When Is an Ellipse Inscribed In a Quadrilateral Tangent at the Midpoint of Two or More Sides

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I. Introduction

Among all ellipses inscribed in a triangle, T, the midpoint, or Steiner, ellipse is interesting and well-known [2]. It is the unique ellipse tangent to T at the midpoints of all three sides of T and is also the unique ellipse of maximal area inscribed in T. What about ellipses inscribed in quadrilaterals, Q? Not surprisingly, perhaps, there is not always a midpoint ellipse—i.e., an ellipse inscribed in Q which is tangent at the midpoints of all four sides of Q. In fact, in [1] it was shown that if there is a midpoint ellipse, then Q must be a parallelogram. That is, if Q is not a parallelogram, then there is no ellipse inscribed in Q which is tangent at the midpoints of all four sides of Q; But can one do better than four sides of Q? In other words, if Q is not a parallelogram, is there an ellipse inscribed in Q which is tangent at the midpoint of three sides of Q? In Theorem 1 we prove that the answer is no. In fact, unless Q is a trapezoid (a quadrilateral with at least one pair of parallel sides), or what we call a midpoint diagonal quadrilateral (see the definition below), then there is not even an ellipse inscribed in Q which is tangent at the midpoint of two sides of Q (see Lemmas 3 and 4).

Definition: A convex quadrilateral, Q, is called a midpoint diagonal quadrilateral (mdq) if the intersection point of the diagonals of Q coincides with the midpoint of at least one of the diagonals of Q.

A parallelogram, P, is a special case of an mdq since the diagonals of P bisect one another. In [5] we discussed mdq’s as a generalization of parallelograms in a certain sense related to tangency chords and conjugate diameters of inscribed ellipses.

What about uniqueness? If Q is an mdq, then the ellipse inscribed in Q which is tangent at the midpoint of two sides of Q is not unique. Indeed we prove (Lemma 3) that in that case there are two such ellipses. However, if Q is a trapezoid, then the ellipse inscribed in Q which is tangent at the midpoint of two sides of Q is unique (Lemma 4).

Is there a connection with tangency at the midpoint of the sides of Q and the ellipse of maximal area inscribed in Q as with parallelograms? In [3] we showed that the midpoint ellipse for a parallelogram also turns out to be the unique ellipse of maximal area inscribed in Q. For trapezoids, we prove (Lemma 4) that the unique ellipse of maximal area inscribed in Q is the unique ellipse tangent to Q at the midpoint of two sides of Q. However, for mdq’s, the unique ellipse of maximal area inscribed in Q need not be tangent at the midpoint of any side of Q.

We use the notation \( Q(A_1, A_2, A_3, A_4) \) to denote the quadrilateral with vertices \( A_1, A_2, A_3, \) and \( A_4 \), starting with \( A_1 \) = lower left corner and going clockwise. Denote the sides of \( Q(A_1, A_2, A_3, A_4) \) by \( S_1, S_2, S_3, \) and \( S_4 \), going clockwise and starting with the leftmost side, \( S_1 \), and denote the diagonals of \( Q(A_1, A_2, A_3, A_4) \) by \( D_1 = A_1A_3 \) and \( D_2 = A_2A_4 \).
We note here that there are two types of mdq’s: Type 1, where the diagonals intersect at the midpoint of $D_2$ and Type 2, where the diagonals intersect at the midpoint of $D_1$; Mdq’s of types 1 and 2, respectively, are illustrated below.

Given a convex quadrilateral, $Q = Q(A_1, A_2, A_3, A_4)$, which is not a parallelogram, it will simplify our work below to consider quadrilaterals with a special set of vertices. In particular, there is an affine transformation which sends $A_1, A_2$, and $A_4$ to the points $(0,0), (0,1)$, and $(1,0)$, respectively. It then follows that $A_3 = (s,t)$ for some $s,t > 0$; Summarizing:

$$Q_{s,t} = Q(A_1, A_2, A_3, A_4), \quad (1)$$
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\[ A_1 = (0,0), A_2 = (0,1), A_3 = (s,t), A_4 = (1,0). \]

Since \( Q_{st} \) is convex, \( s + t > 1 \); Also, if \( Q \) has a pair of parallel vertical sides, first rotate counterclockwise by 90\(^\circ\), yielding a quadrilateral with parallel horizontal sides. Since we are assuming that \( Q \) is not a parallelogram, we may then also assume that \( Q_{st} \) does not have parallel vertical sides and thus \( s \neq 1 \). Note that any trapezoid which is not a parallelogram may be mapped, by an affine transformation, to the quadrilateral \( Q_{s1} \); Thus we may assume that \( (s,t) \in G \), where

\[ G = \{(s,t) : s,t > 0, s + t > 1, s \neq 1 \}. \]  

(2)

The following result gives the points of tangency of any ellipse inscribed in \( Q_{st} \), which is not a parallelogram may be mapped, by an affine transformation, to the quadrilateral \( Q_{s1} \); Thus we may assume that

\[ (s,t) \in G, \]

where

\[ G = \{(s,t) : s,t > 0, s + t > 1, s \neq 1 \}. \]  

(2)

The following lemma gives necessary and sufficient conditions for \( Q_{st} \) to be an mdq.

**Lemma 1:**

(i) \( Q_{st} \) is a type 1 midpoint diagonal quadrilateral if and only if \( s \neq t \).

(ii) \( Q_{st} \) is a type 2 midpoint diagonal quadrilateral if and only if \( s + t = 2 \).

**Proof:** The diagonals of \( Q_{st} \) are \( D_1 : y = \frac{t}{s}x \) and \( D_2 : y = 1 - x \), and they intersect at the point

\[ P = \left( \frac{s}{s+t}, \frac{t}{s+t} \right). \]

The midpoints of \( D_1 \) and \( D_2 \) are \( M_1 = \left( \frac{s}{2}, \frac{t}{2} \right) \) and \( M_2 = \left( \frac{1}{2}, \frac{1}{2} \right) \), respectively. Now
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\[ M_s = P \iff \frac{s}{s+t} = \frac{1}{2} \quad \text{and} \quad \frac{t}{s+t} = \frac{1}{2}, \text{both of which hold if and only if} \quad s = t; \text{That proves (i)}; \]
\[ M_t = P \iff \frac{s}{s+t} = \frac{1}{2} s \quad \text{and} \quad \frac{t}{s+t} = \frac{1}{2} t, \text{both of which hold if and only if} \quad s + t = 2. \text{That proves (ii)}. \]

The following lemma shows that affine transformations preserve the class of mdq’s. We leave the details of the proof to the reader.

**Lemma 2:** Let \( T : R^2 \to R^2 \) be an affine transformation and let \( Q \) be a midpoint diagonal quadrilateral. Then \( Q' = T(Q) \) is also a midpoint diagonal quadrilateral.

### II. Main Results

The following result shows that among non-trapezoids, the only quadrilaterals which have inscribed ellipses tangent at the midpoint of two sides are the mdq’s.

**Lemma 3:** Let \( Q \) be a convex quadrilateral in the \( xy \) plane which is not a trapezoid.

(i) There is an ellipse inscribed in \( Q \) which is tangent at the midpoint of two or more sides of \( Q \) if and only if \( Q \) is a midpoint diagonal quadrilateral, in which case there are two such ellipses.

(ii) There is no ellipse inscribed in \( Q \) which is tangent at the midpoint of three sides of \( Q \).

**Proof:** By Lemma 2 and standard properties of affine transformations, we may assume that \( Q = Q_{s,t} \), the quadrilateral given in (1) with \((s,t) \in G\); The midpoints of the sides of \( Q_{s,t} \) are given by \( MP_s = \left(0, \frac{1}{2}\right) \in S_s \), \( MP_s' = \left(s, \frac{1}{2} + t\right) \in S_s \), \( MP_t = \left(\frac{1}{2}, t\right) \in S_t \), and \( MP_t = \left(1, \frac{1}{2}\right) \in S_t \); Now let \( E_0 \) denote an ellipse inscribed in \( Q_{s,t} \), and let \( P_j \in S_j, j = 1, 2, 3, 4 \) denote the points of tangency of \( E_0 \) with the sides of \( Q_{s,t} \); By Proposition 1(ii),

\[ P_1 = MP_1 \iff \frac{qt}{q(t-s) + s} = \frac{1}{2}, \quad (4) \]
\[ P_2 = MP_2 \iff \frac{(1-q)s}{q(t-1)(s+t) + s} = \frac{1}{2}, \quad (5) \]
\[ \text{and} \quad \frac{t(s+q(t-1))}{q(t-1)(s+t) + s} = \frac{1+t}{2}, \quad (6) \]
\[ P_3 = MP_3 \iff \frac{s + q(t-1)}{q(s + t - 2) + 1} = \frac{1+s}{2}, \quad (7) \]
\[ \text{and} \quad \frac{(1-q)t}{q(s + t - 2) + 1} = \frac{t}{2}, \quad (8) \]
\[ P_4 = MP_4 \iff q = \frac{1}{2}. \quad (9) \]

Equations (4) and (9) each have the unique solutions \( q_1 = \frac{s}{s+t} \) and \( q_4 = \frac{1}{2} \), respectively. The system of equations in (5) and (6) has unique solution \( q_2 = \frac{s}{t^2 + st + s-t} \), and the system of equations in (7) and (8) has unique solution \( q_3 = \frac{1}{s+t} \); It is trivial that \( q_1, q_3, q_4 \in J = (0,1) \); Since \((s,t) \in G, t(s+t-1) > 0\), which implies that \( q_2 \in J \). We now check which pairs of midpoints of sides of \( Q_{s,t} \) can be points of tangency of \( E_0 \); Note that different values of \( q \) yield distinct inscribed ellipses by the one-to-one correspondence between ellipses inscribed in \( Q_{s,t} \) and points \( q \in J \).

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(a) \( S_1 \) and \( S_2 \): \( q_1 = q_2 \Leftrightarrow \frac{s}{t^2 + st + s - t} - \frac{s}{s + t} = 0 \Leftrightarrow \frac{st(s + t - 2)}{ts + t^2 + s - t}(s + t) = 0 \Leftrightarrow s + t = 2. \)

(b) \( S_1 \) and \( S_3 \): \( q_1 = q_3 \Leftrightarrow \frac{1}{s + t} - \frac{s}{s + t} = 0 \Leftrightarrow \frac{s - 1}{s + t} = 0 \), which has no solution since \( s \neq 1. \)

(c) \( S_2 \) and \( S_3 \): \( q_2 = q_3 \Leftrightarrow \frac{1}{s + t} - \frac{s}{t^2 + st + s - t} = 0 \Leftrightarrow \frac{(t - s)(s + t - 1)}{ts + t^2 + s - t}(s + t) = 0 \Leftrightarrow s = t. \)

(d) \( S_1 \) and \( S_4 \): \( q_1 = q_4 \Leftrightarrow \frac{1}{2} - \frac{s}{s + t} = 0 \Leftrightarrow \frac{1}{2} - \frac{s - t}{s + t} = 0 \Leftrightarrow s = t. \)

(e) \( S_2 \) and \( S_4 \): \( q_2 = q_4 \Leftrightarrow \frac{1}{2} - \frac{s}{s + t} = 0 \Leftrightarrow \frac{1}{2} - \frac{s + t - 2}{s + t} = 0 \Leftrightarrow s + t = 2. \)

That proves that there is an ellipse inscribed in \( Q_{st} \) which is tangent at the midpoints of \( S_1 \) and \( S_2 \) or at the midpoints of \( S_3 \) and \( S_4 \) if and only if \( s + t = 2 \), and there is an ellipse inscribed in \( Q_{st} \) which is tangent at the midpoints of \( S_2 \) and \( S_3 \) or at the midpoints of \( S_1 \) and \( S_4 \) if and only if \( s = t \). Furthermore, if \( s \neq t \) and \( s + t \neq 2 \), then there is no ellipse inscribed in \( Q_{st} \) which is tangent at the midpoint of two sides of \( Q_{st} \). That proves (i) by Lemma 1. To prove (ii), to have an ellipse inscribed in \( Q_{st} \) which is tangent at the midpoint of three sides of \( Q_{st} \), those three sides are either \( S_1, S_2, \) and \( S_3 \); \( S_1, S_2, \) and \( S_4 \); \( S_1, S_3, \) and \( S_2 \); \( S_2, S_3, \) and \( S_4 \); By (a)-(f) above, that is not possible.

For trapezoids inscribed in \( Q \) we have the following result.

**Lemma 4:** Assume that \( Q \) is a trapezoid which is not a parallelogram. Then

(i) There is a unique ellipse inscribed in \( Q \) which is tangent at the midpoint of two sides of \( Q \), and that ellipse is the unique ellipse of maximal area inscribed in \( Q \).

(ii) There is no ellipse inscribed in \( Q \) which is tangent at the midpoint of three sides of \( Q \).

**Proof:** Again, by affine invariance, we may assume that \( Q = Q_{s,t} \), the quadrilateral given in (1) with \( t = 1 \).

Note that \( 0 < s \neq 1 \). Now let \( E_0 \) denote an ellipse inscribed in \( Q_{s,t} \). Letting \( MP_j \in S_j, j = 1, 2, 3, 4 \) denote the corresponding midpoints of the sides and using Proposition 1(ii) again, with \( t = 1 \), we have

\[ P_1 = MP_1 \Leftrightarrow \frac{q}{(1 - s)q + s} = \frac{1}{2}, \quad (10) \]

\[ P_2 = MP_2 \Leftrightarrow (1 - q)s = \frac{s}{2}, \quad (11) \]

\[ P_3 = MP_3 \Leftrightarrow \frac{s}{(s - 1)q + 1} = \frac{1 + s}{2} \quad \text{and} \quad (12) \]

\[ \frac{1 - q}{(s - 1)q + 1} = \frac{1}{2}, \quad (13) \]

\[ P_4 = MP_4 \Leftrightarrow q = \frac{1}{2}. \quad (14) \]

The unique solution of the equations in (11) and in (14) is \( q = \frac{1}{2} \in J \); The unique solution of the equation in (10) is \( q = \frac{s}{1 + s} \in J \), and the unique solution of the system of equations in (12) and (13) is \( q = \frac{1}{1 + s} \in J \);

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We now check which pairs of midpoints of sides of \( Q_{1,1} \) can be points of tangency with \( E_0 \):

(a) \( q = \frac{1}{2} \) gives tangency at the midpoints of \( S_2 \) and \( S_4 \).

(b) \( S_1 \) and \( S_2 \) or \( S_1 \) and \( S_4 \):
\[
\frac{s}{1 + s} = \frac{1}{2} \iff s = 1.
\]

(c) \( S_3 \) and \( S_2 \) or \( S_3 \) and \( S_4 \):
\[
\frac{1}{1 + s} = \frac{1}{2} \iff s = 1.
\]

(d) \( S_1 \) and \( S_3 \):
\[
\frac{s}{1 + s} = \frac{1}{1 + s} \iff s = 1.
\]

Since we have assumed that \( s \neq 1 \), the only way to have an ellipse inscribed in \( Q_{1,1} \) which is tangent at the midpoint of two sides of \( Q_{1,1} \) is if those sides are \( S_2 \) and \( S_4 \) and \( q = \frac{1}{2} \). That proves that there is a unique ellipse inscribed in \( Q_{1,1} \) which is tangent at the midpoint of two sides of \( Q_{1,1} \). Now suppose that \( E_0 \) is any ellipse with equation \( Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \), and let \( a \) and \( b \) denote the lengths of the semi-major and semi-minor axes, respectively, of \( E_0 \). Using the results in [7], it can be shown that
\[
a^2 b^2 = \frac{4s^2}{\Delta^3},
\]
where \( \Delta = 4AC - B^2 \) and \( \delta = CD^2 + AE^2 - BDE - F \Delta \). By Proposition 1(i), then, with \( t = 1 \) and after some simplification, we have
\[
a^2 b^2 = f(q) = \frac{s}{4} q(1 - q); \quad q = \frac{1}{2}
\]
and thus gives the ellipse of maximal area inscribed in \( Q_{1,1} \). That proves the rest of (i). (ii) now follows easily and we omit the details.

Remark: It can be shown [5] that if \( Q \) is a trapezoid which is not a parallelogram, then \( Q \) cannot be an mdq. Thus the only quadrilaterals Lemmas 3 and 4 have in common are parallelograms.

Since a convex quadrilateral which is not a parallelogram either has no two sides which are parallel, or is a trapezoid, the following theorem follows immediately from Lemma 3(ii) and Lemma 4(ii).

Theorem: Suppose that \( Q \) is a convex quadrilateral which is not a parallelogram. Then there is no ellipse inscribed in \( Q \) which is tangent at the midpoint of three sides of \( Q \).

Examples: (1) Let \( Q \) be the quadrilateral with vertices \((0,0), (0,1), (2,4), \) and \((1,1)\); It follows easily that \( Q \) is a type 1 midpoint diagonal quadrilateral. The ellipse with equation
\[
10 \left( x - \frac{2}{3} \right)^2 - 10 \left( x - \frac{2}{3} \right) \left( y - \frac{4}{3} \right) + 4 \left( y - \frac{4}{3} \right)^2 = \frac{5}{3}
\]
is tangent to \( Q \) at \( \left( \frac{1}{2}, 0 \right) \) and at \( \left( \frac{1}{2}, \frac{1}{2} \right) \), the midpoints of \( S_1 \) and \( S_1 \), respectively. The ellipse with equation
\[
54 \left( x - \frac{4}{5} \right)^2 - 54 \left( x - \frac{4}{5} \right) \left( y - \frac{8}{5} \right) + 16 \left( y - \frac{8}{5} \right)^2 = \frac{27}{5}
\]
is tangent to \( Q \) at \( \left( \frac{5}{2}, \frac{5}{2} \right) \) and at \( \left( \frac{3}{2}, \frac{5}{2} \right) \), the midpoints of \( S_2 \) and \( S_3 \), respectively. One can show that neither of these ellipses is the ellipse of maximal area inscribed in \( Q \). See Figure 1 below.
(2) Let $Q$ be the trapezoid with vertices $(0,0), (0,1), (2,1)$, and $(1,1)$. The ellipse with equation
\[
\left( x - \frac{5}{4} \right)^2 - 3 \left( x - \frac{5}{4} \right) \left( y - \frac{1}{2} \right) + \frac{25}{4} \left( y - \frac{1}{2} \right)^2 = 1
\]
is tangent to $Q$ at $(2,1)$ and at $\left( \frac{1}{2}, 0 \right)$, the midpoints of $S_2$ and $S_4$, respectively. See Figure 2 below.

References