Conjugacy Classes and Action of $\Delta(3,4,k)$ on $PL(F_q)$

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Abstract: The triangle group $\Delta(3,4,k)$ can be defined as $<r,s; r^3 = s^4 = (rs)^k = 1>$, where $r, s$ are the generators of the group. In this paper, we have discussed conjugacy classes that arises from the actions of $\Delta(3,4,k)$ on $PL(F_q)$. Here, $F_q$ is a finite field for any prime $q$ and $PL(F_q) = F_q \cup \infty$. A relation between conjugacy classes of a homomorphism and parameters of $F_q$ has also drawn by using computer coding scheme.

Keywords: Conjugacy classes, Linear-fractional transformations, Parameterization and Non-degenerate homomorphism.

I. Introduction

It is well known [2, 3] that $\Delta' = \Delta^3(2, Z)$ is the group of linear-fractional transformations of the form $z \rightarrow \frac{az + b}{cz + d}$ where $a, b, c, d \in Z$, $ad - bc \neq 0$. This group is generated by $r, s$ satisfying the relations $r^3 = s^4 = 1$. (1.1)

It is also proved in [2, 3] that if a linear-fractional transformation $t$ inverts both $r$ and $s$, that is, $t^2 = (rt)^2 = (st)^2 = 1$, then we get an extended group $\Delta'' = G^{3,4}(2, Z)$ which is again a group of transformations having form

$z \rightarrow \frac{az + b}{cz + d}, a, b, c, d \in Z$

The defining relations of this extended group are:

$\Delta' < r, s, t; r^3 = s^4 = t^2 = (rt)^2 = (st)^2 = 1>$. (1.2)

Thus we can define the group $G^{3,4}(2, q)$ as the group of linear-fractional transformations of the form $z \rightarrow \frac{az + b}{cz + d}$ where $a, b, c, d \in F_q$ and $ad - bc \neq 0$. We can also define a group $G^{3,4}(2, q)$ as a subgroup of $G^{3,4}(2, Z)$ such that $ad - bc$ is a non-zero square in $F_q$ [5]. It is well known in [7, 8] that triangle group $\Delta(k, l, m)$ is finite precisely when $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} > 1$, and infinite in case of $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} \leq 1$. $\Delta(2,4, k)$ is infinite for $k \geq 4$, whereas for $k = 1, 2, 3$ triangle group $\Delta(2,4, k)$ is $C_2, D_0, S_4$ respectively [8, 9]. A general description of triangle group $\Delta(3,4, k)$ having representation $<r, s; r^3 = s^4 = (rs)^k = 1>$ can be found in [1, 4, 6]. It is also known that by adjoining an involution $t$, which inverts both $r$ and $s$, the groups $\Delta(3,4, k)$ can be extended to the triangle groups $\Delta(3,4, k) = < r, s, t; r^3 = s^4 = (rs)^k = t^2 = (rt)^2 = (st)^2 = 1>$. The triangle group $\Delta(3,4, k)$ is of index 2 in $\Delta(3,4, k)$ and so is normal in $\Delta' = G^{3,4}(2, Z)$.

II. Parameters of Conjugacy Classes for $\Delta' = G^{3,4}(2, Z)$

Let $\alpha: G^*(2, Z) \rightarrow G^*(2, q)$ be a homomorphism. Choose $r = r\alpha, s = s\alpha$ and $t = t\alpha$ in $G^*(2, q)$ satisfying $r^3 = s^4 = t^2 = (rt)^2 = (st)^2 = 1$. (2.1)

This homomorphism $\alpha$ is termed as ‘non-degenerate’ if $r$ and $s$ have same orders as that of $(r)\alpha$ and $(s)\alpha$ respectively. It means none of the generators $r, s$ lies in kernel of $\alpha$ so that their images $\bar{r} = r\alpha, \bar{s} = s\alpha$ are of orders $3$ and $4$ respectively.

If a natural map $GL(2,q) \rightarrow G^*(2,q)$ maps matrix $M$ to an element $g$ of $G^*(2,q)$, then $\theta = (\text{trace}(M))^2/\text{det}(M)$ is called invariant of conjugacy class of $g$. It can be pertained as parameter of element $g$ or of conjugacy class. Actions of $G(2,Z)$ on $PL(F_q)$, via $\alpha$, will be considered so that $g$ be taken as $(rs)\alpha = \bar{r} \bar{s}$. Hence, $\theta$ is the parameter of the class containing $\bar{r} \bar{s}$. We can also establish a relation between $\alpha$ and $\theta \in F_q$. It can be proved very easily that if $R$ and $S$ are two non-singular $2 \times 2$ matrices corresponding to the generators $\bar{r}$ and $\bar{s}$ of $\Delta'$ with $\text{det}(RS) = 1$ and $\text{trace}(RS) = m_2$, then $RS$ satisfy the following characteristic equation:

$$(RS)^2 - m_2 RS + 1 = 0$$
\[(RS)^2 = m_2RS - I \quad (2.2)\]

Multiplying both sides of this equation by \(S\), we get:
\[(RS)^3 = m_2(RS)^2 - (RS)I \quad (2.3)\]

By putting equation (2.2) in equation (2.3), we obtain
\[(RS)^3 = (m_2^2 - 1) - m_2I \quad (2.4)\]

On recursion, we get
\[(RS)^k = [(k - 1)0) m_2^{k-1} - (k - 2 0 \) m_2^{k-2} - (k - 3 1) m_2^{k-3} + \ldots ] RS - [(k - 2 0) m_2^{k-2} - (k - 3 1) m_2^{k-3} + \ldots ] I \quad (2.5)\]

For some important result is necessary to prove Theorem 3.2.

**Lemma 3.1:** For a non-singular \(2 \times 2\) matrix, if its trace is zero then it represents an involution provided its entries are from \(F_q\).

**Theorem 3.2:** Let \(r, s\) be any two elements of \(G^{3,4}(2, q)\) and \(R, S\) be their corresponding matrices respectively, then \(m_2 = \sqrt{2}m_2 - 1 = 0\), where \(m_2\) is the trace of matrix \(RS\).

**Proof:** Consider two elements \(r, s\) of \(G^{3,4}(2, q)\), such that order of \(r\) is 3 whereas that of \(s\) is 4. Let \(R = [r_1, r_2, r_3, r_4]\) and \(S = [s_1, s_2, s_3, s_4]\) be their corresponding matrices and are the elements of \(GL(2, q)\). Since \(r^3 = 1\), so \(R^3\) will be a scalar matrix and its determinant will be a square in \(F_q\). Since, for any matrix \(M, M^3 = \lambda I\) if and only if \((\text{trace}(M))^2 = \text{det}(M)\), so we may assume that \(\text{trace}(R) = r_1 + r_4 = -1\). Replacing \(R\) by a suitable scalar, we can also assume that \(\text{det}(R) = 1\). Thus \(R = [r_1, r_2, r_3, r_1 - 1]\). Therefore we have \(\text{det}(R) = -r_1^2 - r_4 - kr_4^2 = 1\), where \(\text{det}(R) = 1\), so
\[1 + r_1^2 + r_4 + kr_4^2 = 0 \quad (3.1)\]

As \(r^3 = 1\) and \(\text{trace}(R) = -1\), so every element of \(GL(2, q)\) with trace equal to \(-1\) has up to scalar multiplication, a conjugate of the form \([0 1 - 1 1]\). Therefore, we can assume that \(R\) has the form \([0 1 - 1 1]\). Similarly, \(S = s_1 ks_3 s_3 - s_1 s_2 s_3^2 - 1\) giving \(\text{det}(S) = -s_1 s_2 s_3^2 s_3^2 = 1\), so that
\[1 + s_1^2 + \sqrt{2} s_1 + k s_3^2 = 0 \quad (3.2)\]

Consider an invertible element \(t\) in \(G^{3,4}(2, q)\) such that it satisfies the relation:
\[t^2 = (rt)^2 = (st)^2 = 1 \quad (3.3)\]

Let \(T = [t_1, t_2, t_3, t_4]\) be a matrix representing \(t\). Then, since \(t\) is an involution, therefore \(t_4 = -t_1\) yields \(T = [t_1, t_2, t_3, -t_1]\). Let \(RT\) be the matrix representing \(rt\) of \(G^{3,4}(2, q)\). Then \(RT = [kt_3, -kt_1, t_3, t_1 + t_2]\), which again by Lemma 3.1, and \((rt)^2 = 1\), implies that
\[t_3 + t_2 = -kt_3 \quad (3.4)\]

Similarly, if \(ST\) is a matrix that represents an element \(st\) of \(G^{3,4}(2, q)\), then we get
\[ST = [s_1 t_1 + s_2 t_3, s_1 t_2 - s_2 t_4, s_3 t_1 + t_3 (\sqrt{2} - s_1) s_3 t_2 - t_1 (\sqrt{2} - s_1)\].\]
Since \(st\) is also an involution therefore by the arguments given above, we have \(s_1 t_1 + s_2 t_3 + s_3 t_2 - t_1 (\sqrt{2} - s_1) = 0\), which together with equation (3.4) yields
\[2s_1 t_1 + s_2 t_3 - s_3 t_1 - k s_3 t_3 - \sqrt{2} t_1 = 0.\]
That is,
\[t_1 (2s_1 - s_3 + \sqrt{2}) + t_3 (s_2 - k s_3) = 0 \quad (3.5)\]

Now for a non-singular matrix \(T\), we must have \(\text{det}(T) \neq 0\), that is
\[t_1^2 + t_1 t_3 + k t_3^2 \neq 0 \quad (3.6)\]

Therefore, necessary and sufficient conditions for the existence of \(t\) in \(G^{3,4}(2, q)\) are the equations (3.4), (3.5) and (3.6). Hence \(t\) exists in \(G^{3,4}(2, q)\) unless \(kt_3^2 - t_1^2 + t_1 t_3 = 0\). If both \(2s_1 - s_3 + \sqrt{2}\) and \(s_2 - k s_3\) are equal to zero, then the existence of \(t\) is trivial. If not, then \(t_1 / t_3 = -(s_2 - k s_3) / (2s_1 - s_3 - \sqrt{2})\), and so equation (3.6) is equivalent to \((s_2 - k s_3)^2 - (2s_1 - s_3 + \sqrt{2})(2s_1 + \sqrt{2} k - s_2) \neq 0\). Thus \(t\) exists in \(G^{3,4}(2, q)\) satisfying equation (3.3) unless \((s_2 - k s_3)^2 = (2s_1 - s_3 + \sqrt{2})(2s_1 + \sqrt{2} k - s_2)\). Which after simplification gives
\[(s_2 - k s_3)(s_2 - k s_3 + 2s_1 + \sqrt{2}) = -4k + 2s_3 - 2 \quad (3.7)\]
Conjugacy Classes and Action of $Δ(3,A,k)$ on $PL(F_q)$

Now $RS = \left\{ ks_3 k(\sqrt{2} - s_1) s_1 - s_3 s_2 - s_3 - \sqrt{2} + s_1 \right\}$, this implies that the $tr(RS) = s_1 + s_2 + k s_3 - \sqrt{2}$. Let $tr(RS) = m_2$. Also, using equation (3.7), we have $det(RS) = k(s_2 s_3 - \sqrt{2} s_1 + s_1^2)$. Since $det(RS) = 1$. So $k = -1$. Hence we have

$$1 = \sqrt{2}s_1 - s_1^2 - s_3 s_3 \quad (3.8)$$

Also, we have

$$m_2 = s_1 + s_2 - s_3 - \sqrt{2} \quad \quad (3.9)$$

Substituting $k = -1$ and values from equations (3.8) and (3.9) in equation (3.7), we get,

$$m_2^2 - \sqrt{2}m_2 + 2 = 3$$

$$m_2^2 - \sqrt{2}m_2 - 1 = 0. \quad (3.10)$$

**Theorem 3.3:** Let $g$ be any non-trivial element of $G^{±3,4}(2, q)$, such that order of both $g$ and its dual not equal to 2, then $g$ is the image of $rs$ under some non-degenerate homomorphism of $\mathcal{I}$ into $G^{±3,4}(2, q)$.

**Proof:** To prove this result, we show by using theorem 3.2, that every non-trivial element of $G^{±3,4}(2, q)$ is the product of two elements, one having order 3 whereas other of order 4. In fact we must find elements $r \underline{s}$ and $\underline{s}$ belong to $G^{±3,4}(2, q)$ and satisfy the relations (2.1), too.

For this, consider the elements $r \underline{s}$ and $\underline{s}$ of $G^{±3,4}(2, q)$ represented by the matrices $R = [r_1 k r_3 r_3 - r_1 - 1]$, $S = [s_1 k s_3 s_3 - \sqrt{2} - s_1]$ and $T = [0 \ 0 \ 0 \ 1]$, where $r_1, r_3, s_1, s_3, k$ are in $F_q$, with $k \neq 0$, so that

$$1 + r_1 + r_1^2 + k r_3^3 = 0. \quad (3.11)$$

Further, let assume the determinant of $S$ be equal to 1, we have

$$1 + k s_3^2 + s_3^2 + \sqrt{2}s_1 = 0. \quad (3.12)$$

We take $r \underline{s}$ in a given conjugation class. A matrix representing $r \underline{s}$ is given by

$$RS = \left[ r_1 s_1 + kr_3 s_3 k r_3 s_3 + k r_3(-\sqrt{2} - s_1) r_3 s_3 - s_3 (r_1 + 1) k r_3 s_3 - r_1 (-\sqrt{2} - s_1) + \sqrt{2} + s_1 \right]$$

Its trace, which we denote by $m_2$, is given by

$$m_2 = tr(RS) = 2 k r_3 s_3 + r_1 (2 s_1 + \sqrt{2}) + (s_1 + \sqrt{2}). \quad (3.13)$$

As determinant of $R$ and $S$ is 1, therefore $det(RS) = det(R)det(S) = 1$. Hence, we have

$$RST = \left[ kr_3 s_3 - \sqrt{2} k r_3 s_3 - k r_1 s_3 - k s_3 r_3 + \sqrt{2} s_1 + \sqrt{2} + s_1 - k r_3 s_3 + k r_3 s_3 + k s_3 \right].$$

So, $trace(RST) = k(2 r_1 s_3 - 2 r_3 s_1 + s_3 - \sqrt{2} r_3)$. Let $trace(RST) = km_3$, then

$$m_3 = 2 r_1 s_3 - 3 (2 s_1 + \sqrt{2}) + s_1. \quad (3.14)$$

Hence, we have

$$m_3^2 + km_3^2 - \sqrt{2}m_2 - 1 = 0. \quad (3.15)$$

Since $g = r \underline{s}$ (or its dual $r \underline{s}$) are not of order 2, so we must have $(r \underline{s})^2 \neq 1$ and $(r \underline{s})^2 \neq 1$. Thus by lemma 3.1, the traces of the matrices $RS$ and $RST$ are not equal to zero. Hence $m_2 \neq 0$, and $m_3 \neq 0$, so that $\theta = m_2^2 \neq 0$; and it is sufficient to show that we can choose $r_1, r_3, s_1, s_3, k$ in $F_q$ so that $m_2^2$ is indeed equal to $\theta$.

From equation (3.15), we have $km_3^2 = 1 - m_2^2 + \sqrt{2}m_2$. If $m_2^2 - \sqrt{2}m_2 \neq 1$, we can select the value of $k$ as per same argument.

**Theorem 3.4:** For any non-degenerate homomorphism $a\alpha$ and its dual $\alpha'$, 

$$\theta + \phi = 1 + \sqrt{2}m_2,$$

where $\theta$ and $\phi$ are the parameters of $a\alpha$ and $\alpha'$ respectively.

**Proof:** Consider a non-degenerate homomorphism $a\alpha' \to G^{±3,4}(2, q)$ satisfies the relations $r\alpha = r, s\alpha = \varepsilon$ and $t\alpha = \varepsilon$ and $\alpha'$ is its dual. Consider the matrices $R = [r_1 k r_3 r_3 - r_1 - 1]$, $S = [s_1 k s_3 s_3 - \sqrt{2} - s_1]$ and $T = [0 \ 0 \ 0 \ 1]$, representing the elements $r \underline{s}$ and $\underline{s}$ of $G^{±3,4}(2, q)$ respectively. By lemma 3.1, $trace(RS) = trace(RST) = 0$ if and only if $(r \underline{s})^2 = (r \underline{s})^2 = 1$. As $det(RS) = 1$, so we can assume that parameter $\theta$ (say) of $r \underline{s}$ equals to $m_2^2$. Also since $trace(RST) = km_3$ and $det(RST) = k$ (since $det(R) = 1$, $det(S) = 1$ and $det(T) = k$), we get the parameter $\phi$ of $r \underline{s}$ equals to $km_3^2$. Therefore, we have $\theta + \phi = m_2^2 + km_3^2$.

DOI: 10.9790/5728-1601022328 www.iorsjournals.org 25 | Page
Conjugacy Classes and Action of $\Delta(3,4,k)$ on $PL(F_q)$

Substituting the value of $m_2^2$ from equation (3.15), we get $\theta + \phi = 1 + \sqrt{2}m_2$. Hence if $\theta$ is the parameter of the non-degenerate homomorphism $\alpha$, then $\phi = 1 + \sqrt{2}m_2 - \theta$ is the parameter of the dual $\alpha'$ of $\alpha$.

**Corollary 3.5:** If $t$ inverts both $r$ and $s$, then order of $rs$ is 12.

**Proof:** From theorem 3.2, we have $m_2^2 = 1 + \sqrt{2}m_2$. After rearranging this result, we get

$$m_2^2 - 1 = \sqrt{2}m_2 \quad (3.16)$$

Taking square on both sides of equation (3.16), we get

$$m_2^4 - 2m_2^2 + 1 = 2m_2^2 \quad (3.17)$$

Replacing $m_2^2$ by $\theta$ in equation (3.17), we get

$$\theta^2 - 3\theta + 1 = 0 \quad (3.18)$$

From table 1 given below, it is evident that this is the corresponding equation for $k = 12$. Hence order of $rs$ is 12.

**Table 1:** Minimal Equations satisfied by $\theta$

<table>
<thead>
<tr>
<th>Triangle Group $\Delta(3,4,k)$</th>
<th>Minimal Equation satisfied by $\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta(3,4,1)$</td>
<td>$\theta - 4 = 0$</td>
</tr>
<tr>
<td>$\Delta(3,4,2)$</td>
<td>$\theta = 0$</td>
</tr>
<tr>
<td>$\Delta(3,4,3)$</td>
<td>$\theta - 1 = 0$</td>
</tr>
<tr>
<td>$\Delta(3,4,4)$</td>
<td>$\theta - 2 = 0$</td>
</tr>
<tr>
<td>$\Delta(3,4,5)$</td>
<td>$\theta - 3\theta + 1 = 0$</td>
</tr>
<tr>
<td>$\Delta(3,4,6)$</td>
<td>$\theta - 3 = 0$</td>
</tr>
<tr>
<td>$\Delta(3,4,7)$</td>
<td>$\theta - 5\theta^2 + 6\theta - 1 = 0$</td>
</tr>
<tr>
<td>$\Delta(3,4,8)$</td>
<td>$\theta^2 - 4\theta + 2 = 0$</td>
</tr>
<tr>
<td>$\Delta(3,4,9)$</td>
<td>$\theta^2 - 6\theta^2 + 9\theta - 1 = 0$</td>
</tr>
<tr>
<td>$\Delta(3,4,10)$</td>
<td>$\theta^2 - 5\theta + 5 = 0$</td>
</tr>
<tr>
<td>$\Delta(3,4,11)$</td>
<td>$\theta^2 - 9\theta^4 + 28\theta^3 - 35\theta^2 + 15\theta - 1 = 0$</td>
</tr>
<tr>
<td>$\Delta(3,4,12)$</td>
<td>$\theta^2 - 4\theta + 1 = 0$</td>
</tr>
<tr>
<td>$\Delta(3,4,13)$</td>
<td>$\theta^2 - 11\theta^5 + 45\theta^4 - 84\theta^3 + 70\theta^2 - 21\theta + 1 = 0$</td>
</tr>
<tr>
<td>$\Delta(3,4,14)$</td>
<td>$\theta^2 - 7\theta^2 + 14\theta - 7 = 0$</td>
</tr>
<tr>
<td>$\Delta(3,4,15)$</td>
<td>$\theta^4 - 9\theta^3 + 26\theta^2 - 24\theta + 1 = 0$</td>
</tr>
</tbody>
</table>
IV. Computational Approach to Calculate Conjugacy Classes

Flowchart and Algorithm

Following flowchart and algorithm help us to develop a computer coding scheme for drawing relation between homomorphism and parameters of conjugacy classes.

Figure 1: Flow Chart

1. Input integer values $k$, set $i = 0$.
2. For $i < k$. If $i$ is prime, calculate $g_k(\theta) = f_i(\theta)$
3. Otherwise calculate divisors for $k$
4. Calculate $g_k(\theta) = \frac{f_i(\theta)}{g_i(d_1, d_2, ..., d_n)(\theta)}$
5. Add $g_k(\theta)$ to the list.
6. Display list in table form.

Coding Scheme

Following code written in Java programming language will generate the conditions in form of equations $f(\theta) = 0$ for the existence of triangle groups $\Delta(3A,k)$ for $1 \leq k \leq n$ as shown in table 1 for $1 \leq k \leq 15$.

(* Get Input from user *)

$k = \text{Input[ Enter the value of } k];$

(* Initializedenominator to be used when Kis prime *)

$mylist = \text{Range}[k];$

$resultlist = \text{List}[];$
Conjugacy Classes and Action of $\Delta(3A, k)$ on $PL(F_q)$

\[ \text{denom} = 1; \]
\[ \text{finalResult} = 1; \]
\[ r = 2; \]
\[ (* \text{Functionthatimplementsthestheformula} *) \]
\[ r = \sqrt{u}; \]
\[ S\text{olver}[k_\ldots]: \sum_{n=1}^{(k+1)/2} (-1)^{n+1} \left( \frac{(k-n)!}{((k-n)-(n-1))! (n-1)!} \right) (r)^{k-(2n-1)}; \]
\[ (* \text{Loopfrom1toinputRange} *) \]
\[ \text{For}[i = 1, i \leq k, i + +, \]
\[ (* \text{checkkforprimecondition} *) \]
\[ \text{If}[i == 1, \text{finalResult} = \theta - 4, \]
\[ \quad \text{If}[\text{PrimeQ}[i], \]
\[ \quad \quad (* \text{IfKisPrime} *) \]
\[ \quad \text{finalResult} = \text{solver}[i], (* g_k(\theta) = f_k(\theta) *) \]
\[ \quad \text{divof} K = \text{Divisors}[i]; (* \text{IfKisNotPrime} *) \]
\[ \quad \text{length} = \text{Length}[\text{divof} K]; \]
\[ \quad \text{newlist} = \text{Delete}[\text{divof} K,\{(1),(-1)}]; (* \text{GetDivisorsof} K *) \]
\[ \quad \text{length}2 = \text{Length}[\text{newlist}]; \]
\[ \quad \text{Do}[\text{denom} = \text{denom} \star \text{solver}[\text{Part}[\text{newlist},n]],\{n,1,\text{length2},1\}; (* g_k(\theta) = f_k \frac{\theta}{g_k[d_1,d_2,d_3,...]}(\theta) *) \]
\[ \quad \text{finalResult} = \frac{\text{solver}[i]}{\text{denom}}; \]

References


Conjugacy Classes and Action of $\Delta(3,4,k)$ on $PL(F_q)$

