Frame operator of K-frame in n-Hilbert space

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Abstract: In this paper we describe some properties of frame in n-inner product spaces. Characterizations between K-frames and quotient operators in n-inner product space are given.

Key Word: Frame, K-frame, quotient operator, n-inner product space, n-normed space.

I. Introduction

Reconstruction of functions using a family of elementary functions were first introduced by Gabor [6] in 1946. Later in 1952, Duffin and Schaeffer were introduced frames in Hilbert spaces in their fundamental paper [5], they used frames as a tool in the study of non-harmonic Fourier series. After some decades, frame theory was popularized by Daubechies, Grossman, Meyer [3]. A frame for a separable Hilbert space is a generalization of such an orthonormal basis and this is such a tool that also allows each vector in the space to be written as a linear combination of elements from the frame but, linear independence among the frame elements is not required. Several generalizations of frames namely, K-frame [8], Fusion frame [2], G-frame [14], etc. have been introduced in recent times. K-frames for a separable Hilbert space were introduced by Lara Gavruta. K-frame is more generalization than the ordinary frame and many properties of ordinary frame may not hold for such generalization of frame.

The concept of 2-inner product space was introduced by [4]. S. Gahler [7] introduced the notion of 2-normed space. H. Gunawan and Mashadi [9] developed the generalization of 2-norm space for n ≥ 2. The generalization of 2-inner product space for n ≥ 2 was developed by A. Misiak [13]. The notion of a frame in a n-inner product space has been presented by P. Ghosh and T. K. Samanta [10] and they also studied frame in tensor product of n-inner product spaces [11]. The author also presented K-frame and some its properties in n-Hilbert space [12].

In this paper, some properties of frame in n-Hilbert space are going to be established. We give a relationship between K-frame and quotient operators in n-Hilbert space.

Throughout this paper, X will denote a separable Hilbert space with the inner product <·,·> and B(X) denote the space of all bounded linear operator on X. We also denote R(T) for range set of T, N(T) for null space of T where T ∈ B(X) and l² denote the space of square summable scalar-valued sequences.

II. Preliminaries

Definition 2.1. [1] A sequence {f_i} of elements in X is said to a frame for X if there exist constants A, B > 0 such that

\[A \|f\|^2 \leq \sum_{i=1}^{\infty} |<f,f_i>|^2 \leq B \|f\|^2 \text{ for all } f \in X.\]

The constants A, B are called frame bounds. If the collection \{f_i\} satisfies

\[\sum_{i=1}^{\infty} |<f,f_i>|^2 \leq B \|f\|^2 \text{ for all } f \in X,\]

then it is called a Bessel sequence.

Definition 2.2. [1] Let \{f_i\} be a frame for X. Then the operator defined by \(T: l^2 \to X, T((c_i)) = \sum_{i=1}^{\infty} c_i f_i\) is called pre-frame operator and its adjoint operator given by \(T^*: X \to l^2, f \mapsto \{<f,f_i>\}\) is called the analysis operator. The frame operator is given by \(S: X \to X, S f = T T^* f = \sum_{i=1}^{\infty} <f,f_i> f_i\).

Definition 2.3. [8] Let K: X → X be a bounded linear operator. Then a sequence \{f_i\} in X is said to be a K-frame for X if there exist constants A, B > 0 such that

\[A \|K f\|^2 \leq \sum_{i=1}^{\infty} |<f,f_i>|^2 \leq B \|f\|^2 \text{ for all } f \in X.\]

Definition 2.4. [8] Let U, V : X → X be two bounded linear operator with N(U)⊂N(V). Then a linear operator \(T = [U/V]\) given by
Frame operator of K-frame in n-Hilbert space

Definition 2.5. [9] A real valued function $\|, \ldots, \| : X^n \rightarrow R$ is called a n-norm on $X$ if the following conditions hold:

(I) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

(II) $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutations of $x_1, x_2, \ldots, x_n$,

(III) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|$ for all $\alpha \in K$,

(IV) $\|x + y, x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|y, x_2, \ldots, x_n\|$. 

The pair $(X, \|, \ldots, \|)$ is then called a linear n-normed space.

Definition 2.6. [13] Let $n \in N$ and $X$ be a linear space of dimension greater than or equal to $n$ over the field $K$, where $K$ is the real or complex numbers field. A function $<, \ldots, , > : X^{n+1} \rightarrow K$ is satisfying the following five properties:

I. $<x_1, x_1 | x_2, \ldots, x_n> \geq 0$ and $<x_1, x_1 | x_2, \ldots, x_n> = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent.

II. $<x_i | x_2, \ldots, x_n> = <x_i, x_2, \ldots, x_n> = 0$ for every permutation $(i(2), \ldots, i(n))$ of $(2, \ldots, n)$,

III. $<x_i | x_2, \ldots, x_n> - <x, y | x_2, \ldots, x_n> = <x_i | x_2, \ldots, x_n> - <x, y | x_2, \ldots, x_n>$ for all $\alpha \in K$,

IV. $<x + y, z | x_2, \ldots, x_n> = <x, z | x_2, \ldots, x_n> + <y, z | x_2, \ldots, x_n>$, is called an n-inner product on $X$ and the pair $(X, <, \ldots, , >)$ is called n-inner product space.

Theorem 2.7. [13] For n-inner product space $(X, <, \ldots, , >)$, 

| $<x, y | x_2, \ldots, x_n>$ | $\geq 0$ | $\leq \|x, x_2, \ldots, x_n\| \|y, x_2, \ldots, x_n\|$ hold for all $x, y, x_2, \ldots, x_n \in X$. 

Theorem 2.8. [13] For every n-inner product space $(X, <, \ldots, , >)$, 

$\|x_1, x_2, \ldots, x_n\| = \sqrt{<x_1, x_1 | x_2, \ldots, x_n>}$ defines a n-norm for which

$<x, y | x_2, \ldots, x_n> = \frac{1}{4} (\|x + y, x_2, \ldots, x_n\|^2 - \|x - y, x_2, \ldots, x_n\|^2)$,

$\|x + y, x_2, \ldots, x_n\|^2 + \|x - y, x_2, \ldots, x_n\|^2 = 2 (\|x, x_2, \ldots, x_n\|^2 - \|y, x_2, \ldots, x_n\|^2)$ hold for all $x, y, x_2, \ldots, x_n \in X$. 

Definition 2.9. [9] A sequence $\{x_n\}$ in a linear n-normed space $X$ is said to be convergent to some $x \in X$ if for every $x_2, \ldots, x_n \in X$, $\lim_{n \rightarrow \infty} x_n = x$ and it is called a Cauchy sequence if $\lim_{n \rightarrow \infty} x_n = x$ for every $x_2, \ldots, x_n \in X$. The space $X$ is said to be complete if every Cauchy sequence in this space is convergent in $X$. A n-inner product space is called n-Hilbert space if it is complete with respect to its induce norm.

Note 2.10. [10] Let $L_F$ denote the linear subspace of $X$ spanned by the non-empty finite set $F = \{a_2, a_3, \ldots, a_n\}$, where $a_2, a_3, \ldots, a_n$ are fixed elements in $X$. Then the quotient space $X/L_F$ is a normed linear space with respect to the norm $\|x + L_F\| = \|x, a_2, \ldots, a_n\|$ for every $x \in X$. Let $M_F$ be the algebraic complement of $L_F$, then $X$ can be expressed as the direct sum of $L_F$ and $M_F$. Define $<x, y >_F = <x, y | a_2, \ldots, a_n>$. Then $<_-, _>_F$ is a semi-inner product on $X$ and this semi-inner product induces an inner product on $X/L_F$, which is given by

$<x + L_F, y + L_F>_F = <x, y >_F = <x, y | a_2, \ldots, a_n> \forall x, y \in X$. 

By identifying $X/L_F$ with $M_F$ in an obvious way, we obtain an inner product on $M_F$. Now for every $\in M_F$, we define $\|x\|_F = <x, x>_F$ and it can be easily verify that $(M_F, \|, \|_F)$ is a norm space. Let $X_F$ be the completion of the inner product space $M_F$.

For the remaining part of this paper, $(X, <, \ldots, , >)$ is consider to be a n-Hilbert space and $I$ will denote the identity operator on $X_F$.

Definition 2.11. [10] A sequence $\{f_i\}$ of elements in $X$ is said to be a frame associated to $(a_2, \ldots, a_n)$ for $X$ if there exist constants $0 < A \leq B < \infty$ such that

$T = \frac{U}{V} : R(V) \rightarrow R(U), T(Vx) = Ux$ is called quotient operator.
Frame operator of K-frame in n-Hilbert space

The constants $A, B$ are called frame bounds. If the collection $\{f_i\}$ satisfies
$$\sum_{i=1}^{\infty} |< f, f_i | a_2, \ldots, a_n >|^2 \leq B \| f, a_2, \ldots, a_n \|^2$$
for all $f \in X$. If it is called a Bessel sequence associated to $(a_2, \ldots, a_n)$ in $X$ with bound $B$.

Theorem 2.12. [10] Let $\{f_i\}$ be a sequence in $X$. Then $\{f_i\}$ is a frame associated to $(a_2, \ldots, a_n)$ with bounds $A$ and $B$ if and only it is a frame for the Hilbert space $X_F$ with bounds $A$ and $B$.

Definition 2.13. [10] Let $\{f_i\}$ be a Bessel sequence associated to $(a_2, \ldots, a_n)$ for $X$. Then the bounded linear operator defined by $T_F : l^2 \rightarrow X_F , T_F((c_i)) = \sum_{i=1}^{\infty} c_i f_i$ is called pre-frame operator and its adjoint operator given by $T_F^* : X_F \rightarrow l^2 , T_F^* f = \{ < f, f_i | a_2, \ldots, a_n > \}$ is called the analysis operator.

The frame operator is given by $S_F : X_F \rightarrow X_F , S_F f = \sum_{i=1}^{\infty} < f, f_i | a_2, \ldots, a_n > f_i$.

The frame operator $S_F$ is bounded, positive, self-adjoint and invertible.

Definition 2.14. [12] Let $K$ be a bounded linear operator on $X_F$. Then a sequence $\{f_i\}$ of elements in $X$ is said to be a $K$-frame associated to $(a_2, \ldots, a_n)$ for $X$ if there exist constants $0 < A \leq \infty$ such that
$$A \| f, a_2, \ldots, a_n \|^2 \leq \sum_{i=1}^{\infty} | < f, f_i | a_2, \ldots, a_n >|^2 \leq B \| f, a_2, \ldots, a_n \|^2$$
for all $f \in X_F$.

This can be written as $A \| K^* f, a_2, \ldots, a_n \|^2 \leq \sum_{i=1}^{\infty} | < f, f_i | a_2, \ldots, a_n >|^2 \leq B \| f, a_2, \ldots, a_n \|^2$.

III. Some properties of frame in n-Hilbert space

Theorem 3.1. Let $Y$ be closed subspace of $X_F$ and $P_Y$ be the orthogonal projection on $Y$. Then for a sequence $\{f_i\}$ in $X_F$ the following hold:

(i) If $\{f_i\}$ is a frame associated to $(a_2, \ldots, a_n)$ for $X$ with frame bounds $A,B$ then $\{P_Y f_i\}$ is a frame for $Y$ with the same bounds.

(ii) If $\{f_i\}$ is a frame for $Y$ with frame operator $S_Y$, then $P_Y f = \sum_{i=1}^{\infty} f, S_Y^{-1} f_i | a_2, \ldots, a_n > f_i$ for all $f \in X_F$.

Proof: By the definition of orthogonal projection of $X_F$ onto $Y$, we get
$$P_Y f = \begin{cases} f & \text{if } f \in Y \\ 0 & \text{if } f \notin Y \end{cases}$$

(i) Suppose $\{f_i\} \subseteq X_F$ associated to $(a_2, \ldots, a_n)$ for $X$ with frame bounds $A,B$. Then $\{f_i\}$ is a frame for $X_F$ with frame bounds $A,B$. So,
$$A \| f \|^2 \leq \sum_{i=1}^{\infty} | < f, f_i | a_2, \ldots, a_n >|^2 \leq B \| f \|^2$$
for all $f \in X_F$.

So by (1), the above inequality can be write as
$$A \| f \|^2 \leq \sum_{i=1}^{\infty} | < f, f_i | a_2, \ldots, a_n >|^2 \leq B \| f \|^2$$
for all $f \in Y$.

(ii) Let $\{f_i\}$ be a frame associated to $(a_2, \ldots, a_n)$ for $X$ with frame operator $S_Y$. Then it is easy to verify that $f = \sum_{i=1}^{\infty} f, S_Y^{-1} f_i | a_2, \ldots, a_n > f_i$ for all $f \in Y$. Therefore by (1), we get $P_Y f = \sum_{i=1}^{\infty} f, S_Y^{-1} f_i | a_2, \ldots, a_n > f_i$ for all $f \in Y$. Now, if $f$ belongs to the orthogonal complement of $Y$ then $< f, S_Y^{-1} f_i | a_2, \ldots, a_n > = 0$ and $P_Y f = 0$ if $f$ belongs to the orthogonal complement of $Y$. Therefore, $P_Y f = \sum_{i=1}^{\infty} f, S_Y^{-1} f_i | a_2, \ldots, a_n > f_i$ for all $f \in X_F$. This completes the proof.

Note 3.2. Let $\{f_i\}$ be a frame associated to $(a_2, \ldots, a_n)$ for $X$ and $f \in X_F$. If for some $\{c_i\} \subseteq l^2, f = \sum_{i=1}^{\infty} c_i f_i$, then
$$\sum_{i=1}^{\infty} | c_i |^2 = \sum_{i=1}^{\infty} | < f, S_Y^{-1} f_i | a_2, \ldots, a_n > |^2 + \sum_{i=1}^{\infty} | c_i - < f, S_Y^{-1} f_i | a_2, \ldots, a_n > |^2.$$

Theorem 3.3. Let $\{f_i\}$ be a frame associated to $(a_2, \ldots, a_n)$ for $X$ with pre-frame operator $T_F$. Then the pseudo-inverse of $T_F$ is described by $T_F^* : X_F \rightarrow l^2 , T_F^* f = \{ < f, S^{-1}_F f_i | a_2, \ldots, a_n > \}$.

Proof. By the Theorem (2.12), $\{f_i\}$ be a frame for $X_F$. Then for $f \in X_F$ has a representation $f = \sum_{i=1}^{\infty} c_i f_i$, for some $\{c_i\} \subseteq l^2$ and this can be written as $T_F((c_i)) = f$. By note (3.2), the frame coefficient $< f, S^{-1}_F f_i | a_2, \ldots, a_n >$ have minimal $l^2$-norm among all the sequences representing $f$. Hence, the above equation has a unique solution of minimal norm namely, $T_F^* f = \{ < f, S^{-1}_F f_i | a_2, \ldots, a_n > \}$.
Theorem 3.4. Let \( \{ f_i \} \) be a frame associated to \((a_2, \ldots, a_n)\) for \( X \), then the optimal frame bounds \( A, B \) are given by \( A = \| S_F^{-1} \|^{-1} = \| T_F \|^2, B = \| S_F \| = \| T_F \|^2 \), where \( T_F \) is the pre-frame operator, \( T_F^* \) is the pseudo-inverse of \( T_F \) and \( S_F \) is the corresponding frame operator.

Proof. By the definition, the optimal upper frame bound is given by 
\[
B = \sup \left\{ \sum_{i=1}^n | < f, f_i > a_2, \ldots, a_n > | \right\} = \sup \left\{ < S_f f, f > a_2, \ldots, a_n > | \right\} = \| S_F \|. \quad \text{(1)}
\]
Therefore, \( B = \| S_F \| = \| T_F \|^2 \). We know that the dual frame \( S_F^{-1} \) has frame operator \( S_F^{-1} \) and the optimal upper bound is \( A = A^{-1} \). So by the above similar process \( A^{-1} = \| S_F^{-1} \|^{-1} \) and this implies that \( A = \| S_F^{-1} \|^{-1} \). Now, from the Theorem (3.3), we obtain \( \| S_F^{-1} \| = \sup \left\{ \| T_F^* f \| \right\} \) for all \( f \). Thus, \( A = \| S_F^{-1} \|^{-1} = \| T_F \|^2 \). This completes the proof.

IV. Frame operator for \( K \)-frame

Theorem 4.1. Let \( \{ f_i \} \) be a Bessel Sequence associated to \((a_2, \ldots, a_n)\) for \( X \) with frame operator \( S_F \) and \( K \) be a bounded linear operator on \( X_F \). Then \( \{ f_i \} \) is a \( K \)-frame associated to \((a_2, \ldots, a_n)\) for \( X \) if and only if the quotient operator \( T = \begin{pmatrix} K^*/S_F & \frac{1}{S_F} \end{pmatrix} \) is bounded.

Proof. Let \( \{ f_i \} \) be a \( K \)-frame associated to \((a_2, \ldots, a_n)\) for \( X \). Then there exist positive constants \( A, B \) such that 
\[
A \| K^* f \|_F^2 = \sum_{i=1}^n | < f, f_i > a_2, \ldots, a_n > |^2 \leq B \| f \|_F^2 \quad \text{for all } f \in X_F. \quad \text{(2)}
\]
Since \( S_F \) is the corresponding frame operator, we can write 
\[
< S_f f, f > a_2, \ldots, a_n > = \sum_{i=1}^n | < f, f_i > a_2, \ldots, a_n > |^2 \quad \text{for all } f \in X_F. \quad \text{(3)}
\]
By (3), the inequality (2) can be written as 
\[
A \| K^* f \|_F^2 \leq \sum_{i=1}^n | < f, f_i > a_2, \ldots, a_n > |^2 \leq B \| f \|_F^2 \quad \text{for all } f \in X_F. \quad \text{(4)}
\]
Let us now define the operator \( T = \begin{pmatrix} K^*/S_F & \frac{1}{S_F} \end{pmatrix} : R_+ S_F^2 f \rightarrow R(K^* \), by \( T \left( \frac{1}{S_F^2} f \right) = K^* f \forall f \in X_F \).

Now, let \( f \in N \left( \frac{1}{S_F^2} \right) \). Then \( \frac{1}{S_F^2} f = \theta \) implies that \( \left\| \frac{1}{S_F^2} f \right\|_F^2 = 0 \), so by (4), \( A \| K^* f \|_F^2 = 0 \). This implies that \( K^* f = \theta \) implies \( f \in N(K^*) \) and this implies that \( N \left( \frac{1}{S_F^2} \right) \subseteq N(K^*) \). This shows that the quotient operator \( T \) is well-defined. Also for all \( f \in X_F \), 
\[
\left\| T \left( \frac{1}{S_F^2} f \right) \right\|_F = \| K^* f \|_F \leq \frac{1}{\| S_F \|} \| S_F^2 f \|_F. \quad \text{Hence, } T \text{ is bounded.}
\]
Conversely, suppose that the quotient operator \( T \) is bounded. Then there exists \( B > 0 \) such that 
\[
\left\| T \left( \frac{1}{S_F^2} f \right) \right\|_F \leq B \| S_F^2 f \|_F \quad \forall f \in X_F. \quad \text{(5)}
\]
This implies that \( \| K^* f \|_F \leq B \| S_F^2 f \|_F \)
\[
= B \| S_F f, f > a_2, \ldots, a_n > | \quad \text{[ since } S_F^2 \text{ is also self-adjoint]}
\]

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Also, \( \{ f_i \} \) be a Bessel Sequence associated to \((a_2, \ldots, a_n)\) in \(X\), so there exists \(C > 0\) such that
\[
\sum_{i=1}^{\infty} |< f, f_i | a_2, \ldots, a_n >|^2 \leq B \| f, a_2, \ldots, a_n \|^2 \quad \text{for all } f \in X_F
\] (6).
Hence, from (5) and (6), \( \{ f_i \} \) is a \( K \)-frame associated to \((a_2, \ldots, a_n)\) for \(X\).

**Theorem 4.2.** Let \( \{ f_i \} \) is a \( K \)-frame associated to \((a_2, \ldots, a_n)\) for \(X\) with the frame operator \(S_F\) and \(T\) be a bounded linear operator on \(X_F\). Then the following are equivalent:

1. Let \( \{ T f_i \} \) is a \(TK\)-frame associated to \((a_2, \ldots, a_n)\) for \(X\).
2. \( U = \left( \frac{(TK)^*}{\frac{1}{S_F^2} T^*} \right) \) is bounded.
3. \( V = \left( \frac{(TK)^*}{(TS_F^2 T^*)^\frac{1}{2}} \right) \) is bounded.

**Proof.** (1)\(\Rightarrow\) (2) Suppose \( \{ T f_i \}_{i=1}^{\infty} \) is a \(TK\)-frame associated to \((a_2, \ldots, a_n)\) for \(X\). Then there exists constant \(A, B > 0\) such that
\[
A \| (TK)^* f \|^2 \leq \sum_{i=1}^{\infty} |< f, T f_i | a_2, \ldots, a_n >|^2 \leq B \| f \|^2 \quad \forall \ f \in X_F
\] (7)
Since \(S_F\) is the corresponding frame operator, we can write
\[
< S_F f, f | a_2, \ldots, a_n > = \sum_{i=1}^{\infty} |< f, f_i | a_2, \ldots, a_n >|^2 \quad \forall \ f \in X_F
\]
Now,
\[
\sum_{i=1}^{\infty} |< f, T f_i | a_2, \ldots, a_n >|^2 = \sum_{i=1}^{\infty} |< T f, f_i | a_2, \ldots, a_n >|^2 = < S_F (T^* f), T^* f | a_2, \ldots, a_n >
\]
\[
= \frac{1}{S_F^2} (T^* f) . S_F^2 (T^* f) | a_2, \ldots, a_n > = \left\| \frac{1}{S_F^2} (T^* f) \right\|^2_F
\]
Let us now consider the quotient operator,
\[
\left( (TK)^*/S_F^2 T^* \right) : R (\frac{1}{S_F^2} T^*) \rightarrow R ( (TK)^* ) \quad \text{by} \quad \left( \frac{1}{S_F^2} T^* \right) f \rightarrow (TK)^* f \quad \forall \ f \in X_F
\]
From (7), we can write
\[
A \| (TK)^* f \|^2 \leq \left\| \frac{1}{S_F^2} (T^* f) \right\|^2_F \quad \forall \ f \in X_F
\]
\[
\Rightarrow \| (TK)^* f \|^2 \leq \frac{A}{B} \left\| \frac{1}{S_F^2} (T^* f) \right\|^2_F \quad \forall \ f \in X_F
\]
This shows that the quotient operator \( (TK)^*/S_F^2 T^* \) is bounded.

(2)\(\Rightarrow\) (3) suppose that the quotient operator \( (TK)^*/S_F^2 T^* \) is bounded.

Then there exists constant \(B > 0\) such that
\[(\langle T K \rangle f, f \rangle)^\frac{1}{2} \leq B \left\| \frac{1}{2} (T^* f) \right\|^2_{\mathcal{F}} \quad \forall f \in X_{\mathcal{F}} \quad (8)\]

Now for such \(f \in X_{\mathcal{F}}\), we have

\[\left\| \frac{1}{2} (T^* f) \right\|^2_{\mathcal{F}} = <S_{\mathcal{F}} T^* f, T^* f| a_2, ..., a_n> = <T S_{\mathcal{F}} T^* f, f| a_2, ..., a_n> = \frac{1}{2} (T^* S_{\mathcal{F}} T^* f) \right\|^2_{\mathcal{F}} \quad (9)\]

From (8) and (9), we get

\[\left\| (T K)^* f \right\|_{\mathcal{F}} \leq B \left\| (T S_{\mathcal{F}} T^*) \right\|^\frac{1}{2} f) \right\|^2_{\mathcal{F}} \quad \forall f \in X_{\mathcal{F}}.\]

Hence, the quotient operator \([T K] / (T S_{\mathcal{F}} T^*)\) is bounded.

(3) \(\Rightarrow\) (1) suppose the quotient operator \([T K] / (T S_{\mathcal{F}} T^*)\) is bounded.

Then there exists constant \(B > 0\) such that

\[\left\| (T K)^* f \right\|_{\mathcal{F}} \leq B \left\| (T S_{\mathcal{F}} T^*) f \right\|^\frac{1}{2} \quad \forall f \in X_{\mathcal{F}}. \quad (10)\]

It is easy to verify that \(T S_{\mathcal{F}} T^*\) is self-adjoint and positive and hence the square root \(T S_{\mathcal{F}} T^*\) of exists. Now, for each \(f \in X_{\mathcal{F}}\), we have

\[\sum_{i=1}^{\infty} |<f, T f_i | a_2, ..., a_n>|^2 = \sum_{i=1}^{\infty} |<T^* f, f_i | a_2, ..., a_n>|^2 = \left< S_{\mathcal{F}} (T^* f), T^* f | a_2, ..., a_n> = \left< S_{\mathcal{F}} T^* f, S_{\mathcal{F}} T^* f | a_2, ..., a_n> = \left< \left(\left(\frac{1}{2} S_{\mathcal{F}} T^*\right)^* S_{\mathcal{F}} T^* f, f | a_2, ..., a_n> = \left< T S_{\mathcal{F}} T^* f, f | a_2, ..., a_n> = \left\| (T S_{\mathcal{F}} T^*) \right\|^\frac{1}{2} f \right\|_{\mathcal{F}} \right\| \quad (11)\]

From (10) and (11),

\[\frac{1}{B} \left\| (T K)^* f \right\|_{\mathcal{F}} \leq \sum_{i=1}^{\infty} |<f, T f_i | a_2, ..., a_n>|^2 \quad \forall f \in X_{\mathcal{F}}\]

On the other hand, since \(\{f_i\}_{i=0}^{\infty}\) is a K-frame associated to \((a_2, ..., a_n)\),

\[\sum_{i=1}^{\infty} |<f, T f_i | a_2, ..., a_n>|^2 = \sum_{i=1}^{\infty} |<T^* f, f_i | a_2, ..., a_n>|^2 \leq C \|T^* f \right\|^2_{\mathcal{F}}\]

Hence, \(\{T f_i\}_{i=1}^{\infty}\) is a TK-frame associated to \((a_2, ..., a_n)\), for \(X\).
References