Mathematical Sanctity of the Golden Ratio

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Abstract: The frequency of appearance of the Golden Ratio (Φ) in nature implies its importance as a cosmological constant and sign of being a fundamental characteristic of the Universe. Except than Leonardo Da Vinci’s ‘Monalisa’ it appears on the sunflower seed head, flower petals, pinecones, pineapple, tree branches, shell, hurricane, tornado, ocean wave, and animal flight patterns. It is also very prominent on human body as it appears on human face, legs, arms, fingers, shoulder, height, eye-nose-lips, and all over DNA molecules and human brain as well. It is inevitable in ancient Egyptian pyramids and many of the proportions of the Parthenon. But very few of us are aware of the fact that it is part and parcel for constituting black hole’s entropy equations, black hole’s specific heat change equation, also it appears at Komar Mass equation of black holes and Schwarzschild–Kottler metric - for null-geodesics with maximal radial acceleration at the turning point of orbits [1, 2, 3, 4]. But here in this paper the discussion is limited to the exhibition of mathematical aptitude of Golden Ratio a.k.a. the Divine Proportion.

Keywords: Golden Ratio, Divine Proportion, Cosmological Constant, Fundamental Constant of Nature

I. Introduction

By definition, two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities. Let’s say, straight line A is divided into two segments, into B and C in such manner that:

\[
\frac{A}{B} = \frac{B}{C} = \Phi; \text{ or, } B \cdot B = A \cdot C; \text{ So, } A = B + C.
\]

Now if we divide this equation by \(C \cdot C\), [ie, C-Square], we will find that, \(B \cdot B = B \cdot C + C \cdot C\). Which means that, \((B/C)^2 = B/C + 1\); ie, \(\Phi^2 = \Phi + 1\) or, \([\Phi^2 - \Phi - 1] = 0\). Solving this quadratic equation will give us \(\Phi = 1.618033988749895...\) the most irrational number which we denote by the Greek alphabet Phi.

Golden Ratio can be expressed in so many different ways. One of the most common expression is given below:

\[
\Phi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}}
\]

From this expression it can be formulated into \(\Phi = (1 + \frac{1}{\Phi})\), that is \(\Phi^2 = (\Phi + 1)\) or, \([\Phi^2 - \Phi - 1] = 0\).

Also another most common expression of Golden Ratio is:

\[
\Phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}
\]

From this expression it can be formulated into \(\Phi = \sqrt{\Phi + 1}\); that is, \(\Phi^2 = (\Phi + 1)\) or, \([\Phi^2 - \Phi - 1] = 0\).
II. Golden Ratio Φ in Arithmetic Numerals

As we can see the quadratic equation $[Φ^2 - Φ - 1] = 0$ gives the root value equal to the golden ratio, it can be written as $Φ = [(Φ^2 - Φ - 1)]$. Hence, $Φ = (Φ + 1)(Φ - 1)$. i.e. $1.618034 \times 0.618034$. Another interesting fact about that equation is $(Φ + 1) = 2.618034 = Φ^2 & (Φ - 1) = 0.618034 = \frac{1}{Φ}$. So, $[Φ^2 \frac{1}{Φ}] = Φ$.

So, $[Φ^2 - Φ - 1] = 0$

Alternate Solution - 1:

Given, $[Φ^2 - Φ - 1] = 0$

Alternate Solution - 2:

Given, $[Φ^2 - Φ - 1] = 0$

Alternate Solution - 3:

Given, $[Φ^2 - Φ - 1] = 0$

So, the reason for bringing up this mysterious sequence is it has an uncanny relationship with the Golden ratio Φ.

III. Golden Ratio Φ in Algebra

It has been observed that Golden Ratio appears at Fibonacci sequence as well. The Fibonacci sequence $F_n$ is such that each number is the sum of the two preceding ones, starting from 0 & 1; that is, $F_n = F_{n-1} + F_{n-2}$. So, $F_0 = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... F_n, F_{n+1}, F_{n+2}, ...$ (up to infinity). One of the most frequently rediscovered facts about the Fibonacci sequence is if we tabulate these numbers in a column, shifting the decimal point one to the right for each successive number, the sum equals $1/F_{12}$, 1/89, as indicated below:

<table>
<thead>
<tr>
<th>Sum of:</th>
<th>0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td></td>
</tr>
<tr>
<td>0.0002</td>
<td></td>
</tr>
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</tr>
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<tr>
<td>0.00000013</td>
<td></td>
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<tr>
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<td></td>
</tr>
<tr>
<td>0.00000055</td>
<td></td>
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<tr>
<td>0.0000000089</td>
<td></td>
</tr>
<tr>
<td>... ETC...</td>
<td></td>
</tr>
<tr>
<td>0.01123595505618...</td>
<td>= 1/89</td>
</tr>
</tbody>
</table>

Another fun fact of Fibonacci Sequence is (Last digit of $F_0$), (Last digit of $F_9$), (Last digit of $F_{62}$), ... (up to infinity) = Fibonacci Sequence itself. So, the reason for bringing up this mysterious sequence is it has an uncanny relationship with the Golden ratio Φ.
It has been observed that, the golden ratio can be approximated by a process of successive dividing of each term in the Fibonacci sequence by the previous term. And with each successive division, the result comes closer and closer to \( \Phi \), i.e., \( F_{n+1}/F_n = \Phi \). For example, 89/55 = 1.6181818181... Very close to \( \Phi \); as shown in the graph above. Because, let, \( F_n = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... \), A, B, C, ... (up to infinity). Say, B/A = X. So, C/B \( \approx \) X, as well. Hence, B/A = C/B. But, C = (A + B). That is, B/A = (A + B)/B. Or, B/A = (A/B + 1). Which means, X = (1/X + 1). Or, \( X^2 = X + 1 \); i.e, \( X^2 - X - 1 = 0 \). Hence, X = \( \Phi \).

Again, by forming matrix, we can say that, \( \begin{bmatrix} C \\ B \end{bmatrix} = \begin{bmatrix} A + B \\ B \end{bmatrix} \). Here say, \( \mathcal{E} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \). So, the characteristic equation will be, \( \lambda^2 - \lambda - 1 = 0 \); where \( \lambda \) is eigenvalue of \( \mathcal{E} \), & \( I \) is a \( (2 \times 2) \) identity matrix.

\[
\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 0;
\]

\[
\begin{bmatrix} 0 & 0 \\ 0 & -\lambda \end{bmatrix} = 0;
\]

\[
\begin{bmatrix} 1 - \lambda \\ 1 - \lambda \end{bmatrix}
\]

\[
\begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 - \lambda & 1 \\ 1 & 1 - \lambda \end{bmatrix} = 0; \quad \Rightarrow \quad [(-\lambda)(1 - \lambda) - (1 \times 1)] = 0. \]

So, \( [\lambda^2 - \lambda - 1] = 0 \). And here \( \mathcal{E} \) is a \( (2 \times 2) \) binary matrix. And similar to this matrix, the highest probability of any non-trivial eigenvalues that show up in binary matrices is (like this one), \( \Phi \). Furthermore the quadratic equation, \( [\Phi^2 - \Phi - 1] \) = 0; can be represented as, \( \frac{1}{\Phi} = (\Phi + 1) \) = 0. And again, \( \frac{1}{\Phi} = (\Phi - 1) \) = 0. \[2\]

IV. Golden Ratio \( \Phi \) in Trigonometry

From the inception of the idea of Golden Ratio, mathematicians all across the globe attempted to come up with equations correlating \( \pi \) and \( \Phi \). Personally, I figured two pi-phi relations: (i) \( 6\Phi^2 \approx 5\pi \) & (ii) \( \Phi \approx \frac{7\pi}{5\pi} \), by myself. But nothing beats the pi-phi relation \( \cos(\pi/5) = \Phi/2 \). Here beneath goes the mathematical evidence of the claim.

Let’s say, \( a = \cos(\pi/5) \) and \( b = \cos(2\pi/5) \). Hence, \( b = \cos(2\pi/5) = \cos(\pi/5 + \pi/5) \). Which means, term ‘b’ can be expressed as; \( b = \cos(\pi/5) \cos(\pi/5) - \sin(\pi/5) \sin(\pi/5) = \cos^2(\pi/5) - \sin^2(\pi/5) = \cos^2(\pi/5) - [1 - \cos^2(\pi/5)] \).

That is, \( b = 2\cos^2(\pi/5) - 1 = 2[\cos(\pi/5)]^2 - 1 \). i.e, \( b = 2a^2 - 1 \) \[\text{equation (i)}\].

Again, \( \cos(4\pi/5) = \cos[(2\pi - \pi)/5] = \cos(\pi - \pi/5) = \cos(\pi) \cos(\pi/5) - \sin(\pi) \sin(\pi/5) = \cos(\pi) = -a \).

As we know that, \( \cos(\pi) = 0 \) and \( \cos(\pi) = -1 \). Hence, \( -a = \cos(4\pi/5) = \cos(2\pi/5 + 2\pi/5) \). Hence, we can say that, \( -a = \cos(2\pi/5) \cos(2\pi/5) - \sin(2\pi/5) \sin(2\pi/5) = \cos^2(2\pi/5) - \sin^2(2\pi/5) = \cos^2(2\pi/5) - [1 - \cos^2(2\pi/5)] \).

That is, \( -a = 2\cos^2(2\pi/5) - 1 = 2[\cos(2\pi/5)]^2 - 1 \). i.e, \( -a = 2b^2 - 1 \) \[\text{equation (ii)}\].

Now if we deduce eqn. (ii) from eqn. (i), we get that; \( b + a = (2a^2 - 1) - (2b^2 - 1) = 2a^2 - 2b^2 + 1 \). That is \( (a + b) = 2a^2 - 2b^2 = 2(a^2 - b^2) = 2(a + b)(a - b) \). That is, \( (a - b) = (b - b)/(a + b) ) = 1/2 \) or, \( b = a - 1/2 \)

Putting this value in equation (i) gives us \( a - 1/2 = 2a^2 - 1 \), or, \( 2a^2 - a - 1 + 1/2 = 0 \), that is \( 2a^2 - a - 1/2 = 0 \).

So, that is, \( 4a^2 - 2a - 1 = 0 \). If we would put the value of \( b \) in equation (ii), we would’ve got, \( -a = 2(a - 1/2)^2 - 1 \).

That is to say, \( 2(a^2 - a + 1/4) - 1 + a = 0 \). Which means, \( 4a^2 - 2a - 1 = 0 \). The same. So, \( a = \cos(\pi/5) = 1 + \sqrt{5}/4 = \Phi/2 \).

Based on the concept of Pythagoras a right-angle triangle was made known as the Kepler Triangle, which is named after the German mathematician and astronomer Johannes Kepler (1571–1630). The edge lengths in a precise geometric progression in which the common ratio is \( \sqrt{\Phi} \); and the geometric progression goes like 1: \( \sqrt{\Phi} : \Phi \). Here, length of the hypotenuse of the right-angle triangle is \( \Phi \) and so the other two arms have lengths of 1 and \( \sqrt{\Phi} \). So, Pythagoras \( (\Phi)^2 = (\sqrt{\Phi})^2 + 1 \); or, \( \Phi^2 = \Phi + 1 \) i.e, \( \Phi^2 - \Phi - 1 = 0 = 0 \) [5].

The picture shown beside is a Kepler triangle. If \( \Theta \) is the angle between hypotenuse \( \Phi \) & base 1, then following relations can be drawn as well:

(i) \( \sin(\Theta) = \sqrt{\Phi}/\Phi = 1/\sqrt{\Phi} \) (ii) \( \cos(\Theta) = 1/\Phi \) (iii) \( \tan(\Theta) = \sqrt{5}/\Phi \). Hence, we can say that, \( \Theta = \sin^{-1}(1/\sqrt{\Phi}) = \cos^{-1}(1/\Phi) = \tan^{-1}(\sqrt{5}/\Phi) = 0.9 \quad \text{rad} = 51.83^\circ \). So, the other angle is \( (180 - 90 - 51.83^\circ) = 37.17^\circ = 2\pi/3 \) rad (roughly). Another interesting fact of this diagram is, here we have a circle with a diameter of \( \Phi \) and we have a square with sides of \( \sqrt{\Phi} \). Though it is not possible to square a circle, we can see Sketch of the “Vitruvian Man” by Leonardo Vinci shows these two geometric figures have perimeter very close to each other. So, the circle and the square have closely equal perimeter. Now, perimeter of the square is four times its arms, viz. \( 4\sqrt{\Phi} \). And perimeter of the circle is \( 2\pi \times \text{radius} = \pi \times \text{Diameter} = \pi \cdot \Phi \). Hence, we can say, \( \pi \cdot \Phi = 4\sqrt{\Phi} \), i.e, \( \pi = 4/\sqrt{\Phi} \). It fit for an error that’s less than 0.1%. Which brings us to another pi-phi relationship.
V. Golden Ratio Φ in Geometry

Here is one way to draw a rectangle with the Golden Ratio [6]:

First draw a square of unit length, that is the length is one. Place a dot at half way along one side & draw a line from that point to an opposite corner. So, the line will have a length of $\sqrt{(1)^2 + (\frac{1}{2})^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$.

Now either we add this value with $\frac{1}{2}$ or we deduct this value from $\frac{1}{2}$ to get the golden ratio.

We need to prove that $\angle A = \angle B = \angle C = \angle D = \angle E$. And all five arms. Besides, the assumptions will be all the five arms of the ‘polygon’ will have equal length. Let’s suppose ABCDE is the polygon; then $AB = BC = CD = DE = AE$.

Hence, $\angle A = \angle B = \angle C = \angle D = \angle E$. And all the angles of the pentagon are equal to be: $\theta = \{\pi - 2\pi / 5\} = \{3\times \pi / 5\} = \{3\times 36\} = 108\degree$. Having said that, it is noticeable that a perfect pentagon will inscribe inside a circle, and the five points will divide the circle into $360\degree / 5 = 72\degree$. Noticeably, $72\degree + 108\degree = 180\degree$.

Also $72\degree = (36\times 2)$, $108\degree = (36\times 3)$ & $180\degree = (36\times 5)$. Also from trigonometric expression we derived $\cos(\pi / 5) = \cos(36\degree) = \Phi / 2$. $AB = BC = CD = DE = AE$ & $\angle A = \angle B = \angle C = \angle D = \angle E = 108\degree$. As well as, $AO = BO = CO = DO = EO$, where O is the center of the circle. Join B & E. Line BE intersects line AO at point N. So, AN ⊥ BE, as well as $NE = NB = 1/2BE$. Join A & C. Line AC intersects line BE at point P and line BO at point Q. Also, $BO\perp AC$; which means, $AQ = CQ = 1/2AC$. We need to prove that, $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \Phi$.

Now at triangle AEB; $AE = BE$. As, $\angle BAE = 108\degree$, so other two angles; $\angle AEB = \angle BEA = (180\degree - 108\degree) / 2 = 72\degree / 2 = 36\degree$. In $\triangle AEN$; $\angle AEN = 90\degree$, $\angle NAE = 108\degree / 2 = 54\degree$. So $\angle AEN$ will be equal to $(180\degree - 90\degree - 54\degree) = 36\degree$. As, $\angle AEN = \angle AEB$. So, in $\triangle AEN$; $\cos(\angle AEN) = \cos(36\degree) = \Phi / 2$. $AB = BC = CD = DE = AE$ & $\angle A = \angle B = \angle C = \angle D = \angle E = 108\degree$. As well as, $AO = BO = CO = DO = EO$, where O is the center of the circle. Join B & E. Line BE intersects line AO at point N. So, $AN \perp BE$, as well as $NE = NB = 1/2BE$. Join A & C. Line AC intersects line BE at point P and line BO at point Q. Also, $BO \perp AC$; which means, $AQ = CQ = 1/2AC$. We need to prove that, $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \Phi$.

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Golden ratio can be expressed geometrically via an equilateral triangle inscribed inside a circle as well. In the figure below ΔABC is an equilateral triangle inscribed in a circle with center G and radius of AG = BG = CG. Now extend AG, that intersects BC at point X, & extend BG, that intersects AC at point Y, and extend CG, that intersects AB at point Z. Join Z & Y and extend in both directions to intersect the circle at point B' & C'. From this construction we will see that, ZY/B'Z = ZY/C'Y = B'Y/ZY = C'Z/ZY = Φ.

Let us assume, ZY = a, & YC' = b. We need to prove that a/b = (a + b)/a.

Now, AB = BC = CA & AG = BG = CG = 2GX = 2GY = 2GZ as well as AX = BY = CZ. So, AX┴BC, BY┴AC & CZ┴AB. Also we know that, points X, Y, and Z are midpoints of BC, CA, and AB respectively viz, AZ = BZ = BX = CX = CY = AY = ½AB = ½BC = ½CA. Now BC||B'C'. So, ΔAZY,<AZY = <ABY &<CZA = <BAC. So, ΔAZY is also equilateral as all these angles are 60° viz AZ = ZY = AY = a. Now, as product of segments of two intersecting cords of a circle are equal.

There are other ways to prove it as well like in ΔABP; <ABP = 60°, <APB = 90°; hence, <APB = 60°. Now sin<APB = AB/AP. Say the arms of the equilateral triangles are of length and radius of circle is R. Hence, sin<APB = sin60° = a/2R; viz, R = a/(2sin60°). Now in ΔABC, we see, sin<ABX = sin60° = √3/2 = AX/AB = (AG + GX)/AB = (R + r)/a = 3R/2a; viz, R = a/√3.

Now, PX' = (AP - AX') = (AP - AZ.sin<AZX) = (2R + r)/a = 3R/2a; viz, R = a/√3. Also in ΔABX, we see, sin<ABX = sin60° = √3/2 = AX/AB = (AG + GX)/AB = (R + r)/a = 3R/2a; viz, R = a/√3. Now, PX' = (AP - AX') = (AP - AZ.sin<AZX) = (2R + r)/a = 3R/2a; viz, R = a/√3. Also in ΔABX, we see, sin<ABX = sin60° = √3/2 = AX/AB = (AG + GX)/AB = (R + r)/a = 3R/2a; viz, R = a/√3.

There is also another construction of the golden ratio can be expressed geometrically via an equilateral triangle inscribed inside a circle as well. In the figure below ΔABC is an equilateral triangle inscribed in a circle with center G and radius of AG = BG = CG. Now extend AG, that intersects BC at point X, & extend BG, that intersects AC at point Y, and extend CG, that intersects AB at point Z. Join Z & Y and extend in both directions to intersect the circle at point B' & C'. From this construction we will see that, ZY/B'Z = ZY/C'Y = B'Y/ZY = C'Z/ZY = Φ.

Let us assume, ZY = a, & YC' = b. We need to prove that a/b = (a + b)/a. So, AX = BY = CZ as well as AX = BY = CZ. So, AX┴BC, BY┴AC & CZ┴AB. Also we know that, points X, Y, and Z are midpoints of BC, CA, and AB respectively viz, AZ = BZ = BX = CX = CY = AY = ½AB = ½BC = ½CA. Now BC||B'C'.

Now extend AG, that intersects BC at point X, & extend BG, that intersects AC at point Y, and extend CG, that intersects AB at point Z. Join Z & Y and extend in both directions to intersect the circle at point B' & C'. From this construction we will see that, ZY/B'Z = ZY/C'Y = B'Y/ZY = C'Z/ZY = Φ.

Let the radius of both circles r. In ΔBCD, (BD)^2 = (CD)^2 + (BC)^2. But, AC = BC = ½AB = DE = ½CD = r. So, (BD)^2 = (2r)^2 - (r)^2. Viz, (BD)^2 = (2r)^2 + (r)^2. Suggests, BD = √5r. Now in here, BE = BD + DE. So, BE = (√5r + r). Again AB = 2r. So, the ratio BE/AB = (√5r + r)/2r = (1 + √5)/2 = 1.618034 = Φ. Not only this but also this concept can be nicely modified into a construction with four circle which is shown in the diagram below (left). As well as another most straightforward construction of the golden ratio with this concept has been devised by Nguyen Thanh Dung shown in the diagram below (right) [7].
Tran Quang Hung [7] has devised another configuration of a 1x3 rectangle with a circle that produces golden ratio. But there is not an immediately obvious relation between the case of 1x2 and 1x3 rectangles. If we consider the arms of the square of the diagram to be a, then the red plus blue line becomes equal to $\sqrt{(3a)^2 + (a)^2} = a\sqrt{10}$; while the blue one becomes equal to the diameter $\sqrt{(a)^2 + (a)^2} = a\sqrt{2}$ (1st-bottom-left). So, the ratio becomes equal to $\sqrt{5}$.

In all these three diagrams the above-mentioned relationship can be observed. And as we know that the value of $\Phi = \frac{1 \pm \sqrt{5}}{2}$, therefore, golden ratio is found in all these three diagrams as well. Golden ratio has also observed in the constructions that involves a rhombus and a regular hexagon. Before going into that discussion it will be better to converse about another very elegant way of obtaining the golden ratio, offered in a (2002) article by K. Hofstetter [7]. It's shown in the diagram provided below. Here, it will be convenient to denote $S(R)$ the circle with center $S$ through point $R$. For the construction, let $A$ and $B$ be two points. Circles $A(B)$ and $B(A)$ intersect in $C$ and $D$ and cross the line $AB$ in points $E$ and $F$. Circles $B(E)$ and $A(F)$ intersect in $X$ and $Y$, as in the diagram. Because of the symmetry, points $X$, $D$, $C$, $Y$ are collinear. The fact is $CX/CD = \Phi$.

Assume for simplicity that $AB = 2$. Then $CD = 2\sqrt{3}$, & $CX = \sqrt{15} + \sqrt{3}$. Hence, the ratio of $CX$ & $CD$:

$$(CX)/(CD) = (\sqrt{15} + \sqrt{3})/2\sqrt{3}$$

$$= (\sqrt{5} + 1)/2 = \Phi.$$ 

Notice that the whole construction can be accomplished with compass only. This much simplicity as well as diversity has made golden ratio this much widespread and this is the reason of calling it in different other names like the golden mean or golden section (Latin: section aurea). Similarly some other names include extreme mean ratio, medial section, divine cut proportion, divine section (Latin: section divina), golden cut, golden proportion and golden number [5]. Hence, now we will discuss how this ratio has also observed in constructions involving a rhombus and a regular hexagon.
Let, ABCD is a rhombus with 2AC=BD. The inscribed circle has a center O. Also, E and F are the points of intersection of the circle with BD. Then, the point F divides DE in the golden ratio.

Now, let M be the point of tangency of (O) with AD. So, OM┴AD. Hence, ΔMOD & ΔAOD are similar as we can see <ADO = <MDO. Therefore, (MD)/(OD) = (OD)/(OA) = 2; viz. MD=2(OM) =EF.

From the property of a tangent, DM^2 = DF·DE.

Or, EF^2 = DF·DE.
Or, (FD)/(FE) = (EF)/(ED), as required.

Tran Quang Hung [7] has devised another configuration of golden ratio Φ in a hexagon. Square ABHG is constructed outside the hexagon ABCDEF. A circle with center at A, radius AH cuts EF at I in golden ratio as shown in the diagram below.

Say AB = BC = CD = DE = EF = AF = GA = BH = a
Therefore, AH = AI = √2a. Set α=<FAI, β=<AIF.

Now by applying the Law of Sines in ΔAIF:

AF/sinβ = FI/sinα = AI/sin120°

Thus we can say: sinβ = 3/(2√3) = 6/(4√3) = √6/4.

Hence, cosβ = 1 - (sinβ)^2 = 1 - 6/16 = √10/4.

Now, observe that (α+β) = 60°, so we can imply that sinα = sin(60° - β) = sin60°·cosβ - cos60°·sinβ.

Thus, sinα = (√3 / 2)·sinβ = 6/(4√3)·√6/4 = √3/(4√2) = 1/(2√2).

Now

AF/sinβ = FI/sinα; or, (2√3)a/3 = FI/(√6/4) = FI/(√5 - 1)

Or, FI/a = (√5 - 1)/2 viz, a/Fl = FE/FI = (√5 + 1)/2 = Φ.

Tran Quang Hung in 2015 has found once more the golden ratio in a combination of a semicircle, a square, & a right isosceles triangle [7]. Given a right isosceles triangle ABC and its circumcircle, inscribed a square DEFG with a side FG along the hypotenuse AB. Let the side DE extended beyond E intercept the circumcircle at P. Then the point E divides DP in the golden ratio.

From the similarity of the isosceles right triangles DEC and AEF, we have (DE)/(CE) = (AE)/(EF). It thus follows; DE^2 = DE·EF = AE·EC. If the line DE intersects the semicircle again at Q, then EQ=DP. By the intersecting chords theorem, AE·CE = EP·EQ = EP·DP. Therefore, DE^2 = EP·DP, meaning that E divides DP in the golden ratio.

We are going to conclude our discussion for this segment with an example of Tran Quang Hung [7]. Let ABC be an equilateral triangle inscribed in circle (O). D is reflection of A through BC. MN is diameter of (O) parallel to BC. AD meets (O) again at P. Then, circle (D) and passing through B, C divides PM, PN in golden ratio.
Here, OB=OC = OA = ON = OM = OP = PD = PC = PB = R (say). So, BC=CD=BD = AB = AC = DL = DK = √\(3\)R and, NP=MP = √\(R^2 + R^2\) = √\(2\)R.In ΔDLP, <DPL=135°.Say, PL=x. For a triangle with sides a,b,c & angle \(\mu\) opposite to c, as per law of cosine \(c^2 = a^2 + b^2 - 2ab \cos(\mu)\); viz, DL² = PD² + PL² - 2.PD.PL.cos135°; so, (√\(3\)R)² = (R)² + (x)² - 2.R.x.cos(180° - 45°). Now, as \(\cos45° = \frac{1}{\sqrt{2}}\); so, \((x)^2 + \sqrt{2}.R(x) - 2(R)^2 = 0\); or \(2x = -\sqrt{2}.R + \sqrt{2}R^2 + 8R^2\)

With positive value, \(x = \frac{\sqrt{10}-\sqrt{2}}{2}.R = \frac{\sqrt{5}-1}{\sqrt{2}}.R = PL\).

Now, NL = NP – PL = √\(2\)R - \(\frac{\sqrt{5}-1}{\sqrt{2}}.R\) = \(\frac{3-\sqrt{5}}{\sqrt{2}}.R\).

Now by taking ratio, PL/NL = \(\frac{\sqrt{5}-1}{\sqrt{2}}.R/\frac{3-\sqrt{5}}{\sqrt{2}}.R\) = \(\frac{\sqrt{5}-1}{3-\sqrt{5}}\).

Thus we can prove that, PL/NL = \(\frac{\sqrt{5}+1}{2}\) = Φ.

The following (right) construction of the golden ratio \(\Phi\) has appeared in the Mathematical Gazette, volume 101, number 551, July 2017, page 303 constructed by John Molkach [7]. There are two unit circles (A) & (B). The circle (O) has a 2R diameter of AB and tangent to both circles. Vertical segments AC & BF are tangent to circle B & circle A, respectively. So, AB = AC = BF = 1. CF crosses (O) in D and E, as shown in the diagram below. John proves that CE = Φ.

To prove that let us consider, CE=x.

Then, by Intersecting Secants rules

\[ CA^2 = CD \times CE. \]

Or, \((1)^2 = (x−1).x\) Viz \(x^2 - x = 1=0\)

VI. Golden ratio in Fractals

It is not so much that the golden ratio is “related to a fractal,” as fractal patterns are based on any number. Fractal patterns created using golden ratio, however, are optimized in a way that does not occur with any other number. As an example, in the image below the fractal pattern expands using the golden ratio. According to Mario Livio [8]: some of the greatest mathematical minds of all ages, from Pythagoras and Euclid in ancient Greece, through the medieval Italian mathematician Leonardo of Pisa & the Renaissance astronomer Johannes Kepler, to the present-day scientific figures such as Oxford physicist Roger Penrose, have spent endless hours over this simple ratio and its properties. The Biologists, musicians, historians, architects, psychologists, artists and even mystics have pondered debated the basis of its ubiquity and appeal. In fact, it is fair to say that golden ratio has inspired thinkers of all disciplines like no other number in mathematics.

VII. Conclusion

Mathematicians since Euclid have studied the properties of the golden ratio, including its appearance in dimensions of a regular pentagon and in a golden rectangle, which may be cut into a square and a smaller rectangle with that of the same aspect ratio. The golden ratio has also been used to analyze the proportions of natural objects as well as man-made systems such as financial markets, in some cases based on dubious fits to data. The golden ratio appears in some patterns in nature, including the spiral arrangement of leaves and other plant parts. Some twentieth-century artists and architects, including Le Corbusier and Salvador Dalí, have proportioned their works to approximate the golden ratio especially in the form of the golden rectangle, in which the ratio of the longer side to the shorter is the golden ratio believing this proportion to be aesthetically pleasing.
A Fibonacci spiral which approximates the golden spiral, using Fibonacci sequence square sizes up to 55. The spiral is drawn starting from the inner 1×1 square and continues outwards to successively larger squares.

Figure: Fibonacci Spiral Drawn by Nafish Sarwar Islam using MATLAB

References

[3]. Salvatore Giandinoto, “Superluminal Transportation of High Energy Particles Through Wormholes Using the Phi-Based Solution to the Schrödinger Wave Equation, the theorem of Residues and the Cauchy Integral Formula”, Journal Published on 2007
[6]. Weblink: https://www.mathsisfun.com/numbers/golden-ratio.html; collected on 2019
[7]. Weblink: https://www.cut-the-knot.org/do_you_know/GoldenRatio.shtml, collected on 2019