induced subgraphs there is no $C_6$ and no $F(L)$ where $L \in L_1$. Here $L_1 = \{K_3, P_4, K_{1,3}, C_4\}$*
***Proof.***Let $G$ be a graph hereditarily strong dominated by $D_1$. Suppose $G$ has $C_6$ or $F(L)$ as its induced subgraph. Neither $C_6$ nor $F(L)$ has strong domination in $D_1$. Let $G$ be a graph which is minimal non-$D_1$-strong dominated with non of the given subgraphs. If there is a cut point in $G$ then by cut point lemma which states that for a compact class $D$ of connected graphs, a graph $G$ with atleast one cutpoint is minimal non-$D$-strong dominated if it is isomorphic to some $F(L)$ where $L$ is a graph minimal not in $D$, then $G = F(L)$. Otherwise let $x$ be an arbitrary vertex of $G$. Here $Gx$is connected. Since $G$ is minimal non-$D_1$-strong dominating graph. It is enough to prove the following lemma.

**Lemma 1.2.***Suppose that G is a non-$D_1$-strong dominated star cutset free graph having no induced $F(C_4)$, $F(P_1)$, $F(K_{1,3})$, $C_6$. Then Gx= H is also non $D_1$-strong dominating graph.*

***Proof.***Suppose $H$ has some strong dominating induced subgraph$D \in D_1$. If $D = abca$ $P_3$ then since $G$ is connected $x$ has a neighbor $y$ in $H$ and $y$ has a neighbor in $D$. If $y$ has only one neighbor in $D$ then $G$ has a strong dominating $P_4$. But there is no $P_4$ because omitting any one of the end vertex, the remaining $P_3$ is not strong dominating. So leaves on end vertices are put. Hence a $P_6$ or $C_6$ is obtained depending on the adjacency of the leaves put. Here $P_6$ or $C_6$ is equal to $F(P_4)$ which is forbidden. Suppose $y$ has two neighbors in $D$ and these are $a$ and$c$. Then a $C_4$ is induced by $\{a, b, c, y\}$ is strong dominating. But there is no strong dominating $C_4$. Since $y$ is in strong dominating set it has a private strong dominated vertex $x$. If we remove $c$ we get a $P_4$ which is not strong dominating in $G$. Thus $c$ has a leaf. If we remove $b$, $G$ is not strong dominated by the resulting graph. Similarly if $a$ is removed the resulting graph is not strong dominating. Thus $a$ and$b$ have leaf. Therefore we get a $F(C_4)$ as an induced subgraph which is forbidden. Thus whenever there is some vertex $y$ adjacent to $x$ but not adjacent to the middle vertex of $P_3$ the lemma is proved.

**Lemma 1.3.***Let x, b be two vertices at distance two in a star-cutset free graph G. If every neighbor of x is the neighborhood of b then all vertices different from x and b are adjacent to both x and b.*

***Proof.***If $x$ has a non neighbor different from $b$, then we get a contradiction that $b$ and its neighbors adjacent to $x$ form a star cutset. Applying the above lemma to the previous induced graph we get y has neighbor $x$ which is adjacent to $b$. Thus if $D = P_3$ the graph $H$ is not strong dominating. If $D = P_2 = \{a, b\}$, assume there is no strong dominating induced $P_3$ in $H$. Then every vertex in $H$ is adjacent to both $a$ and$b$. Then $\{a, y\}$ inducing $P_2$ strong dominates $G$. Thus a contradiction. If $D = P_1$, applying the same procedure as for $P_2$ we get a contradiction. Therefore $H$ has no strong dominating subgraph in $D_1$ which contradicts that $G$ is minimal non-$D_1$-strong dominated graph.

**Theorem 1.4.***A graph G is hereditarily strong dominated by $D_2 = \{\{K_n/n > 1\} \cup P_3\}$ iff among its induced subgraphs there is no $C_6$ and no $F(L)$ where $L \in L_2$.*

***Proof.***Let $G$ be hereditarily strong dominated by $D_2$. Neither $C_6$ nor $F(L)$ has strong dominating subgraph in $D_2$ where $L \in L_2$ and $L_2 = \{K_4 - e, C_4, P_4, K_{1,3}\}$. Let $G$ be a minimal non-$D_2$-strong dominated graph which does not contain the given subgraphs and $x \in V(G)$. Here $G$ has no cut points and has no star cutset. Thus $H = Gx$is connected, $x$ has some neighbor $y$ in $H$ and $y$ has some neighbor in the strong dominating induced subgraph$D$ in $D_2$ of $H$. Suppose $D$ is a $P_l$ or $P_2$ or $P_3$ proof is same as in theorem 1.1. Suppose $D$ is a clique of size atleast 3. Then the graph $G_{n,i}$defined as a graph obtained from $K_{n+1}$ by removing $i$ edges incident to the vertex $y$ where $0 < i < n$ and $G_{n,0} = K_n$, is not strong dominating.

**Lemma 1.5.***Let $n > 2$, D be a partial Ieaf graph of $G_{n,i}$such that $D = G_{n,i}$ or y has a leaf and if there is any leaf on any neighbor of y then all the non neighbors have leaves. Such a D cannot, be Strong dominating in C.*

***Proof.***For $n = 2$, $D$ is one of the graphs $P_2$, $P_3$, $P_4$, $P_5$ or $P_3$ with pendants to each of its vertices in theorem 1.1 we have already discussed all these graphs. Suppose the lemma is valid for $n = 2$ to $n - 1$, we prove that it is valid for $n$. Let D be a strong dominating subgraph satisfying the conditions and $n > 3$. For $i = 0$ the lemma is true. Suppose $i > 2$, we can put leaves on two non neighbors$s$, $t$ as omitting one of them $D$ is not strong dominating by induction process. Similarly a second leaf can be put. But we obtain $F(PW)$ as an induced subgraph since $y$ has atleast one neighbor in $G_{n,i}$. This contradicts the definition of $G$. Therefore $i = 1$.

Also we prove that $n-i = 1$ Since$i < n-1$, we can prove $n < 1$. Suppose there existsatleast two neighbors. Here we put a leaf on $q$ if it does not have because $D - y$ is a clique and cannot be strong dominating. We have therefore there exists only one non-neighbor. We may put leaf on it. Now take a neighbor of $y$ and delete it, the remaining graph cannot strong dominate. Similarly we put leaf on this neighbor. Likewise leaf may be put on some other neighbors. But by this way we have constructed $F(K_4 - e)$ which is forbidden. Thus $n - i = 1$. Therefore $n = 2$ but we have assumed $n > 3$.

**Theorem 1.6.***In a connected graph, each connected subgraph has some con nected strong dominating subgraph with atmost four vertices if among the in ducedsubgraphs of G, there is no $C_7$ and no $F(L)$ for any connected five vertex graph L.*

***Proof.***Let $G$ be hereditarily strong dominated by $D_3$. Neither $C_6$ nor $F(L)$ has strong dominating subgraph in $D_3$ where $L \in L_3$.

$L = \{$Connected five vertex graph $\}$and $D_3 = \{$ Connected graphs with at-most four vertices$\}$. Let $G$ be a minimal non-$D_3$-strong dominated graph. Since some four vertex graph strong dominates $H = Gx,V(H) \cup \{y\}$ strong dominates $G$. Thus it is enough to prove that no connected five vertex strong dominating subgraph exists in $G$ and no truncated leaf-graph of any connected five vertex graph can be strong dominating $G$. We show that by induction on the number of leaves in decreasing order, namely, if there does not exist any strong dominating truncated leaf-graph of a connected five vertex graph, with a given number of leaves, then there does not exist any with one fewer leaf. The maximal truncated leaf-graph $F(D)$ of $D$, cannot be strong dominating in $G$ because it is excluded as an induced subgraph.

Consider some truncated leaf-graph and some missing leaf in it, namely some non-cutting vertex $u$ where we have not put leaf. $T = Du$ is connected, the given partial leaf-graph of $T$ cannot strong dominate. Therefore a leaf $u_1$ is put on $u$. Thus we have obtained that a further leaf can be attached to the original truncated leaf-graph. Next we prove that no strong dominating induced subgraph can be a partial 2 1eaf-graph of a connected four vertex graph. Take a partial leaf-graph of a connected four vertex graph T, and there is a leaf $u_j$on $u$. Omitting $u_l$, the partial leaf-graph is not strong dominating. Thus $u_1$ has a private strong dominated vertex and we obtain a strong dominating partial 2 leaf-graph which is a contradiction. Finally we prove that when G is a 2-connected, minirnal non-$D_3$-strong dominated graph, and $L$ a partial 2 1eaf graph of a connected three-vertex graph. Then $L$ cannot be strong dominating in $G$. Suppose it is false, that is if the partial 2 1eaf-graph $L$ of the connected four vertex graph $T$ has a leaf on every non cutting vertex, then the proof is done because $L$ contains an $F(D)$ where $D$ is the five vertex graph obtained from $T$ by adding the middle vertex of the 2-leaf. Otherwise there is some non-cutting vertex $U$ of $T$ that has no leaf. Omitting $u$, the resulting partial 2 leaf-graph of some connected three-vertex graph is contradicting. There are two possibilities for a connected three-vertex graph namely, the triangle and the $P_3$. Now we show that all their partial 2 1eaf-graphs are not strong dominating. For that we prove that no induced path can be strong dominating. $P_5$ is not strong dominating because omission of any one of the end vertex results in a $P_4$ which is not strong dominating. Thus end vertices have leaves. Therefore we obtain a $P_7$ or a $C_7$ depending on the adjacency of the leaves. Similarly it can be proved that $P_6$ is not strong dominating . The other partial 2 Ieaf-graph are either one of $P_5$, $P_6$ or they are isomorphic to $P_5$, $P_6$ after deletion of one of the end vertex. We prove that there is no strong dominating $F(K_{1,3})$ because $K_{1,3}$ is not strong dominating therefore there exists some vertex $r$ not strong dominated by $K_{1,3}$. It is strong dominated by the leaves on $K - 1, 3$. We may suppose that it has two neighbors. We prove that there is no strong dominating bull in $G$. A bull is $K_3$ with two of the vertices having leaves. Removal of a particular vertex results in a graph which is already proved to be not strong dominating. This holds for all the vertices of the bull. Thus the proof of 2-connected, minimal non-$D_3$-strong dominated graph $G$ and $L$ partial 2 1eaf-graph of a connected three-vertex graph. Then $L$ cannot be strong dominating in $G$.

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