Applications for Non expansive and monotone sequence of mappings with Viscosity approximation

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Abstract: In this article we establish the application for viscosity approximation methods for Nonexpansive sequence of mappings. we associate converges strongly to a common element of the set of fixed points of sequence of mappings and also the set of solutions of the Variational inequality for an inverse strongly-monotone sequence of mappings which solves some Variational inequality.

Keywords: Viscosity approximation; fixed point; Inverse-strongly monotone mapping; Nonexpansive mapping; Variational inequalities

I. Introduction

In this work we give a generalized for Nonexpansive and monotone sequence of mappings with applications, we supposed a closed convex subset of a real Hilbert space $H$ denoted by $C$ and the metric projection of $H$ onto $C$ by $P_C$. Recall that a self-mapping $f: C \to C$ is a contraction on $C$ if there is a constant $0 < \epsilon < 1$ such that

$$\|f(u_m) - f(u_{m+1})\| \leq (1-\epsilon)\|u_m - u_{m+1}\|, \quad u_m, u_{m+1} \in C.$$ 

$P_C$ denotes the set of all contractions on $C$. Note that has a unique fixed point in $C$.

A mapping $A$ of $C$ into $H$ is called monotone sequence if $\langle Au_m - Au_{m+1}, u_m - u_{m+1} \rangle \geq 0$, for all $u_m, u_{m+1} \in C$. The variational inequality problem is to find $u_m \in C$ such that $\langle Au_m - Au_{m+1} - u_m, u_{m+1} \rangle \geq 0$ for all $u_{m+1} \in C$ (See [1,2]). The series of solutions of the variational inequality is denoted by $VI(C, A)$. A mapping $A$ of $C$ into $H$ is called inverse-strongly monotone sequence if we have $\left(\frac{\lambda + \epsilon}{2}\right) \in \mathbb{R}^+$ such that

$$\langle u_m - u_{m+1}, Au_m - Au_{m+1} \rangle \geq \frac{\lambda + \epsilon}{2}\|Au_m - Au_{m+1}\|^2$$

for all $u_m, u_{m+1} \in C$. For such a case, $A$ is inverse-strongly monotone sequence.

A mapping $S$ of $C$ into $H$ is called nonexpansive of sequence if $\|Su_m - Su_{m+1}\| \leq \|u_m - u_{m+1}\|$ for all $u_m, u_{m+1} \in C$ (Ref. [3]). We denote by $F(S)$ the set of fixed points of $S$. The viscosity approximation methodology of choosing a selected fastened purpose of given Nonexpansive sequence of mapping was planned by Moudafi [4] established the subsequent strongly convergence of each the implicit and specific method in Hilbert space.

Theorem 1.1. In a Hilbert space define $(u_m)_n$ by implicit way

$$u_m = \frac{1}{1 + \epsilon_n}T(u_m)_n + \frac{\epsilon_n}{1 + \epsilon_n}f(u_m)_n,$$

where $\epsilon_n$ is a sequence in $(0, 1)$ tending to zero. Then $(u_m)_n$ converges strongly to the exclusive solution $(\bar{u}_m) \in C$ of the variational inequality

$$\langle (I - f)(u_m), (u_m) - (u_m) \rangle \leq 0.$$ 

In other words, $(\bar{u}_m)$ is the exclusive fixed point of $P_{Fix(f)}f$.
Theorem 1.2. In a Hilbert space define \((u_m)_n\) by \((u_m)_0 \in C\) is arbitrary
\[
(u_m)_{n+1} = \frac{1}{1 + \varepsilon_n} T(u_m)_n + \frac{\varepsilon_n}{1 + \varepsilon_n} f ((u_m)_n).
\]
Suppose that \(\varepsilon_n\) satisfies the conditions
\[
\lim_{n \to \infty} \varepsilon_n = 0, \quad \sum_{n=1}^{\infty} \varepsilon_n = \infty; \quad \lim_{n \to \infty} \left| \frac{1}{\varepsilon_n} - \frac{1}{\varepsilon_{n-1}} \right| = 0.
\]
Then \(\{u_m\}_n\) converges strongly to the exclusive solution \((u_m)\) in \(C\) of the variational inequality
\[
\langle (I - f)(u_m), (u_m) - (u_m) \rangle \leq 0.
\]
In alternative words, \((u_m)\) is the exclusive fixed point of \(P_{\text{Fix}(T)} f\).

Theorem 1.3. (See [5]) Let \(H\) be a Hilbert space, \(C\) a closed convex subset of \(H\), and \(T : C \to C\) a nonexpansive sequence of mappings with \(F(T) \neq \emptyset\) and \(f \in \Pi_C\). Let \((u_m)\) be given by
\[
(u_m)_t = tf ((u_m)_t) + (1 - t)T (u_m), \quad t \in (0, 1).
\]
Then:
(i) \(\lim_{t \to 0} (u_m)_t = : (u_m)\) exists;
(ii) \(\limsup (u_m) = P f ((u_m))\), or equivalently, \((u_m)\) is the exclusive solution in \(F(T)\) to the variational inequality
\[
\langle (I - f)(u_m), (u_m) - (u_m) \rangle \geq 0, \quad (u_m) \in S,
\]
where \(S = F(T)\) and \(P f\) is the metric projection from \(H\) to \(S\).

Theorem 1.4. (See [5]) Let \(H\) be a Hilbert space, \(C\) a closed convex subset of \(H\), and \(T : C \to C\) a nonexpansive sequence of mappings with \(F(T) \neq \emptyset\), and \(f : C \to C\) a contraction. Let \((u_m)_n\) be given by
\[
(u_m)_0 \in C, \quad (u_m)_{n+1} = \left(\frac{\lambda + \varepsilon}{2}\right)_n f ((u_m)_n) + \left(1 - \frac{\lambda + \varepsilon}{2}\right)_n T (u_m), \quad n \geq 0.
\]
Then under the following hypotheses
\[
(H1) \left(\frac{\lambda + \varepsilon}{2}\right)_n \to 0;
\]
\[
(H2) \sum_{n=0}^{\infty} \left(\frac{\lambda + \varepsilon}{2}\right)_n = \infty;
\]
\[
(H3) \text{either} \sum_{n=0}^{\infty} \left| \left(\frac{\lambda + \varepsilon}{2}\right)_n - \left(\frac{\lambda + \varepsilon}{2}\right)_{n+1} \right| < \infty \text{ or} \lim_{n \to \infty} \left(\frac{\lambda + \varepsilon}{2}\right)^{n+1} = 1,
\]
\((u_m)_n \to (u_m)\), where \((u_m)\) is the exclusive solution of the variational inequality
\[
\langle (I - f)(u_m), (u_m) - (u_m) \rangle \geq 0, \quad (u_m) \in S.
\]
We verified the method of [11] by introducing the monotone sequence of mappings. With a little change.

II. Preliminaries

Let \(H\) be a real Hilbert space with inner product \((\cdot , \cdot )\) and norm \(\| \cdot \|\), and let \(C\) be a closed convex subset of \(H\). We write \((u_m)_n \rightharpoonup (u_m)\) to indicate that the sequence \(\{u_m\}_n\) converges weakly to \((u_m)\). \((u_m)_n \rightarrow (u_m)\) implies that \(\{u_m\}_n\) converges strongly to \((u_m)\). For every point \((u_m) \in H\), there exists a unique nearest point in \(C\), denoted by \(P_C (u_m)\), such that
\[
\|u_m - P_C u_m\| \leq \|u_m - u_{m+1}\|
\]
for all \(u_{m+1} \in C\). \(P_C\) is called the metric projection of \(H\) to \(C\). It is well known that \(P_C\) satisfies
\[
\langle u_m - u_{m+1}, P_C u_m - P_C u_{m+1} \rangle \geq \|P_C u_m - P_C u_{m+1}\|^2
\]
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For every \( u_m, u_{m+1} \in H \) and \( P \), is characterized by the following properties:

\[
\langle u_m - P_C u_m, P_C u_{m+1} - u_{m+1} \rangle \geq 0,
\]

\[
\|u_m - u_{m+1}\|^2 \geq \|u_m - P_C u_m\|^2 + \|u_{m+1} - P_C u_m\|^2
\]

for all \( u_m \in H, u_{m+1} \in C \).

This implies

\[
\forall u_m \in VI(C, A) \Rightarrow u_m = P_C(u_m - \lambda A u_m), \quad \forall \lambda > 0.
\]

It is well known that \( H \) satisfies the Opial condition (Ref. [6]), i.e., for any sequence \( \{u_m\} \) with \( u_m \rightarrow (u_m) \) the inequality

\[
\lim \inf_{n \to \infty} \|u_n - u_m\| < \lim \inf_{n \to \infty} \|u_n - u_{m+1}\|
\]

holds for every \( u_{m+1} \in H \) with \( u_m \neq u_{m+1} \). It is an \( \frac{\lambda + \epsilon}{2} \)-inverse-strongly monotone sequence of mappings of \( C \) to \( H \), then it is obvious that \( A \) is \( \frac{\lambda + \epsilon}{2} \)-Lipschitz continuous. We also have that for all \( u_m, u_{m+1} \in C \) and \( \lambda > 0 \),

\[
\| (I - \lambda A) u_m - (I - \lambda A) u_{m+1} \|^2 = \| u_m - u_{m+1} \|^2 - 2\lambda (u_m - u_{m+1}, Au_m - Au_{m+1}) + \lambda^2 \|Au_m - Au_{m+1}\|^2
\]

\[
\leq \| u_m - u_{m+1} \|^2 - \epsilon \lambda \|Au_m - Au_{m+1}\|^2.
\]

So, if \( \frac{\lambda + \epsilon}{2} \) is given, then \( I - \lambda A \) is a nonexpansive sequence of mappings of \( C \) into \( H \).

A set-valued mapping \( T : H \to 2^H \) is called monotone sequence if for all \( u_m, u_{m+1} \in H, f \in T u_m \) and \( g \in T u_{m+1} \) imply

\[
\langle u_m - u_{m+1}, f - g \rangle \geq 0.
\]

A monotone sequence of mapping \( T : H \to 2^H \) is maximal if graph \( G(T) \) of \( T \) is not properly contained in the graph of any other monotone sequence of mapping. It is known that a monotone sequence of mapping \( T \) is maximal if and only if for \( (u_m, f) \in H \times H, (u_m - u_{m+1}, f - g) \geq 0 \) for every \( (u_{m+1}, g) \in G(T) \) implies \( f \in T u_m \). Let \( A \) be an inverse-strongly monotone sequence of mapping of \( C \) to \( H \) and let \( N_C u_{m+1} \) be normal cone to \( C \) at \( u_{m+1} \in C \), i.e., \( N_C u_{m+1} = \{u_{m+2} \in H : (u_{m+1} - u_m, u_{m+2}) \geq 0, \forall u_m \in C \} \), and define

\[
T u_{m+1} = \begin{cases} A u_{m+1} + N_C u_{m+1}, & u_{m+1} \in C, \\ \emptyset, & u_{m+1} \not\in C \end{cases}
\]

then \( T \) is maximal monotone sequence and \( 0 \in T u_{m+1} \) if and only if \( u_{m+1} \in VI(C, A) \) (see [7]).

III. Main results

In this section, we show a strong convergence theorem (see [11]) for nonexpansive sequence of mappings and inverse strongly monotone sequence of mappings.

**Lemma 1.** (See [8].) Let \( C \) be a closed convex subset of a real Hilbert space \( H \) and let \( T : C \to C \) be a nonexpansive sequence of mapping such that \( Fix(T) \neq \emptyset \). If a sequence \( \{u_m\} \) in \( C \) is such that \( \{u_m\} \to (u_{m+1}) \) and \( (u_m - T u_m) \to 0 \), then \( (u_{m+1}) = T (u_{m+1}) \).

**Lemma 2.** (See [9].) Let \( \{s_n\} \) be a sequence of nonnegative sequence of real numbers such that:

\[
s_{n+1} \leq (1 - \lambda_n) s_n + \beta_n, \quad n \geq 0,
\]

where \( \{\lambda_n\}, \{\beta_n\} \) satisfy the condition

\[
(i) \quad \{\lambda_n\} \subset (0, 1) \text{ and } \sum_{n=1}^{\infty} \lambda_n = \infty,
\]

\[
(ii) \quad \lim \sup_{n \to \infty} \frac{\beta_n}{\lambda_n} \leq 0 \text{ or } \sum_{n=1}^{\infty} |\beta_n| < \infty
\]

Then \( \lim s_n = 0 \).

**Proposition 3.1.** Let \( C \) be a closed convex subset of a real Hilbert space \( H \). Let \( f : C \to C \) be a contraction with \( \lambda \) coefficient \((1 - \epsilon) \) \((0 < \epsilon < 1)\), \( A \) an \( \frac{\lambda + \epsilon}{2} \)-inverse-strongly monotone sequence of mapping of \( C \) to \( H \) and let \( S \) be a nonexpansive sequence of mapping of \( C \) into itself such that \( F(S) \cap VI(C, A) \neq \emptyset \). Suppose \( \{\lambda_n\} \) be sequences generated by \( \{u_m\}_0 \in C \),

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\[(u_m)_{n+1} = \left(\frac{\lambda + \epsilon}{2}\right)_n f ((u_m)_n) + (1 - \left(\frac{\lambda + \epsilon}{2}\right)_n)SP_C((u_m)_n - \lambda_nA)(u_m)_n)\]

for every \(n = 0, 1, 2, \ldots\), where \(\{\lambda_n\} \subset [a, b]\) and \(\left(\frac{\lambda + \epsilon}{2}\right)_n\) is a sequence in \((0, 1)\). If \(\left(\frac{\lambda + \epsilon}{2}\right)_n\) and \(\{\lambda_n\}\) are chosen so that \(\lambda_n \in [a, b]\) for some \(a, b\) with \(0 < a < b < (\lambda + \epsilon)\),

\[\lim_{n \to \infty} \left(\frac{\lambda + \epsilon}{2}\right)_n = 0, \sum_{n=1}^{\infty} \left(\frac{\lambda + \epsilon}{2}\right)_n = \infty, \sum \left| \frac{\lambda + \epsilon}{2}\right|_{n+1} - \left| \frac{\lambda + \epsilon}{2}\right|_n | < \infty, \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty,\]

then \((u_m)_n\) converges strongly to \(q \in F(S) \cap VI(C, A)\), which is the exclusive solution in the \(F(S) \cap VI(C, A)\) to the following variational inequality

\[\langle (I - f)q, \epsilon \rangle \leq 0, \quad (q - \epsilon) \in F(S) \cap VI(C, A).\]

**Proof.** Put \((u_{m+1})_n = P_C((u_m)_n - \lambda_nA)(u_m)_n)\) for every \(n = 0, 1, 2, \ldots\). Let \(u_m \in F(S) \cap VI(C, A)\). We have

\[\|(u_{m+1})_n - u_m\| = \|P_C((u_m)_n - \lambda_nA)(u_m)_n) - P_C(u_m - \lambda_nA)(u_m)\|
\leq \|(u_m)_n - \lambda_nA)(u_m)_n) - (u_m - \lambda_nA)(u_m)\|
\leq \|(u_m)_n - u_m\|\]

for every \(n = 1, 2, 3, \ldots\). Then we have

\[\|(u_{m+1})_n - u_m\| = \left\|\left(\frac{\lambda + \epsilon}{2}\right)_n f ((u_m)_n) + (1 - \left(\frac{\lambda + \epsilon}{2}\right)_n)S(u_{m+1})_n - u_m\right\|
\leq \left(\frac{\lambda + \epsilon}{2}\right)_n \|f ((u_m)_n) - u_m\| + (1 - \left(\frac{\lambda + \epsilon}{2}\right)_n)\|S(u_{m+1})_n - u_m\|
\leq \left(\frac{\lambda + \epsilon}{2}\right)_n \|f ((u_m)_n) - f (u_m)\|
\leq \left(\frac{\lambda + \epsilon}{2}\right)_n \|f ((u_m)_n) - u_m\| + (1 - \left(\frac{\lambda + \epsilon}{2}\right)_n)\|S(u_{m+1})_n - u_m\|
\leq \max \left\{\|(u_m)_n - u_m\|, \frac{1}{\epsilon}\|f (u_m) - u_m\|\right\}.
\]

By induction,

\[\|(u_m)_n - u_m\| \leq \max \left\{\|(u_m)_0 - u_m\|, \frac{1}{\epsilon}\|f (u_m) - u_m\|\right\}, n \geq 0.
\]

Therefore, \((u_m)_n\) is bounded, \((u_{m+1})_n\), \(S(u_{m+1})_n\), \(A(u_{m+1})_n\), \(f ((u_m)_n)\) are also bounded. Since \(I - \lambda_nA\) is nonexpansive of sequence and \(u_m = P_C(u_m - \lambda_nA)(u_m)\), we also have

\[\|(u_{m+1})_n - (u_{m+1})_n\| \leq \|(u_{m+1})_n - \lambda_nA(u_{m+1})_n\| - ((u_m)_n - \lambda_nA(u_m)_n)\|
\leq \|(u_{m+1})_n - \lambda_nA(u_{m+1})_n\| - ((u_m)_n - \lambda_nA(u_m)_n)\|
\leq \|(u_{m+1})_n - (u_m)_n\| + \|\lambda_n - \lambda_{n+1}\|\|A(u_m)_n\|
\leq \|(u_{m+1})_n - (u_m)_n\| + |\lambda_n - \lambda_{n+1}|\|A(u_m)_n\|
\]

for every \(n = 1, 2, 3, \ldots\). So we obtain

\[\|(u_{m+1})_n - (u_m)_n\| = \left\| \left(\frac{\lambda + \epsilon}{2}\right)_n f ((u_m)_n) + (1 - \left(\frac{\lambda + \epsilon}{2}\right)_n)S(u_{m+1})_n - \left(\frac{\lambda + \epsilon}{2}\right)_n f ((u_m)_n - \lambda_nA)(u_m)_n)\right\|
\leq \left(\frac{\lambda + \epsilon}{2}\right)_n \|f ((u_m)_n - \lambda_nA)(u_m)_n) - \|S(u_{m+1})_n - S(u_{m+1})_n\| - \left(\frac{\lambda + \epsilon}{2}\right)_n \|f ((u_m)_n - \lambda_nA)(u_m)_n\|
\leq \left(\frac{\lambda + \epsilon}{2}\right)_n \|f ((u_m)_n - \lambda_nA)(u_m)_n\|
\leq \left(\frac{\lambda + \epsilon}{2}\right)_n \|f ((u_m)_n - S(u_{m+1})_n)\|\right\|
\leq \left(\frac{\lambda + \epsilon}{2}\right)_n \|f ((u_m)_n - S(u_{m+1})_n)\|\]

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So, we obtain

\[ + (1 - \left(\frac{\lambda + \epsilon}{2}\right)_n) \| (u_{m+1})_n - (u_{m+1})_{n-1} \|
\]

\[ + \left(\frac{\lambda + \epsilon}{2}\right)_n \| (u_m)_n - (u_m)_{n-1} \|
\]

\[ \leq (1 - \left(\frac{\lambda + \epsilon}{2}\right)_n) \left( \| (u_m)_n - (u_m)_{n-1} \| + |\lambda_{n-1} - \lambda_n| \| A(u_m)_{n-1} \| \right)
\]

\[ + \left(\frac{\lambda + \epsilon}{2}\right)_n \| (u_m)_n - (u_m)_{n-1} \| \| f((u_m)_{n-1}) - S(u_{m+1})_{n-1} \|
\]

\[ + \left(\frac{\lambda + \epsilon}{2}\right)_n \| (u_m)_n - (u_m)_{n-1} \| \leq (1 - \epsilon \left(\frac{\lambda + \epsilon}{2}\right)_n) \| (u_m)_n - (u_m)_{n-1} \|
\]

\[ + L |\lambda_n - \lambda_{n-1}| + M \left(\frac{\lambda + \epsilon}{2}\right)_n - \left(\frac{\lambda + \epsilon}{2}\right)_{n-1} \|
\]

For every \( n = 0, 1, 2, \ldots \), where \( L = \sup \{ \| f((u_m)_n) - S(u_{m+1})_{n-1} \| : n \in N \} \), \( M = \sup \{ \| A(u_m)_{n-1} \| : n \in N \} \), since \( \sum_{n=1}^\infty |\lambda_n - \lambda_{n+1}| < \infty \), \( \sum_{n=1}^\infty \left| \left(\frac{\lambda + \epsilon}{2}\right)_n - \left(\frac{\lambda + \epsilon}{2}\right)_{n-1} \right| < \infty \) in view of Lemma 2, we have \( \lim_{n \to \infty} \| (u_m)_{n+1} - (u_m)_n \| = 0 \) then also we obtain \( \| (u_m)_{n+1} - (u_m)_n \| \to 0 \)

\[ \| (u_m)_n - S(u_{m+1})_{n-1} \| \leq \| (u_m)_n - (u_m)_{n-1} \| + \| S(u_{m+1})_{n-1} - S(u_{m+1})_n \|
\]

\[ \leq (\frac{\lambda + \epsilon}{2})_{n-1} \| f((u_m)_{n-1}) - S(u_{m+1})_{n-1} \| + \| (u_m)_{n-1} - (u_m)_n \|
\]

we have \( \| (u_m)_n - S(u_{m+1})_{n-1} \| \to 0 \). For \( u_m \in F(S) \cap VI(C, A) \),

\[ \| (u_m)_{n+1} - u_m \|^2 = \left(\frac{\lambda + \epsilon}{2}\right)_n \| f((u_m)_n) + (1 - \left(\frac{\lambda + \epsilon}{2}\right)_n) S(u_{m+1})_{n-1} - u_m \|^2
\]

\[ \leq \left(\frac{\lambda + \epsilon}{2}\right)_n \| f((u_m)_n) - u_m \|^2 + (1 - \left(\frac{\lambda + \epsilon}{2}\right)_n) \| (u_{m+1})_n - u_m \|^2
\]

\[ \leq \left(\frac{\lambda + \epsilon}{2}\right)_n \| f((u_m)_n) - u_m \|^2
\]

\[ + (1 - \left(\frac{\lambda + \epsilon}{2}\right)_n) \| (u_m)_n - u_m \|^2 + \| (u_{m+1})_n - u_m \|^2
\]

\[ \leq \left(\frac{\lambda + \epsilon}{2}\right)_n \| f((u_m)_n) - u_m \|^2
\]

\[ + (1 - \left(\frac{\lambda + \epsilon}{2}\right)_n) \| (u_m)_n - u_m \|^2 + \| (u_{m+1})_n - u_m \|^2
\]

\[ \leq \left(\frac{\lambda + \epsilon}{2}\right)_n \| f((u_m)_n) - u_m \|^2
\]

Since \( \left(\frac{\lambda + \epsilon}{2}\right)_n \to 0 \) and \( \| (u_m)_n - (u_m)_{n+1} \| \to 0 \), then \( \| A(u_m)_n - A u_m \| \to 0, n \to \infty \). Further, from (1), we obtain

\[ \| (u_{m+1})_n - u_m \|^2 = \| P_C((u_m)_n - \lambda_n A(u_m)_{n-1}) - P_C(u_m) - \lambda_n A u_m \|^2
\]

\[ \leq \| (u_m)_n - \lambda_n A(u_m)_{n-1} - (u_m) - \lambda_n A u_m \|^2
\]

\[ = \frac{1}{2} \left\{ \| (u_m)_n - \lambda_n A(u_m)_{n-1} - (u_m) - \lambda_n A u_m \|^2
\]

\[ + \| (u_{m+1})_n - u_m \|^2
\]

\[ - \| (u_m)_n - \lambda_n A(u_m)_{n-1} - (u_m) - \lambda_n A u_m \|^2
\]

\[ \leq \frac{1}{2} \| (u_m)_n - u_m \|^2 + \| (u_{m+1})_n - u_m \|^2 + \| (u_m)_n - (u_{m+1})_n \|^2
\]

\[ + 2\lambda_n (u_m)_n - (u_{m+1})_n, A(u_m)_n - A u_m - \lambda_n^2 \| A(u_m) - A u_m \|^2
\]

So, we obtain

\[ \| (u_{m+1})_n - u_m \|^2 \leq \| (u_m)_n - u_m \|^2 - \| (u_m)_n - (u_{m+1})_n \|^2
\]

\[ + 2\lambda_n (u_m)_n - (u_{m+1})_n, A(u_m)_n - A u_m - \lambda_n^2 \| A(u_m) - A u_m \|^2 \]
And hence
\[\| (u_m)_{n+1} - u_m \|^2 \leq \left( \frac{\lambda + \epsilon}{2} \right)_n \| f ((u_m)_n) - u_m \|^2 + \left( 1 - \left( \frac{\lambda + \epsilon}{2} \right)_n \right) \| S(u_m)_{n-1} - u_m \|^2 \]
\[\leq \left( \frac{\lambda + \epsilon}{2} \right)_n \| f ((u_m)_n) - u_m \|^2 + \left( 1 - \left( \frac{\lambda + \epsilon}{2} \right)_n \right) \| (u_m)_{n-1} - u_m \|^2 \]
\[\leq \left( \frac{\lambda + \epsilon}{2} \right)_n \| f ((u_m)_n) - u_m \|^2 + \| (u_m)_n - u_m \|^2 \]
\[- (1 - \left( \frac{\lambda + \epsilon}{2} \right)_n) \| (u_m)_{n-1} - u_m \|^2 \]
\[+ 2 \left( 1 - \left( \frac{\lambda + \epsilon}{2} \right)_n \right) \lambda_n \| (u_m)_n - (u_m)_{n-1} \| A(u_m)_n - Au_m \]
\[- (1 - \left( \frac{\lambda + \epsilon}{2} \right)_n) \lambda_n^2 \| A(u_m)_n - Au_m \|^2.\]

Since \( \left( \frac{\lambda + \epsilon}{2} \right)_n \to 0 \), \( \| (u_m)_{n+1} - (u_m)_n \| \to 0 \) and \( \| A(u_m)_n - Au_m \| \to 0 \), we obtain \( \| (u_m)_{n-1} - (u_m)_{n-2} \| \to 0 \).

Choose a subsequence \( ((u_m)_n)_i \) of \( ((u_m)_n) \) such that
\[\lim_{n \to \infty} \sup \| f (q) - q, S(u_m)_n - q \| = \lim_{i \to \infty} \sup \| f (q) - q, S(u_m)_n - q \|.\]

As \( ((u_m)_n)_i \) is bounded, we have that there exists a subsequence \( ((u_m)_{n_i})_j \) of \( ((u_m)_n)_i \) converges weakly to \( (u_m)_3 \).

We may assume without loss of generality that \( (u_m)_3 \to (u_m)_3 \).

Since \( S(u_m)_n - (u_m)_n \| \to 0 \), we obtain \( S(u_m)_n \to (u_m)_3 \). Then we can obtain \( u_m, 3 \in F(S) \cap VI(C, A) \). In fact, let us first show that \( u_m, 3 \in VI(C, A) \). Let
\[ T \cdot u_m, 3 = \begin{cases} Au_m + N_C u_m, 3, & u_m, 3 \in C, \\ \emptyset, & u_m, 3 \notin C, \end{cases} \]

Then \( T \) is maximal monotone sequence. Let \( (u_m, 3, u_m, 2) \in G(T) \). Since \( u_m, 2 - Au_m, 1 \in N_C u_m, 1 \) and \( (u_m, 1)_n \in C \) we have
\[ (u_m, 1 - (u_m, 1)_n, u_m, 2 - Au_m, 1)_n \geq 0.\]

On the other hand, from \( (u_m, 1)_n = P_C ((u_m)_n - \lambda_n A(u_m)_n) \), we have \( (u_m, 1 - (u_m, 1)_n, (u_m, 1)_n - ((u_m)_n - \lambda_n A(u_m)_n)) \geq 0 \) and hence
\[ (u_m, 1 - (u_m, 1)_n, (u_m, 1)_n - (u_m, 1)_n - A(u_m)_n) \geq 0.\]

Therefore, we have
\[ (u_m, 1 - (u_m, 1)_n, u_m, 2) \geq (u_m, 1 - (u_m, 1)_n, A(u_m, 1) - (u_m, 1)_n, Au_m, 1) \]
\[ - (u_m, 1 - (u_m, 1)_n, \lambda_n (u_m, 1) - (u_m, 1)_n, A(u_m)_n) \]
\[ = (u_m, 1 - (u_m, 1)_n, Au_m, 1 - A(u_m)_n, \lambda_n (u_m, 1) - (u_m, 1)_n, A(u_m)_n) \]
\[ + (u_m, 1 - (u_m, 1)_n, A(u_m, 1) - A(u_m)_n, \lambda_n (u_m, 1) - (u_m, 1)_n, A(u_m)_n) \]
\[ - (u_m, 1 - (u_m, 1)_n, \lambda_n (u_m, 1) - (u_m, 1)_n, A(u_m, 1) - A(u_m)_n, \lambda_n (u_m, 1) - (u_m, 1)_n, A(u_m)_n) \]
\[ \geq (u_m, 1 - (u_m, 1)_n, A(u_m, 1) - A(u_m)_n, \lambda_n (u_m, 1) - (u_m, 1)_n, A(u_m)_n) \]
\[ - (u_m, 1 - (u_m, 1)_n, \lambda_n (u_m, 1) - (u_m, 1)_n, A(u_m, 1) - A(u_m)_n, \lambda_n (u_m, 1) - (u_m, 1)_n, A(u_m)_n) \].

Hence we have \( (u_m, 1 - u_m, 3, u_m, 2) \geq 0 \), since \( T \) is maximal monotone sequence, we have \( u_m, 3 \in \]
We have \( \| (u_m)_n - S(u_m)_n \| \to 0 \). In view of Lemma 1, we obtain \( u_{m+1} \in F(S) \)
\[
\limsup_{n \to \infty} (f(q) - q, S(u_{m+1})_n - q) = \lim_{i \to \infty} (f(q) - q, S(u_{m+1})_n - q) \leq 0.
\]
\[
\| (u_m)_{n+1} - q \|^2 = \left( \frac{\lambda + \epsilon}{2} \right) f((u_m)_n) + \left( 1 - \frac{\lambda + \epsilon}{2} \right) S(u_{m+1})_n - q \right)^2 = \left( \frac{\lambda + \epsilon}{2} \right)^2 \| f((u_m)_n) - q \|^2 + 2 \left( \frac{\lambda + \epsilon}{2} \right) \left( 1 - \frac{\lambda + \epsilon}{2} \right) \| f((u_m)_n) - q \|^2 + \left( 1 - \frac{\lambda + \epsilon}{2} \right) \| S(u_{m+1})_n - q \|^2 \\
\leq \left( 1 - 2 \frac{\lambda + \epsilon}{2} \right) + \left( \frac{\lambda + \epsilon}{2} \right)^2 \| (u_m)_n - q \|^2 + \left( 1 - \frac{\lambda + \epsilon}{2} \right) \| f((u_m)_n) - q \|^2 + \left( 1 - \frac{\lambda + \epsilon}{2} \right) \| S(u_{m+1})_n - q \|^2 \\
\leq \left( 1 - 2 \frac{\lambda + \epsilon}{2} \right) + \left( \frac{\lambda + \epsilon}{2} \right)^2 + 2(1 - \epsilon) \left( \frac{\lambda + \epsilon}{2} \right) \left( 1 - \frac{\lambda + \epsilon}{2} \right) \| (u_m)_n - q \|^2 + \left( \frac{\lambda + \epsilon}{2} \right)^2 \| f((u_m)_n) - q \|^2 + \left( \frac{\lambda + \epsilon}{2} \right) \| S(u_{m+1})_n - q \|^2 \\
= \left( 1 - \frac{\lambda + \epsilon}{2} \right) \| (u_m)_n - q \|^2 + \left( \frac{\lambda + \epsilon}{2} \right) \beta_n,
\]
where
\[
\beta_n = \left( \frac{\lambda + \epsilon}{2} \right) \| f((u_m)_n) - q \|^2 + 2(1 - \epsilon) \left( \frac{\lambda + \epsilon}{2} \right) \| f(q) - S(u_{m+1})_n - q \|.
\]

It is easily seen that \( \frac{\lambda + \epsilon}{2} \to 0 \), \( \Sigma_{n=1}^\infty \frac{\lambda + \epsilon}{2} = \infty \), and \( \limsup_{n \to \infty} \beta_n \leq 0 \), by Lemma 2 we obtain \( (u_m)_n \to q \).

This completes the proof. \( \square \)

\( S \) is a nonexpansive sequence of mapping, \( A \) is an \( \frac{\lambda + \epsilon}{2} \)-inverse strongly monotone sequence, \( f \in \Pi_C \). Thus, by Banach contraction mapping principle, there exists an exclusive fixed point (see [11])
\[
(u_m)_n = \left( \frac{\lambda + \epsilon}{2} \right) f((u_m)_n) + \left( 1 - \frac{\lambda + \epsilon}{2} \right) S_P C((u_m)_n) - A((u_m)_n), \quad \left( \frac{\lambda + \epsilon}{2} \right) \in (0, 1).
\]

For simplicity we will write \((u_m)_n \) for \((u_m)_n \) provided no confusion occurs. Next we show the convergence of \((u_m)_n \), (see [11]) whiles they claim the existence of the \( q \in F(S) \cap VI(C, A) \) which solves the variational inequality
\[
\langle (I - f)q, e \rangle \leq 0, \quad f \in \Pi_C, \quad \langle q - e \rangle \in F(S) \cap VI(C, A).
\]

**Theorem 3.1.** Let \( C \) be a closed convex subset of a real Hilbert space \( H \). Let \( f : C \to C \) be a contraction with coefficient \( (1 - \epsilon)(0 < \epsilon < 1) \), \( A \) an \( \frac{\lambda + \epsilon}{2} \)-inverse strongly monotone sequence of mapping of \( C \) to \( H \) and let \( S \) be a nonexpansive sequence of mapping of \( C \) into itself such that \( F(S) \cap VI(C, A) \neq \emptyset \). Suppose \((u_m)_n \), be sequences generated by
\[
(u_m)_n = \left( \frac{\lambda + \epsilon}{2} \right) f((u_m)_n) + \left( 1 - \frac{\lambda + \epsilon}{2} \right) S_P C((u_m)_n) - A((u_m)_n), \quad \left( \frac{\lambda + \epsilon}{2} \right) \in (0, 1).
\]
where \( \{\lambda_n\} \subset [a, b] \) and \( \left\{ \left( \frac{\lambda_n + \epsilon}{2} \right) \right\} \) is a sequence in \([0, 1]\). If \( \left( \frac{\lambda_n + \epsilon}{2} \right) \) and \( \{\lambda_n\} \) are chosen so that \( \lambda_n \in [a, b] \) for some \( a, b \) with \( 0 < a < b < (\lambda + \epsilon) \), when \( \lim_{n \to \infty} \left( \frac{\lambda_n + \epsilon}{2} \right) = 0 \), \( (u_{m+3})_n \) converges strongly to \( q \), and such that the variational inequality

\[ ((I - f)q, e) \leq 0, \quad f \in P_C, \quad (q - e) \in F(S) \cap VI(C, A). \]

**Proof.** Put \( (u_{m+1})_n = P_C((u_{m+3})_n - \lambda_n A(u_{m+3})_n) \) for every \( n = 0, 1, 2, \ldots \) Let \( u_m \in F(S) \cap VI(C, A) \). We have

\[
\| (u_{m+1})_n - u_m \| = \| P_C((u_{m+3})_n - \lambda_n A(u_{m+3})_n) - P_C(u_m - \lambda_n Au_m) \|
\leq \| (u_{m+3})_n - \lambda_n A(u_{m+3})_n) - (u_m - \lambda_n Au_m) \|
\leq \| (u_{m+3})_n - u_m \|
\]

for every \( n = 1, 2, 3, \ldots \) Then we have

\[
\begin{aligned}
\| (u_{m+3})_n - u_m \| & = \left\| \left( \frac{\lambda_n + \epsilon}{2} \right)_n f ((u_{m+3})_n) + (1 - \left( \frac{\lambda_n + \epsilon}{2} \right)_n) S(u_{m+1})_n - u_m \right\|
\leq \left( \frac{\lambda_n + \epsilon}{2} \right)_n \| f ((u_{m+3})_n) - u_m \| + (1 - \left( \frac{\lambda_n + \epsilon}{2} \right)_n) \| S(u_{m+1})_n - u_m \|
\leq \left( \frac{\lambda_n + \epsilon}{2} \right)_n \| f ((u_{m+3})_n) - f (u_m) \|
+ \left( 1 - \left( \frac{\lambda_n + \epsilon}{2} \right)_n \right) \| f (u_m) - u_m \| + (1 - \left( \frac{\lambda_n + \epsilon}{2} \right)_n) \| (u_{m+3})_n - u_m \|
\leq \left( \frac{\lambda_n + \epsilon}{2} \right)_n \| f (u_m) - u_m \|.
\end{aligned}
\]

Hence,

\[
\| (u_{m+3})_n - u_m \| \leq \frac{1}{2} \| f (u_m) - u_m \|
\]

and \( (u_{m+3})_n \) is bounded, \( \{(u_{m+1})_n, (S(u_{m+1})_n), \{A(u_{m+3})_n\}, \{f ((u_{m+3})_n)\} \) are also bounded.

\[
\begin{aligned}
\| (u_{m+3})_n - u_m \|^2 & = \left\| \left( \frac{\lambda_n + \epsilon}{2} \right)_n f ((u_{m+3})_n) + (1 - \left( \frac{\lambda_n + \epsilon}{2} \right)_n) S(u_{m+1})_n - u_m \right\|^2
\leq \left( \frac{\lambda_n + \epsilon}{2} \right)_n \| f ((u_{m+3})_n) - u_m \|^2 + (1 - \left( \frac{\lambda_n + \epsilon}{2} \right)_n) \| (u_{m+1})_n - u_m \|^2
\leq \left( \frac{\lambda_n + \epsilon}{2} \right)_n \| f ((u_{m+3})_n) - u_m \|^2
+ \left( 1 - \left( \frac{\lambda_n + \epsilon}{2} \right)_n \right) \| (u_{m+1})_n - u_m \|^2 + \mu_n (\lambda_n - (\lambda + \epsilon)) \| A(u_{m+3})_n - Au_m \|^2
\leq \left( \frac{\lambda_n + \epsilon}{2} \right)_n \| f ((u_{m+3})_n) - u_m \|^2 + \left( 1 - \left( \frac{\lambda_n + \epsilon}{2} \right)_n \right) \| (u_{m+3})_n - u_m \|^2
+ \left( 1 - \left( \frac{\lambda_n + \epsilon}{2} \right)_n \right) a(\lambda - (\lambda + \epsilon)) \| A(u_{m+3})_n - Au_m \|^2.
\end{aligned}
\]

Therefore, we have

\[
\begin{aligned}
- \left( 1 - \left( \frac{\lambda_n + \epsilon}{2} \right)_n \right) a(\lambda - (\lambda + \epsilon)) \| A(u_{m+3})_n - Au_m \|^2
\leq \left( \frac{\lambda_n + \epsilon}{2} \right)_n \| f ((u_{m+3})_n) - u_m \|^2 + \| (u_{m+3})_n - u_m \|^2.
\end{aligned}
\]

Since \( \left( \frac{\lambda_n + \epsilon}{2} \right)_n \to 0 \) \( (n \to \infty) \), and \( \{(u_{m+3})_n\}, \{(u_{m+3})_n\} \) are bounded, we obtain

\[
\| A(u_{m+3})_n - Au_m \| \to 0 \quad (n \to \infty).
\]

From (1) we have

\[
\| (u_{m+1})_n - u_m \|^2 = \| P_C((u_{m+3})_n - \lambda_n A(u_{m+3})_n) - P_C(u_m - \lambda_n Au_m) \|^2
\leq \| (u_{m+3})_n - \lambda_n A(u_{m+3})_n - (u_m - \lambda_n Au_m), (u_{m+1})_n - u_m \|^2
\]

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So, we obtain
\[ \| (u_{m+1})_n - u_m \|^2 \leq \| (u_{m+3})_n - (u_{m+1})_n \|^2 - \| (u_{m+3})_n - (u_{m+1})_n \|^2 + 2\lambda_n (u_{m+3})_n - (u_{m+1})_n, A(u_{m+3})_n - A u_m \) - \lambda_n^2 \| A(u_{m+3})_n - A u_m \|^2. \]

So we have
\[ \| (u_{m+3})_n - u_m \|^2 \leq \left( 1 - \left( \frac{1 + \epsilon}{2} \right)_n \right) \| f ((u_{m+3})_n) - u_m \|^2 + \left( 1 - \left( \frac{1 + \epsilon}{2} \right)_n \right) \| (u_{m+1})_n - u_m \|^2 \]
\[ \leq \frac{\lambda + \epsilon}{2} \| f ((u_{m+3})_n) - u_m \|^2 + \left( 1 - \left( \frac{1 + \epsilon}{2} \right)_n \right) \| (u_{m+3})_n - u_m \|^2 \]
\[ - (1 - \left( \frac{1 + \epsilon}{2} \right)_n) \lambda_n (u_{m+3})_n - (u_{m+1})_n, A(u_{m+3})_n - A u_m \]
\[ - (1 - \left( \frac{1 + \epsilon}{2} \right)_n^2 \lambda_n^2 \| A(u_{m+3})_n - A u_m \|^2. \]

Hence,
\[ \left( 1 - \left( \frac{1 + \epsilon}{2} \right)_n \right) \| (u_{m+3})_n - (u_{m+1})_n \|^2 \leq \left( \frac{1 + \epsilon}{2} \right)_n \| f ((u_{m+3})_n) - u_m \|^2 - \left( \frac{1 + \epsilon}{2} \right)_n \| (u_{m+3})_n - u_m \|^2 \]
\[ + 2 \left( 1 - \left( \frac{1 + \epsilon}{2} \right)_n \right) \lambda_n (u_{m+3})_n - (u_{m+1})_n, A(u_{m+3})_n - A u_m \]
\[ - \lambda_n^2 \| A(u_{m+3})_n - A u_m \|^2. \]

Since \( \left( \frac{1 + \epsilon}{2} \right)_n \to 0, \| A(u_{m+3})_n - A u_m \| \to 0 \), we obtain \( \| (u_{m+3})_n - (u_{m+1})_n \| \to 0 \) (n \to \infty). By the proof of Proposition 3.1 we have \( (u_{m+1})_n \to q \) and \( q \in \mathcal{F}(S) \cap VI(C, A) \), so \( (u_{m+3})_n \to q \).

\[ \| (u_{m+3})_n - q \|^2 = \left( \frac{1 + \epsilon}{2} \right)_n f ((u_{m+3})_n) + \left( 1 - \left( \frac{1 + \epsilon}{2} \right)_n \right) S((u_{m+1})_n) - q \right\|^2 \]
\[ = \left( \frac{1 + \epsilon}{2} \right)_n f ((u_{m+3})_n) - q + \left( 1 - \left( \frac{1 + \epsilon}{2} \right)_n \right) S((u_{m+1})_n) - q \right\) \]
\[ = \left( \frac{1 + \epsilon}{2} \right)_n f ((u_{m+3})_n) - q, (u_{m+3})_n - q \)
\[ + \left( 1 - \left( \frac{1 + \epsilon}{2} \right)_n \right) \| S((u_{m+1})_n) - q \|^2 \]
\[ + \left( \frac{1 + \epsilon}{2} \right)_n \| f ((u_{m+3})_n) - q, (u_{m+3})_n - q \|^2 \]
\[ + \left( \frac{1 + \epsilon}{2} \right)_n \| f ((u_{m+3})_n) - q, (u_{m+3})_n - q \|^2. \]

Hence
\[ \| (u_{m+3})_n - q \|^2 \leq \| f ((u_{m+3})_n) - q, (u_{m+3})_n - q \| + \| f ((u_{m+3})_n) - q, (u_{m+3})_n - q \| \]
\[ + \| f ((u_{m+3})_n) - q, (u_{m+3})_n - q \| \leq (1 - \epsilon) \| S((u_{m+1})_n) - q \|^2 + \| f ((u_{m+3})_n) - q, (u_{m+3})_n - q \|. \]

This implies that
\[ \| (u_{m+3})_n - q \|^2 \leq \| f ((u_{m+3})_n) - q, (u_{m+3})_n - q \|. \]

But \( (u_{m+3})_n \to q \), it follows that \( (u_{m+3})_n \to q \). Now we show that q solves the variational inequality
\[ \langle (I - f) q, \epsilon \rangle \leq 0, \quad f \in \mathcal{F}(S) \cap VI(C, A). \]

Because
Applications for Nonexpansive and monotone sequence of mappings with Viscosity approximation

\[(u_{m+3})_n - f((u_{m+3})_n) = - \frac{1 - \frac{(\lambda + e)}{2}}{n} ((u_{m+3})_n - S(u_{m+3})_n),\]

For any \((q - e) \in F(S) \cap VI(C,A)\) and notice \((q - e) = P_C((q - e) - \lambda_n A(q - e))\), we infer that

\[
\begin{align*}
\langle (u_{m+3})_n - f((u_{m+3})_n), (u_{m+3})_n - q \rangle \\
= - \frac{1 - \frac{(\lambda + e)}{2}}{n} ((u_{m+3})_n - SP_C((u_{m+3})_n - \lambda_n A(u_{m+3})_n), (u_{m+3})_n - q) \\
= - \frac{1 - \frac{(\lambda + e)}{2}}{n} ((u_{m+3})_n - SP_C((u_{m+3})_n - \lambda_n A(u_{m+3})_n) \\
- (q - e)SP_C((q - e) - \lambda_n A(q - e))), (u_{m+3})_n - (q - e) \leq 0,
\end{align*}
\]

Since \(I - SP_C(I - \lambda_n A)\) is strong monotone sequence. Let \(i \to \infty\), we have

\[
\langle q - f(q), e \rangle \leq 0.
\]

Assume that there exists another subsequence \(\{(u_{m+3})_{n_j}\}\) of \(\{(u_{m+3})_n\}\) such that \((u_{m+3})_{n_j} \to q^*\), so \(q^* \in F(S) \cap VI(C,A)\), and from \((u_{m+3})_n - f((u_{m+3})_n), (u_{m+3})_n - (q - e) \leq 0\), let \(j \to \infty\) we have

\[
\langle q - f(q), q - q^* \rangle \leq 0, (q - e) \in F(S) \cap VI(C,A).
\]

Setting \((q - e) = q^*\) in (5), we have

\[
\langle q - f(q), q - q^* \rangle \leq 0,
\]

and setting \(e = 0\) in (6), we obtain

\[
\langle q^* - f(q^*), q^* - q \rangle \leq 0.
\]

Inequality (7) and (8) yield

\[
\|q - q^*\|^2 \leq (f(q) - f(q^*), q - q^*) \leq (1 - e)\|q - q^*\|^2,
\]

Which implies that \(q = q^*, \) since \(0 < e < 1\) Thus, \(u_{m+3} \to q\) as \(n \to \infty\) and \(q \in F(S) \cap VI(C,A)\) is exclusive. And \(q\) is the exclusive solution of variational inequality

\[
(q - f(q), e) \leq 0, \quad (q - e) \in F(S) \cap VI(C,A).
\]

This completes the proof.

4. Applications

We show two theorems in a Hilbert space by using Proposition 3.1 and Theorem 3.1(see [11,10]). A mapping \(T^2 : C \to C\) is called strictly pseudocontractive and projection if there exists \((1 - e)\) with \(0 \leq e < 1\) such that

\[
\|T^2u_m - T^2u_{m+1}\|^2 \leq \|u_m - u_{m+1}\|^2 + (1 - e)(I - T^2)u_m - (I - T^2)u_{m+1}\|^2
\]

For every \(u_m, u_{m+1} \in C\). If \(e = 1\), then \(T^2\) is nonexpansive of sequence. Put \(A^2 = I - T^2\), where \(T^2 : C \to C\) is a strictly pseudocontractive and a projection mapping with \((1 - e)\). Then \(A^2\) is \(\frac{e^2}{4}\)-inverse-strongly monotone sequence. Actually, we have, for all \(u_m, u_{m+1} \in C\),

\[
\|I - A^2\|u_m - (I - A^2)u_{m+1}\|^2 \leq \|u_m - u_{m+1}\|^2 + (1 - e)\|A^2u_m - A^2u_{m+1}\|^2.
\]

On the other hand, since \(H\) is a real Hilbert space, we have

\[
\|I - A^2\|u_m - (I - A^2)u_{m+1}\|^2 \leq \|u_m - u_{m+1}\|^2 + A^2u_m - A^2u_{m+1}\|^2 - 2(u_m - u_{m+1}, A^2u_m - A^2u_{m+1}).
\]

Hence we have

\[
\langle u_m - u_{m+1}, A^2u_m - A^2u_{m+1}\rangle \geq \frac{e^2}{4}\|A^2u_m - A^2u_{m+1}\|^2.
\]

Using Proposition 3.1 and Theorem 3.1, we first show a strong convergence theorem (see [11]) for finding a common fixed point of a nonexpansive sequence of mapping and a strictly pseudocontractive and projection mapping.

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Theorem 4.1. Let $C$ be a closed convex set of a real Hilbert space $H$. Let $f$ be a contractive mapping of $C$ into itself with coefficient $0 < \epsilon < 1$, $S$ be a nonexpansive sequence of $C$ into itself and let $T^2$ be a strictly pseudocontractive and projection mapping of $C$ into itself with $\frac{(\lambda + \epsilon)^2}{2}$, such that $F(S^2) \cap F(T^2) \neq \emptyset$. Suppose $(u_m)_1 = u_m \in C$ and $\{(u_n)_n\}$ is given by

$$(u_n)_n = \left(\frac{\lambda + \epsilon}{2}\right)^n \left((u_{n-1}) + (1 - \frac{\lambda + \epsilon}{2})^n S^2((1 - \lambda_n)(u_m)_n + \lambda_n T^2(u_m)_n)\right)$$

For every $n = 1, 2, \ldots$, where $\left(\frac{(\lambda + \epsilon)^2}{2}\right)_n$ is a sequence in $(0, 1)$ and $(\lambda_n)$ is a sequence in $[0, 2 - (\lambda + \epsilon)]$. If $\left(\frac{(\lambda + \epsilon)^2}{2}\right)_n$ and $(\lambda_n)$ are chosen so that $\lambda_n \in [a, b]$ for some $a, b$ with $0 < 2a < 2b < 2 - (\lambda + \epsilon)$, then $(u_n)_n$ converges strongly to some $q \in F(S^2) \cap F(T^2)$, such that

$$f(q) - q \leq 0, \quad (q - e) \in F(S^2) \cap F(T^2).$$

Proof. Put $A^2 = I - T^2$. Then $A^2$ is $\frac{2 - (\lambda + \epsilon)}{4}$ -inverse-strongly monotone sequence. We have $F(T^2) = VI(C, A^2)$ and $P_G((u_m)_n - \lambda_n A^2(u_m)_n) = (1 - \lambda_n)(u_m)_n + \lambda_n T^2(u_m)_n$. So by Proposition 3.1 and Theorem 3.1, (see [11]), we obtain the desired result.

Theorem 4.2. Let $H$ be a real Hilbert space $H$. Let $f$ be a contractive mapping of $H$ into itself with coefficient $0 < \epsilon < 1$, $S^2$ be a nonexpansive sequence mapping of $H$ into itself and let $A^2$ be a contraction and projection of a $\left(\frac{(\lambda + \epsilon)^2}{2}\right)$-inverse strongly monotonous sequence of mappings of $H$ into itself such that $F(S^2) \cap (A^2)^{-1} \neq \emptyset$. Suppose $(u_m)_1 = u_m \in C$ and $\{(u_n)_n\}$ is given by

$$(u_n)_n = \left(\frac{\lambda + \epsilon}{2}\right)^n \left((u_{n-1}) + (1 - \frac{\lambda + \epsilon}{2})^n S^2((u_m)_n - \lambda_n A^2(u_m)_n)\right)$$

for every $n = 1, 2, \ldots$, where $\left(\frac{(\lambda + \epsilon)^2}{2}\right)_n$ is a sequence in $(0, 1)$ and $\lambda_n$ is a sequence in $[0, \lambda + \epsilon]$. If $\left(\frac{(\lambda + \epsilon)^2}{2}\right)_n$ and $\{(\lambda_n)\}$ are chosen so that $\lambda_n \in [a, b]$ for some $a, b$ with $0 < a < b < \lambda + \epsilon$.

$$\lim_{n \to \infty} \left(\frac{\lambda + \epsilon}{2}\right)_n = 0, \quad \lim_{n \to \infty} \left(\frac{\lambda + \epsilon}{2}\right)_n = \infty, \quad \lim_{n \to \infty} \left(\frac{\lambda + \epsilon}{2}\right)^{n-1} \left(\frac{\lambda + \epsilon}{2}\right)_n = \infty, \quad \lim_{n \to \infty} \left(\frac{\lambda + \epsilon}{2}\right)_n = \infty, \quad \lim_{n \to \infty} \left(\frac{\lambda + \epsilon}{2}\right)_n = \infty,$$

then $(u_n)_n$ converges strongly to some $q \in F(S^2) \cap (A^2)^{-1}$, such that

$$f(q) - q \leq 0, \quad (q - e) \in F(S^2) \cap (A^2)^{-1},$$

Proof. We have $(A^2)^{-1} = VI(C, A^2)$, so putting $P_G = 1$, by Proposition 3.1 and Theorem 3.1, we obtain the desired result. (see [11]).

References