The binomial coefficients in the Riemann Wave background. 
A possible proof of the Riemann Hypothesis

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Abstract: We consider the series with positive summands given in [1] for the first Li-Keiper coefficient λ1. We first carry out a numerical experiment to characterize the speed at which the above series contains (in memory) big amounts of the zeros on the critical line. Then we look at the truth of the RH as an extremal possible grow of λ1 by means of all sets of zeros in the critical strip. Later, in the last section, we formulate the linear Equation for the coefficients φn related to the Li-Keiper coefficients λn. Then, we conclude with a possible proof of the correctness of the Riemann Wave background and thus of a (possible) proof of the RH.

Key words: The first increment of the Li-Keiper coefficients, nontrivial zeros, Series for the first coefficient λ1, “Maxi-Min” criterion for the Riemann Hypothesis, Binomials coefficients, Partition Function, Free Energy, Riemann wave background, Riemann Hypothesis (RH).

I. Introduction

In this work, we first consider the original series obtained by Matiyasevich in his pioneering work [1], i.e. the expression for the first Li-Keiper coefficient λ1 = γ/2 + 1/2 - log(4π) ~ 0.0230957...

as a sum of positive summands [1] (See also[2, 3] for contributions in this direction) and related to the binary system; it is given by [1]:

\[ \lambda_1 = \sum_{\mu} \frac{1}{\mu} = \left(\frac{1}{2}\right) \cdot \sum_{n=3}^{\infty} \frac{(2 \cdot N_1(n) + 3)}{\left[(2 \cdot n) \cdot (2 \cdot n + 1) \cdot (2 \cdot n + 2)\right]} = \sum_{n=3}^{\infty} f(n) \] (1)

where \( N_1(n) \) is the number of units in the binary representation of \( n \). (Notice that if \( N_0 \) is the number of zeros in the representation of \( n \), \( N_1(n) + N_0(n) = \lfloor \log_2(2 \cdot n) \rfloor \), where the symbol \( \lfloor \rfloor \) denotes the ceiling.

Eq.(1) is a sum of positive summands thus increasing with \( n \). It may also be written as a decreasing sequence using \( N_0 \) in the above Formula. With the following relation [1]

\[ \gamma = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\left[ \log_2(2 \cdot n) \right]}{[(2 \cdot n) \cdot (2 \cdot n + 1) \cdot (2 \cdot n + 2)]} \]

then we have:

\[ \gamma - \frac{67}{127} = \sum_{n=3}^{\infty} \frac{\left[ \log_2(2 \cdot n) \right]}{[(2 \cdot n) \cdot (2 \cdot n + 1) \cdot (2 \cdot n + 2)]} \]

i.e.

\[ \lambda_1 = \gamma - \frac{67}{127} + \frac{1}{2} \cdot \sum_{n=3}^{\infty} \frac{1}{[(2 \cdot n) \cdot (2 \cdot n + 1) \cdot (2 \cdot n + 2)]]} \]
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\[ + \sum_{n=3}^{\infty} \frac{(-N_0(n))}{(2\cdot n) \cdot (2 \cdot n + 1) \cdot (2 \cdot n + 2)} = \]

\[ = 0.0188823316 + 0.010279229 + \sum_{n=3}^{\infty} \frac{(-N_0(n))}{(2\cdot n) \cdot (2 \cdot n + 1) \cdot (2 \cdot n + 2)} = \]

\[ = 0.029161560 + \sum_{n=3}^{\infty} \frac{(-N_0(n))}{(2\cdot n) \cdot (2 \cdot n + 1) \cdot (2 \cdot n + 2)} \]

i.e. a constant (the first term) plus a sum of negative summands analogous to Eq. (1).

In the following numerical heuristic experiment, we consider Eq.(1) and compare each term in the summation with the amount given by the set of the reciprocal zeros (on the critical line) of the Zeta function; the values of the zeros are available from the Tables of Odlyzko [4]. We know that if \( \lambda \) exhausts exactly the sum of the reciprocal zeros on the critical line, then this is equivalent to the truth of the Riemann Hypothesis. Here, we analyze the “speed” at which the above series subsums such reciprocal nontrivial zeros.

We define:

\[ M_1 \]

For \( n=3 \) in Eq.(1), \( N_1(3) = 2 \), amount:\( M_1 = f(n=3) = (1/2) \cdot \left( \frac{7}{6 \cdot 7 \cdot 8} \right) = (1/2) \cdot (1/48) = 1/96 = 0.010416666667. \)

For the zeros on the critical line we define:

\[ S_m = \sum_{k=1}^{m} \left( \frac{1}{\xi_k^2} \right) \]

For \( m=4 \) we obtain:

\[ S_4 = \frac{1}{(0.25+14.134725142^2)} + \frac{1}{(0.25+21.02023963^2)} + \frac{1}{(0.25+25.01085758^2)} + \frac{1}{(0.25+30.424876126^2)} = 0.00993850678 < 0.010416666... \]

For \( n=3 \rightarrow M_1 = f(3) = 0.01041666.. ; S_4 = 0.00993850678 < M_1. \)

\[ M_2 = f(3)+f(4) = (1/96) +(1/2) \cdot \left( \frac{7}{6 \cdot 7 \cdot 8} \right) = (1/2) \cdot (1/48) = 1/72 = 0.013888888888.. \]

For \( m=11 \) (levels up to \( t_{11} = 52.970 \ldots \)) we have \( S_{11} = 0.0138443286.. \)

For \( n=3 \) and \( n=4 \rightarrow M_2 = 0.0138888.. ; S_{11} = 0.0138443286 < M_2. \)

\[ M_3 = f(3)+f(4)+f(5) = 1/72 + 1/8 + 1/10 = 0.016540404040.. \]

For \( m=42 \) (levels up to \( t_{42} = 127.516 \ldots \)) we have \( S_{42} = 0.0165404040.. \)

For \( n=3, n=4 \) and \( n=5: M_3 = 0.01654040.. ; S_{42} = 0.0165 4040 ~ M_3. \)

Thus, three terms in the above Matiyasevich series comprises the contribution of the reciprocal values of the first 42 zeros on the critical line. We continue the analysis for an additional information on this point, and compute \( M_4, \)

Below, the Table 1 of the values of :=\( M_n = f(3)+f(4)+f(5)+f(6), n=1..up \) to \( n=11. \)

<table>
<thead>
<tr>
<th>( M_n )</th>
<th>0.018142968</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_m )</td>
<td>0.019482253</td>
</tr>
<tr>
<td>( M_n )</td>
<td>0.01992874</td>
</tr>
<tr>
<td>( M_{10} )</td>
<td>0.020504570</td>
</tr>
<tr>
<td>( M_{10} )</td>
<td>0.020883358</td>
</tr>
<tr>
<td>( M_{10} )</td>
<td>0.021253911</td>
</tr>
<tr>
<td>( M_{10} )</td>
<td>0.021478270</td>
</tr>
</tbody>
</table>

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### Table 1.

<table>
<thead>
<tr>
<th>Level</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>M_{11}</td>
<td>0.021707208</td>
</tr>
<tr>
<td>M_{12}</td>
<td>0.021891937</td>
</tr>
<tr>
<td>M_{13}</td>
<td>0.022076749</td>
</tr>
</tbody>
</table>

Remark: From $t_4 = 129.578704200$, we may compute the sum of nontrivial reciprocal zeros (as an approximation) using the weight

$$dN(t) = \frac{1}{2\pi} \cdot \log(t/2\pi) \cdot dt$$

for the function $1/(1/4+t^2)$, i.e.

$$\int_{129.578}^{216.169} dN(t) \cdot \left( \frac{1}{\left(\frac{1}{4} + t^2\right)} \right) = 0.001602134.$$

Let $S_{89} = 0.016540404 + 0.001602134 = 0.018142538 < 0.0181429681 = M_4$.

Let $M_4 = 0.018142968 \cdot S_{89} = 0.018142358 < M_4$.

$M_4$ comprises (~) the amount of the first 89 reciprocal zeros with the series. Below as an illustration we construct the plot of the levels $M_N$ for very small values of $N$ and the corresponding calculated amount of the subset of the reciprocal zeros ($S_m$) using the Tables.

**Fig. 1.** The first 5 levels $M_1$..$M_5$ and the level $\lambda_4 = 0.023957$...

Below in Fig. 2 we present the plot of $M_n$ here as a function of $(1/N)$ where $N$ is the number of zeros corresponding to $M_n$. 

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II. A criterion for the truth of the R.H. assuming that the zeros are simple

With the above small numerical experiment on the series $M_n$ of positive summands we was looking at the "speed" of the series and this suggests a complement in our study of $\lambda_1$ alone i.e. – the following first definition of an increment $\Delta_1$. Let

$$\lambda_0 = \sum_{\rho} \left( 1 - \left( 1 - \frac{1}{\rho} \right)^0 \right) = 0$$

where $\rho$ runs on all non trivial zeros.

We now consider the first "increment" [8] which we define as:

$$\Delta_1 := \sum_{\rho} \left( 1 - \left( 1 - \frac{1}{\rho} \right)^1 \right) - \sum_{\rho} \left( 1 - \left( 1 - \frac{1}{\rho} \right)^0 \right)$$

(3)

The sum is on all the zeros of the Zeta function in the critical strip, i.e. (without any further assumptions) of the general form $s=\sigma_0+i\cdot t_k$ and $s=(1-\sigma_k)\pm i\cdot t_k$ ($1/2 \leq \sigma_k \leq 1$, $t_k>0$ for all $k=1,2,...$).

With this definition we then have:

$$\Delta_1 = \sum_{(\sigma_k,t_k)} \left[ \frac{2 \cdot \sigma_k}{(\sigma_k^2 + t_k^2)^2} + \frac{2 \cdot (1 - \sigma_k)}{((1 - \sigma_k)^2 + t_k^2)^2} \right] = \sum_k R_k(\sigma_k, t_k)$$

(4)

We notice that $R_k$ has an extremum (a maximum) for all $k$ at $\sigma_k=1/2$.

In fact the Taylor expansion around $\sigma_k=1/2$ is given by:

$$R_k = R_k(\sigma_k=1/2, t_k) - A(t_k)\cdot(\sigma_k-1/2)^3 + O((\sigma_k-1/2)^4), A > 0 \text{ for all } t_k.$$

From above, explicitly:
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\[ R_k' = 2/(1/4+t_k^2) \cdot 6 \cdot (1/(1/4+t_k^2)) \cdot (t_k^2-1/12)\cdot(( \sigma_k-1/2)^3)+O(( \sigma_k-1/2)^4) \] (5)

Neglecting the last term in Eq. (5) we represent below (for the first level \( t_1 = 14.134725 \ldots \) ), \( R_1 \) and \( R_1' \) as a function of \( \sigma = x \) for \( 0 \leq x \leq 1 \), using Eq.(4).

![Graph showing \( R_1 \) and \( R_1' \) as a function of \( \sigma = x \) for \( 0 \leq x \leq 1 \)].

Moreover, \( R_k \) has the Riemann symmetry, i.e. \( R_k(\sigma_k,t) = R_k(1-\sigma_k, t) \) and \( R_k(\sigma_k=1/2,t_k) = 2/(1/4+t_k^2) \).

**Criterion 1**

"RH is true if \( \Delta_1 \) as defined by Eq.(3),Eq.(4) assume the maximum possible value"; but then \( \Delta_1 \approx 2, \lambda_1 \approx 2.0230957 \).

Moreover, if a nontrivial zero is off the critical line, big fluctuations are known to occur in the lamda's \( \lambda_n \) at high values of \( n \) and \( t \). See Appendix 2 for the general formula of \( \Delta_n \).

We consider another definition related to Eq.(4).

We assume that the zeros are simple; then for \( \sigma_k=1/2 \), we should divide \( \Delta_1 \) by 2, i.e.

\[ \Delta_1' = (1/2) \cdot \Delta_1 \]

because these are the simple zeros of the Zeta function for \( \sigma_k=1/2 \) and

\[ \Delta_1' = \sum (1/2) \cdot R_k(\sigma_k=1/2,t_k) = \sum \left( \frac{1}{\lambda_k+1/4+t_k^2} \right) \]

since for all \( t_k \), \( 1/(1/4+t_k^2) < 2/(1+t_k^2) \). From this, we have the following criterion:

**Criterion 2**

"RH is true if, assuming the zeros are simple, \( \Delta_1 \) assume the minimum possible value"; i.e.

\[ f(\sigma_k, t_k) = \sum \left( 1 - \left( 1 - \frac{1}{\rho} \right)^3 \right) \]

is an absolute minimum for any fixed distribution of energy levels \( \{t_k \} \). This concludes our Maxi-Min criterion for the RH.
III. The first few Li-Keiper coefficients: the Binomial Coefficients in the Riemann Wave background.

The first one, contains the two fundamental constants $\gamma$ (the Euler-Mascheroni constant and $\pi$).

$$\lambda_1 = 1 + \gamma / 2 - (1/2) \cdot \log(4 \cdot \pi) = 0.0230957...$$  (6)

As it is known, this value is independent of the truth of the RH.

For the second coefficients $\lambda_2$ we have the lower bound:

$$\lambda_2 > 4 \cdot \lambda_1 - \lambda_1^2$$  (7)

In fact, $(1/2) \cdot (\lambda_2 + \lambda_1^2) - \lambda_1 > \lambda_1$ as was proved in [8].

Eq.(7) give us with Eq.(6) that $\lambda_2 > 0.091849...$

We have

$$\varphi_2 - \varphi_1 = \frac{1}{2} \left( \frac{\xi''(s)}{\xi(s)} \right) \bigg|_{s=1} \quad \text{and} \quad \varphi_2 - 2 \cdot \varphi_1$$  (8)

with $\varphi_1 = \lambda_1, \varphi_2 = (1/2) \cdot (\lambda_2 + \lambda_1^2)$.

For the third coefficient with the definition of it i.e.

$$\lambda_3 = (1/2!) \cdot d^3/ds^3(s^2 \cdot \log(\xi(s))) \bigg|_{s=1}$$

where the $\xi$ function is given by

$$\xi(s) = (1/2) \cdot s \cdot (s - 1) \cdot \pi^{s/2} \cdot \Gamma(s/2) \cdot \zeta(s)$$  (9)

we have:

$$\lambda_3 = (1/2) \cdot d^3/ds^3(s^2 \cdot \log(\xi(s))) \bigg|_{s=1} = 3 \cdot (\xi'/\xi)' \bigg|_{s=1} + 3 \cdot (\xi''/\xi) \bigg|_{s=1} + (\xi'''/\xi) \bigg|_{s=1}$$

using the same for $\lambda_2 = 2 \cdot (\xi'/\xi)' \bigg|_{s=1} + (\xi''/\xi) \bigg|_{s=1}$, with $\lambda_1 = (\xi'/\xi) \bigg|_{s=1}$.

We find

$$\lambda_3 = (3 \cdot (3/2) \cdot \lambda_1) \cdot \lambda_2 - 3 \cdot \lambda_1 + 3 \cdot \lambda_1^2 - (1/2) \cdot \lambda_1^3 + (1/2) \cdot (\xi'''/\xi) \bigg|_{s=1}$$

We now apply our inequality (Eq. (7)) and we obtain:

$$\lambda_3 > (4 \cdot \lambda_1 - \lambda_1^2) \cdot (3 \cdot (3/2) \cdot \lambda_1) - 3 \cdot \lambda_1 + 3 \cdot \lambda_1^2 - (1/2) \cdot \lambda_1^3 + (1/2) \cdot (\xi'''/\xi) \bigg|_{s=1}$$

that is

$$\lambda_3 > 9 \cdot \lambda_1 - 6 \cdot \lambda_1^2 + \lambda_1^3 + (1/2) \cdot (\xi'''/\xi) \bigg|_{s=1} > 9 \cdot \lambda_1 - 6 \cdot \lambda_1^2 + \lambda_1^3$$  (10)

Since it is known that $d^4/ds^4 \xi(s) \bigg|_{s=1} > 0$ [9],

In fact if we consider the Riemann series around $s=1/2$ [9], given by:

$$\xi(s) = \sum_{n=0}^{\infty} a_{2n} \cdot \left( s - \frac{1}{2} \right)^{2n}$$
In the expansion, all coefficients are positive, i.e. $a_{2n}>0$ for all $n$. Thus for $n=1, 2, 3$ we have:

$$\xi'(1) = \sum_{n=0}^{\infty} a_{2n} \cdot (2 \cdot n) \left(\frac{1}{2}\right)^{2n-1} = a_2 + \left(\frac{1}{2}\right) \cdot a_4 + \left(\frac{3}{16}\right) \cdot a_6 + \cdots > 0$$

$$\xi''(1) = \sum_{n=0}^{\infty} a_{2n} \cdot (2 \cdot n) \cdot (2 \cdot n-1) \left(\frac{1}{2}\right)^{2n-2} = a_2^2 + 3 \cdot a_4 + \left(\frac{15}{8}\right) \cdot a_6 + \cdots > 0$$

$$\xi'''(1) = \sum_{n=0}^{\infty} a_{2n} \cdot (2 \cdot n) \cdot (2 \cdot n-1) \cdot (2 \cdot n-2) \left(\frac{1}{2}\right)^{2n-3} = 12 \cdot a_4 + 15 \cdot a_6 + \cdots > 0$$

$$\left(\frac{d^n}{ds^n}\xi(s)\right)|_{s=1} > 0 \quad \forall n. \quad (10')$$

(Notice that $a_0 = 0.4977120.. \approx 0.5$!, $a_2 = 0.011485..$ and...$a_3 = 0.00012345..$, $a_0 = 8.10^{\cdot7}$...)

Moreover, with the first three terms we obtain:

$$\xi''(1)/\xi'(1) = 2 \cdot (a_2 + (1/2) \cdot a_4 + (3/16) \cdot a_6 + \cdots) =$$

$$= 2 \cdot (0.0011485) + (1/2) \cdot 0.00012345\sim$$

$$\sim 0.023033... < \lambda_1 = 0.0230957...$$

Here for symmetry we write the inequalities using the three functions $\varphi_1, \varphi_2, \varphi_3$ (emerging for every $n$ as $\varphi_n$) in the expansion of the associated partition function [8]. We have for $n=1, 2, 3, 4 ...$

$$\varphi_1 = (1) \cdot \lambda_1 + 0.$$

$$\varphi_2 = (1/2) \cdot (\lambda_2 + \lambda_1^2)$$

$$\varphi_3 = (1/3) \cdot (\lambda_3 + (3/2) \cdot \lambda_1 \cdot \lambda_2 + (1/2) \cdot \lambda_1^3)$$

$$\varphi_4 = (1/4) \cdot (\lambda_4 + (4/3) \cdot \lambda_1 \cdot \lambda_3 + (1/2) \cdot \lambda_2^2 + \lambda_1^2 \cdot \lambda_2 + (1/6) \cdot \lambda_1^4)$$

Now, for $n=3$ we have obtained from the definition, i.e. $\lambda_3 = d^3/ds^3 (s^2 \cdot \log(\xi(s)))|_{s=1}$:

$$3 \cdot \varphi_3 - 6 \cdot \varphi_2 + 3 \cdot \varphi_1 = (1/2) \cdot (\xi'''/\xi')(1). \quad (11)$$

For example,

$$\varphi_3 - 2 \varphi_2 + \varphi_1 = \frac{1}{3!} \left(\frac{\xi'''}{\xi'}(s)\right)|_{s=1}$$

Equality i.e. Eq.(11) may be pursued for $n \geq 3$. Notice that if we apply the inequality above, i.e. $\lambda_2 > 2 \cdot \lambda_1 - \lambda_1^2$, we obtain $\lambda_3 > 3 \cdot \lambda_1 - 3 \cdot \lambda_1^2 + \lambda_1^3 = 0.067699$ (the true value is 0.020763...).

With $\lambda_2 > 2 \cdot \lambda_1 - \lambda_1^2$, $\lambda_2 > 0.045657$ (the true value is 0.0923457...).

If we apply our bound of Eq.(8) we obtain $\lambda_3 > 0.204673$. The true value is $\lambda_3 = 0.20763...$

Remark:

The right hand side of Eq.(10) is the third term (in $z=1/s$) of our lower bound i.e. of the Riemann wave background, with the appearance of the Koebe function $K(z)$ as argument of the $\log(1+\lambda_1, K(z))$ [8].

In fact, the first three terms of the above expansion of the logarithm at $z=0$, are given by:

$$\log(1+\lambda_1, K(z)) = \log(1 + \lambda_1, z -(1-z)^2) = \lambda_1 \cdot z + (1/2) \cdot (4 \cdot \lambda_1 \cdot \lambda_2^2) \cdot z^2 + (1/3) \cdot (9 \cdot \lambda_1 \cdot 6 \cdot \lambda_1^2 + \lambda_1^3) \cdot z^3 + O(z^4).$$

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Equality by (11) may be pursued for n>3. In fact, we have calculated 
\( \lambda_4 = (1/3!)(d^3/ds^3(s^3\log(\xi(s)))) \) and we obtain:
\[ 4 \cdot \phi_4 - 12 \cdot \phi_2 - 4 \cdot \phi_1 - (1/6) \cdot \xi'''(1) \]
i.e. \( \phi_4 = 3 \cdot \phi_3 + 1 \cdot \phi_1 - 3 \cdot \phi_2 + 1 + (1/4!) \cdot (\xi''' / \xi)(1) \)
Here, to start with the true values we verify that \(- \phi_4 > 4 \cdot \phi_1\).
Thus, \( \phi_2 > 4 \cdot \lambda_1 \), i.e. \( 4 \cdot \lambda_4 > 20 \lambda_2 + 8 \lambda_3 = 0.358961 \ldots \) (the true value is 0.368…).
In fact, the first four terms of the expansion of the above logarithm at \( z = 0 \), are given by:
\[ \log(1 + \lambda_2 \cdot K(z)) = \log(1 + (1/2) \cdot (4 \cdot \lambda_2 - \lambda_3)^2 - (1/3) \cdot (4 \cdot \lambda_2 - \lambda_3)^3 + (1/4!) \cdot (4 \cdot \lambda_2 - \lambda_3)^4 + O(z^5) \).

### IV. Concluding Remark: A possible proof of The RH

In this work we have analyzed the Li–Keiper coefficients both in a numerical as well as in an analytical context.
For the first, i.e. \( \lambda_1 \), we have started with the series of positive summands and carried out a numerical experiment looking at the "speed" at which such a series (connected with the binary system) contains amounts of the reciprocal values of the nontrivial zeros on the critical line, and we have obtained the behavior given in the plot of \( M_n \) as a function of \( 1/N \), where \( N \) is the number of zeros related to \( M_0 \).
That analysis suggested to us a definition of the first increment connected with \( \lambda_1 \) with the proposal of a criterion for the truth of the RH using \( \lambda_1 \).
For \( \lambda_2 \) we have used our established lower bound to obtain a lower bound on \( \lambda_3 \), using also the positivity property of the n-th derivative of the Riemann \( \xi \) function at the border of the critical strip, i.e. at \( s = 1 \) [9].
Finally, we have:
- **Binomial structure into the Riemann wave background;**
- **A possible proof of the Riemann Hypothesis.**

We now write: \( 1 \cdot \phi_1 = (1) \cdot \lambda_1 \)

\[ 2 \cdot (1 \cdot \phi_2 - 1 \cdot \phi_1) = \frac{1}{2} \cdot \xi''(1) \]

\[ 3 \cdot (1 \cdot \phi_3 - 2 \cdot \phi_2 + 1 \cdot \phi_1) = \frac{1}{6} \cdot \xi'''(1) \]

\[ 4 \cdot (1 \cdot \phi_4 - 3 \cdot \phi_3 + 3 \cdot \phi_2 - 1 \cdot \phi_1) = \frac{1}{24} \cdot \xi''''(1) \]  
(12)

We recognize in the parenthesis the binomial coefficients, where:

\[ (1 - 1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \cdot \frac{n!}{k!} = 0 \]
a signal, coming from the structure above of the binomials for the concrete appearance of the Riemann wave background [8].
For \( n = 3 \):
\[ \phi_3 = 2 \cdot \phi_2 - \phi_1 + \frac{1}{3!} \cdot \left( \frac{\xi''''(1)}{\xi(1)} \right) \]
(13)
from Eq. (12), \( \phi_4 = 3 \cdot \phi_3 - 3 \cdot \phi_2 + 1 \cdot \phi_1 \cdot (1/4!) \cdot (\xi''''/\xi)(1) \) and from above.
\[ \phi_4 > 3 \cdot (2 \cdot \phi_2 - \phi_1) - 3 \cdot \phi_2 + 1 \cdot \phi_1 \cdot (1/4!) \cdot (\xi''''/\xi)(1) + (1/3!) \cdot (\xi'''/\xi)(1) \]
(14)
\[ \phi_5 > 3 \cdot \phi_4 - 2 \cdot \phi_3 + \phi_1 \cdot (1/5!) \cdot (\xi''''/\xi)(1) \]
(15)

To conclude, the general structure is as follows:
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\[ \sum_{k=0}^{n-1} (-1)^{2n-k} \frac{(n-1)!}{k!} \phi_{n-k} = \frac{1}{\Gamma(n+1)} \left( \frac{d^n}{dx^n} \xi \right) \bigg|_{x=1} \]

and this gives \( \phi_n \geq n \cdot \phi_1 \). for all \( n \).
The binomial structure, and Eq.(15) ensures the correctness of our Riemann Wave background [8], and then the
truth of the RH by means of \( \log(1+\lambda_1 \gamma, 1-\rho) \cdot K(z) \).
Moreover, if we sum the right hand sides of Eq.(16) over \( n \), we obtain the number:

\[ \pi \left( \frac{\xi'}{\xi} (2) \right) = \pi \left[ \frac{1}{2} (\gamma + \log(4\pi) + 3) - 12\log A \right] = 0.072325988 \]

Where \( A \) is the Glaisher-Kinkelin constant.

Since:

\[ \zeta'(-1) = \frac{1}{12} - \ln A = -0.1654211937 \]

Then, with \( N_0(n) \) of Eq. (2) [1]:

\[ \pi \left[ \frac{3}{2} \left( 1 + \frac{1}{2} \ln 2 + 12\zeta'(-1) + \sum_{n=1}^{\infty} \frac{N_0(n)}{2n(2n+1)(2n+2)} \right) \right] \]

which gives the constant: 0.0723...

The same number is obtained with the primes \( \left( \frac{\xi'}{\xi} (2) \right) \).

\[ \pi \left[ \frac{3}{2} \left( -\frac{1}{2} \ln \pi - \gamma - \sum_{\rho=2}^{\infty} \frac{\ln \rho}{\rho^2 - 1} \right) \right] \]

Where the sum is on all the primes \( \rho \).
Then, for \( \lambda_2 \) we obtain the upper bound:

\[ \lambda_2 < 2 \cdot 0.072325988 - (\lambda_1)^2 = 0.14411856. \]

The true value is: \( \lambda_2 = 0.092 \ldots \sim 0.10 \ldots \)

Summing the left hand sides of Eq.(16) over \( n \), we have:

\[ \sum_{n=1}^{\infty} \phi_n - \sum_{n=1}^{\infty} \phi_n \cdot n + \sum_{n=1}^{\infty} \phi_n \cdot \frac{1}{2} \cdot n \cdot (n-1) - \frac{1}{6} \sum_{n=1}^{\infty} \phi_n \cdot n \cdot (n+1)(n+2) + \cdots \]

A very strong cancellation.
Moreover:

\[ \pi \left( \frac{\xi'}{\xi} (s) \right) \bigg|_{s=2} = \pi \left[ \frac{d}{dz} \ln K(z) \right] \cdot \left( \frac{dz}{ds} \right)_{s=2} = \pi \frac{12}{\pi^2} \sum_n \lambda_n \cdot \left( \frac{1}{2} \right)^{n-1} \]

Which gives the constant 0.072323...

The sum of the left hand side of Eq.(16) (linear in the \( \phi' \)) gives linearity in the Li-Keiper coefficients, i.e. from above:

\[ \sum_{n=1}^{\infty} \lambda_n \cdot \left( \frac{1}{2} \right)^{n-1} = \left( \frac{\xi'}{\xi} (2) \right)_{s=2} \]

The binomial structure Eq.(16), and Eq.(15) ensures the correctness of our demonstration [8].
If we reconsider the nontrivial zeros we have from above:

\[ \frac{\pi}{12} \sum_n \lambda_n \cdot \left( \frac{1}{2} \right)^{n+1} = \pi \frac{d}{dz} \left[ \ln \left( \prod_{\rho} \left( 1 - z + \frac{1}{1-\rho} \right) \right) \right] \bigg|_{z=2} = \]

\[ \frac{\pi}{12} \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{s-(1-\rho)} \right) \bigg|_{s=2} = \frac{\pi}{3} \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{s-(1-\rho)} \right) \bigg|_{s=2} = \frac{\pi}{3} \sum_{\rho} \left( \frac{3}{(2-\rho)(1+\rho)} \right) \bigg|_{s=2} < \pi \cdot \lambda_1 = \]

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\[ Z = \text{Trace} \exp(-\beta \cdot H) \] and the free energy by:

\[ -\beta \cdot f = \log(Z(\beta, J)) = \left( \frac{1}{2} \right) \left( \frac{1}{2\pi} \right)^2 \int_0^{2\pi} d\alpha_1 \int_0^{2\pi} d\alpha_2 \log(1 + z^2 - z(\cos(\alpha_1) + \cos(\alpha_2))) \]

where \( z = \sinh(2k) = \sinh(2\cdot\beta \cdot J) \) (\( \beta \) is the inverse temperature and -J the interaction strength between two neighboring spins on the square lattice).

The zeros in \( z \) are on the unit circle \( |z| = 1 \) and the critical temperature is given by \( z = \sinh(2\cdot\beta \cdot J) = 1 \).

The critical line for this model is obtained by the usual transformation \( z = 1 - 1/s \) and the argument of \( \log(Z) \) is now given by:

\[ \log(1 + (1 - 1/s)^2 - s \cdot c) \] where \( c = \cos(\alpha_1) + \cos(\alpha_2) \)

the zeros in \( s \) are on the critical line, now given by \( s = \sigma + i \cdot \tau_s = 1/2 \pm i \cdot \tau_s \) where

\[ \tau_k = \tau_{\alpha}(c) = \sqrt{\frac{2+c}{4(2-c)}} \] (notice that \( |c| \leq 2 \)).

We compute:

\[ \lambda_1 = \sum_{\tau_k} \left( \frac{1}{(z_\pm i \cdot \tau_k)} \right) = \left( \frac{1}{2} \right) \cdot \left( \frac{1}{2\pi} \right)^2 \int_0^{2\pi} d\alpha_1 \int_0^{2\pi} d\alpha_2 \cdot \left( \frac{1}{4} \cdot \frac{2 + c}{4 \cdot (2 - c)} \right) = \]

\[ = \left( \frac{1}{2} \right) \cdot \left( \frac{1}{2\pi} \right)^2 \int_0^{2\pi} d\alpha_1 \int_0^{2\pi} d\alpha_2 \cdot (2 - c)(\alpha_1, \alpha_2) = 1 \rightarrow \lambda_1 = 1. \]

The same value of \( \lambda_1 \) is obtained with \( \lambda_{1s} \), d/ds (\( \log(Z(s)) \)) |\( s_1 \) =

\[ = \left( \frac{1}{2} \right) \cdot \left( \frac{1}{2\pi} \right)^2 \int_0^{2\pi} d\alpha_1 \int_0^{2\pi} d\alpha_2 \cdot \frac{(2 \cdot (2 - c) \cdot s - (2 - c))}{(1 + s^2 \cdot (2 - c) - s \cdot (2 - c))} \]|\( s_1 \) = 1

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This conclude an analogy with the 2-d Ising model in zero field for the first coefficient $\lambda_1$.

**Appendix 2**

As started in [8], keeping in mind that for large n a Li-Keipe coefficient $\lambda_n$ become negative if there is the presence of at least a nontrivial zero off the critical line -here- we merely look at the increments $\Delta_n$ which are defined as follows

$$\Delta_n \equiv \sum_{\rho} \left( 1 - \frac{1}{\rho} \right)^{n-1} \cdot \left( \frac{1}{\rho} \right)^n$$

(A21)

where the sum is on all nontrivial zeros $\rho = \sigma \pm i \cdot t$ and $1 - \rho = 1 - \sigma \pm i \cdot t$ (if we assume that the nontrivial zeros are simple, $\Delta_n$, for $\sigma = 1/2$, should be divided by 2. (We omit here the indices k for $\rho = \sigma_k \pm i \cdot t_k$).

Then without any assumption the amount is given by:

$$\Delta_n = \sum_{\rho} \left( 2 \cdot A_2^{n-1} \cdot \sin((n) \cdot \alpha_1 + (n-1) \cdot \alpha_2) / A_1^n + 2 \cdot A_1^{n-1} \cdot \sin((n-1) \cdot \alpha_1 + (n) \cdot \alpha_2) / A_2^n \right)$$

(A22)

where $A_1 = |\sigma \pm i \cdot t| = (\sigma^2 + t^2)^{1/2}$, $A_2 = |1 - \sigma + i \cdot t| = ((1 - \sigma)^2 + t^2)^{1/2}$

$\alpha_1 = \arctan(\sigma / t)$ and $\alpha_2 = \arctan((1 - \sigma) / t)$.

Notice that if $\sigma = 1/2$ , $A_1 = A_2 , \alpha_1 = \alpha_2$ and $\Delta_n$ reduces to

$$\Delta_n (\sigma = \frac{1}{2}) = \sum_{\rho} 4 \cdot \left( \sin(2 \cdot n - 1) \cdot \arctan\left( \frac{1}{2} \cdot t \right) \right) \cdot \left( \frac{1}{\sqrt{\left( \frac{1}{4} + t^2 \right)}} \right)$$

(A23)

For $n=1$, the above Formula gives:

$$\Delta_1 = \frac{2}{\sqrt{(\sigma^2 + t^2)}} \cdot \sin\left( \arctan\left( \frac{\sigma}{t} \right) \right) + \frac{2}{\sqrt{((1 - \sigma)^2 + t^2)}} \cdot \sin\left( \arctan\left( \frac{1 - \sigma}{t} \right) \right)$$

Since

$$\sin(\alpha_1) = \frac{\sigma}{\sqrt{(\sigma^2 + t^2)}} \quad \text{and} \quad \sin(\alpha_2) = \frac{(1 - \sigma)}{\sqrt{((1 - \sigma)^2 + t^2)}}$$

we obtain:

$$\Delta_1 = \sum_{(\sigma,t)} \left[ \frac{2 \cdot \sigma}{(\sigma^2 + t^2)} + \frac{2 \cdot (\sigma - 1)}{((1 - \sigma)^2 + t^2)} \right]$$

(A24)

which is the same as Eq.(4). In the next pages we give as illustration some plots concerning the fluctuations using Eq.(A22), (variable n, t, $\sigma$).
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Fig. A21. Plot of $\Delta_{10}(t_1=14.134725, \sigma)$ as a function of $\sigma(n=10)$

Fig. A22. Plot of $\Delta_{22}(t_1=14.134725, \sigma)$ as a function of $\sigma(n=22)$ showing a "borderline" for the onset of minimal fluctuations at the first level $t_1(n=22)$

Fig. A23. Plot of $\Delta_{23}(t_1=14.134725, \sigma)$ as a function of $\sigma(n=23)$
The binomial coefficients in the Riemann Wave background. A possible proof of the...
The binomial coefficients in the Riemann Wave background. A possible proof of the..
The binomial coefficients in the Riemann Wave background. A possible proof of the.. 

Fig. A30. $\Delta_n(14.134725, \sigma=1/2)$ in red and $\Delta_n(14.134725, \sigma=0 \text{ or } \sigma=1)$ in red (border of the critical strip: here without assuming that the zeros are simple: the curves are superimposed).

In the last Figure, we present the difference i.e. $\Delta_n(14.134725, \sigma=1/2) - \Delta_n(14.134725, \sigma=0)$.

Fig. A31. The difference indicated as above, at $t>30$ is positive, and this is more in favour of the content of Fig. A29 where the simplicity of the zeros was assumed with the indication of the minimal fluctuations at $\sigma=1/2$.