Exact traveling wave solutions for the (2+1)-dimensional Burgers equation via $\exp(-\phi(\eta))$ – expansion method

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Abstract
In this work, the exact traveling wave solutions to the (2+1)-dimensional Burgers equation are studied using the $\exp(-\phi(\eta))$ – expansion method. The traveling wave solutions are expressed in terms of the exponential functions, the hyperbolic functions, the trigonometric functions and the rational functions. The extracted solution plays a significant role in numerous types of scientific investigation such as in nonlinear optics, nuclear physics, magnetic field etc. This method is one of the powerful methods that appear in recent time in establishing some new exact traveling wave solution to the nonlinear partial differential equations. It is shown that the $\exp(-\phi(\eta))$ – expansion method is simple and valuable mathematical instrument for solving nonlinear evolution equation in mathematical physics and engineering.

Keywords: $\exp(-\phi(\eta))$ – expansion method, the (2+1)-dimensional Burgers equation; traveling wave solutions, solitary wave solution.

I. Introduction
There are many physical mechanisms of natural phenomena in this earth is often described by nonlinear evolution equations (NLEEs). It has many wide ranges of application of scientific and engineering fields, as for instance, in fluid mechanics, plasma physics, optical fibers, solid state physics, system identification, and nonlinear opticetc. The numerous applications of analytical solutions to nonlinear partial differential equation indicate that there is a significant demand for better mathematical algorithms with real objects and processes and thus lead to further applications. Therefore, the investigation of the exact traveling wave solutions for nonlinear partial differential equations plays a vital role in the study of nonlinear wave phenomena. In recent decades, there have been significant improvements in the study of exact solutions for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations. As a result, many powerful methods for finding exact solutions of nonlinear partial differential equations, such as, the expansion functions method [1], the modified simple equation method [2], the Jacobi elliptic function expansion method [3-4], the Adomian decomposition method [5], the homogeneous balance method [6-7], the F-expansion method [8], the Backlund transformation method [9], the Darboux transformation method [10], $\exp(-\phi(\xi))$-expansion method [11], the auxiliary equation method [12], the inverse scattering transform [13], the complex hyperbolic function method [14], the $(G'/G)$-expansion method [15-17], the novel $(G'/G)$-expansion method [18-20], the new generalized $(G'/G)$-expansion method [21-23], the $\exp(-\phi(\eta))$ – expansion method [24-27] and so on.

The (2+1)-dimensional Burgers equations is an important class of nonlinear partial differential equation to analyze the basic properties of nonlinear propagation of many physical phenomena, such as amplitude and width of the solitons, solitary wave structure and shock wave structure. A large number of literatures [9-18], where nonlinear Burgers equation is studied and have demonstrated analytical solutions as well as travelling wave solution using different methods. However, the purpose of this research is to extract the new exact solutions of the (2+1)-dimensional Burgers equations using the $\exp(-\phi(\eta))$ – expansion method that appeared in recent time.

II. Description of the $\exp(-\phi(\eta))$ – expansion method
Let us consider a general PDE in the form
\[ F(u, u_t, u_{tt}, u_{ttt}, u_{txx}, \ldots, \ldots) = 0, \ldots \ldots \ldots \ldots \ldots \ldots (1) \]
where $u = u(x,t)$ is an unknown function, $F$ is a polynomial in $u(x,t)$ and its derivatives in which highest order derivatives and nonlinear terms are involved and the superscripts stand for the partial derivatives. In the following, we give the main steps of this methods:

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Step 1: We combine the real variables x and t by a complex variable η

\[ u(x, t) = u(\eta), \quad \eta = x \pm V t \]  

where V is the speed of the traveling wave. The traveling wave transformation \( (2) \) converts Eq. (1) into an ordinary differential equation (ODE) for

\[ R(u, u', u'', \ldots) = 0 \]  

where \( R \) is a polynomial of \( u \) and its derivatives and the superscripts indicate the ordinary derivatives with respect to \( \eta \).

Step 2: Suppose the traveling wave equation of Eq. (3) can be expressed as follows:

\[ u(\eta) = \sum_{i=0}^{N} A_i \exp(-\phi(\eta)) \]  

where \( A_i \) \((0 \leq i \leq N)\) are constant to be determined, such that \( A_N \neq 0 \) and \( \phi = \phi(\eta) \) satisfies the following ordinary equation:

\[ \phi'(\eta) = \exp(-\phi(\eta)) + \mu \exp(\phi(\eta)) + \lambda \]  

Eq. (5) gives the following solutions:

**Family 1:** When \( \mu = 0 \), \( \lambda^2 - 4\mu > 0 \),

\[ \phi(\eta) = \ln(\tanh(2\mu V \eta)) \]  

**Family 2:** When \( \mu = 0 \), \( \lambda^2 - 4\mu < 0 \),

\[ \phi(\eta) = \ln(\tanh(-2\mu V \eta)) \]  

**Family 3:** When \( \mu = 0 \), \( \lambda \neq 0 \), \( \lambda^2 - 4\mu > 0 \),

\[ \phi(\eta) = -\ln(\exp(\frac{\lambda\eta + E}{V})) \]  

**Family 4:** When \( \mu = 0 \), \( \lambda = 0 \), \( \lambda^2 - 4\mu = 0 \),

\[ \phi(\eta) = \ln(\frac{2\mu V \eta}{\eta + E}) \]  

**Family 5:** When \( \mu = 0 \), \( \lambda = 0 \), \( \lambda^2 - 4\mu = 0 \),

\[ \phi(\eta) = \ln(\frac{2\mu V \eta}{\eta + E}) \]  

where \( A_N, \ldots, V, \lambda, \mu \) are constants to be determined latter. \( A_N \neq 0 \), the positive integer N can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (3).

Step 3: We substitute Eq. (4) into Eq. (3) and then we account the function \( \exp(-\phi(\eta)) \). As a result of this substitution, we get a polynomial in the form of \( \exp(-\phi(\eta)) \). We equate all the coefficients of same power of \( \exp(-\phi(\eta)) \) to zero. This procedure yields systems of algebraic equations whichever can be solved to find \( A_N, \ldots \). Substituting the values of \( A_N, \ldots \) into Eq. (4) along with the general solutions of Eq. (5) completes the determination of the solution of Eq. (1).

**III. Application of the method**

In this section, the method is used to construct some new traveling wave solution of the (2+1) dimensional Burgers equation which is very important nonlinear evolution equations in mathematical physics and engineering.

3.1 The (2+1) dimensional Burgers equation

In this section, we will present the method of \( \exp(-\phi(\eta)) \)-expansion method to construct the exact solution and the solitary wave solution of the (2+1) dimensional Burgers equation. Let us consider the Burgers equation,

\[ u_t - uu_x - u_{xx} - u_{yy} = 0 \]  

(11)

We utilize the traveling wave variable Let us consider the (2+1)-dimensional Burgers equation \( u(\eta) = u(x, t) \), \( \eta = x - V t \). Eq. (11) is carried to an ODE

\[ -V u_t - uu' - 2u'' = 0 \]  

(12)

Eq. (12) is integrating, therefore, integrating with respect to \( \eta \) once yields:

\[ V u + \frac{1}{2} u^2 + 2u + P = 0 \]  

(13)

Where P is an integration constant which is to be determined.
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Taking the homogeneous balance between highest order nonlinear term \( u'^2 \) and linear term of the highest order \( u' \) in equation in Eq. (13), we obtain \( N = 1 \). Therefore, the solution of Eq. (13) is of the form:

\[
u(\eta) = A_0 + A_1 \exp(-\phi(\eta)) 
\]

Where \( A_0, A_1 \) are constant to be determined such that \( A_N \neq 0 \), while \( \lambda, \mu \) are arbitrary constants. It is easy to see that

\[
u'(\eta) = -A_1 \exp(-2\phi(\eta)) + \mu \lambda \exp(-\phi(\eta)) 
\]

\[
u^2(\eta) = A_0^2 + 2A_0A_1 \exp(-\phi(\eta)) + A_1^2 \exp(-2\phi(\eta)) 
\]

Substituting \( u, u', u^2 \) into Eq. (13) and then equating the coefficients of \( \exp(-\phi(\eta)) \) to zero, we get

\[
\frac{1}{2} A_1^2 - 2A_1 = 0 
\]

\[
V A_1 + A_0 A_1 - 2A_1 \lambda = 0 
\]

\[
V A_0 + \frac{1}{2} A_0^2 - 2A_1 \lambda + P = 0 
\]

Solving the Eqs. (17)-(19) yields

\[
P = \frac{1}{2} A_0^2 - 2A_0 \lambda + 8 \mu, \quad V = -A_0 + 2 \lambda, \quad A_0 = A_0, \quad A_1 = 4, 
\]

Where \( \lambda, \mu \) are arbitrary constants. Now substituting the values of \( V, A_0, A_1 \) into Eq. (14) yields

\[
u(\eta) = A_0 + 4 \exp(-\phi(\eta)) 
\]

Where \( \eta = x - (A_0 + 2 \lambda) t \)

Now substituting Eqs. (6)-(10) into Eq. (20) respectively, we get the following five traveling wave solutions of the (2+1)-dimensional Burgers equation.

When \( \mu \neq 0, \quad \lambda^2 - 4 \mu > 0 \),

\[
u_1(\eta) = A_0 - \frac{8 \mu}{\sqrt{\lambda^2 - 4 \mu} \tanh \left( \frac{\sqrt{\lambda^2 - 4 \mu}}{2} (\eta + E) \right) + \lambda} 
\]

Where \( \eta = x - (A_0 + \lambda) t \) and \( E \) is an arbitrary constant.

When \( \mu \neq 0, \quad \lambda^2 - 4 \mu < 0 \),

\[
u_2(\eta) = A_0 + \frac{8 \mu}{\sqrt{4 \mu - \lambda^2} \tanh \left( \frac{\sqrt{4 \mu - \lambda^2}}{2} (\eta + E) \right) - \lambda} 
\]

Where \( \eta = x - (A_0 + \lambda) t \) and \( E \) is an arbitrary constant.

When \( \mu = 0, \lambda \neq 0, \quad \lambda^2 - 4 \mu > 0 \),

\[
u_3(\eta) = A_0 + \frac{4 \lambda}{\exp\left( \lambda(\eta + E) \right) - 1} 
\]

Where \( \eta = x - (A_0 + \lambda) t \) and \( E \) is an arbitrary constant.

When \( \mu = 0, \lambda \neq 0, \quad \lambda^2 - 4 \mu = 0 \),

\[
u_4(\eta) = A_0 + \frac{2 \lambda^2 (\eta + E)}{\left( \lambda(\eta + E) \right) + 2} 
\]

Where \( \eta = x - (A_0 + \lambda) t \) and \( E \) is an arbitrary constant.

When \( \mu = 0, \lambda = 0, \quad \lambda^2 - 4 \mu = 0 \),

\[
u_5(\eta) = A_0 + \frac{4}{(\eta + E)} 
\]

Where \( \eta = x - (A_0 + \lambda) t \) and \( E \) is an arbitrary constant.

IV. Graphical representation and physical explanations

In this section we will discuss the physical explanations and graphical representation of the above determined five families of solutions.

4.1 The (2+1)-dimensional Burgers equation

The (2+1)-dimensional Burgers equation demonstrates the coupling between dissipation effect of \( u_{xx}, u_{yy} \) and the convection process of \( u_{t} u_{x} \). Burgers introduced this equation to capture some of the features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion. Solution \( u_1(\eta) \) represents kink waves which are traveling waves. The kink solutions are approach to a constant at infinity. Fig. 1 below the shape of the exact kink-type solution of \( u_1(\eta) \) of the (2+1)-dimensional Burgers equation. Solution \( u_2(\eta), u_3(\eta), u_4(\eta), \) and \( u_5(\eta) \) are the singular kink solution. Figs. 2-5 show the shape of the exact singular Kink-type solution \( u_2(\eta), u_3(\eta), u_4(\eta), \) and \( u_5(\eta) \) of the (2+1)-dimensional Burgers equation.

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Figure 1 shows the kink wave solution $u_1(\eta)$ when $A_0 = 1$, $\mu = 1$, $\lambda = 3$, $E = 1$ and $-10 \leq x, t \leq 10$.

Figure 2 shows the kink wave solution $u_2(\eta)$ when $A_0 = 1$, $\mu = 3$, $\lambda = 1$, $E = 1$ and $-10 \leq x, t \leq 10$.

Figure 3 shows the kink wave solution $u_3(\eta)$ when $A_0 = 1$, $\mu = 0$, $\lambda = 2$, $E = 1$ and $-10 \leq x, t \leq 10$.

Figure 4 shows the kink wave solution $u_4(\eta)$ when $A_0 = 1$, $\mu = 1$, $\lambda = 2$, $E = 1$ and $-10 \leq x, t \leq 10$. 
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Figure 5 shows the wave solution \( u_5(\eta) \) when \( A_0 = 1, \mu = 0, \lambda = 0, E = 1 \) and \(-10 \leq x, t \leq 10\).

V. Conclusion:

In this paper, we have extracted the exact wave solutions for the nonlinear partial differential of (2+1)-dimensional Burgers equations using the \( \exp(-\phi(\eta)) \)-expansion method. The obtained solutions might be useful in analyzing the propagation of gravity waves in ocean, liquid flow, fluid flow in elastic tubes, waves in rivers and lakes in a smaller domain, etc. The efficiency of the \( \exp(-\phi(\eta)) \)-expansion method is more reliable and easier than the other methods to determine the exact solutions of the (2+1)-dimensional Burgers equations. The method might be fundamental for further research of different nonlinear differential equations in theoretical physics, mathematical physics and other branches of nonlinear science.

References

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