Conformally Berwald Finsler Space With special $(\alpha, \beta)$- Metric

Gayathri. K
1. Assistant Professor, REVA University, Kattigenahalli, Bengaluru-560064

Abstract: In this paper, we find the necessary and sufficient conditions for a Finsler space with the metric $L = \alpha - \frac{\beta^2}{\alpha}$ to be a Berwald space and also to be a Berwald space, where $\alpha$ is a Riemannian metric and $\beta$ is a differential one-form. Further, we study the conformal change of Berwald space with the above mentioned special $(\alpha, \beta)$- metric.

Keywords: Finsler Space, Berwald Space, Conformal change.

AMS Subject Classification (2010): 53A20, 53B40, 53C60, 58J60.

I. Introduction

The conformal theory of Finsler spaces has been introduced by M. S. Knebelman in 1929 and later on M. Hashiguchi developed such theory which was based on Matsumoto’s approach to Finsler geometry. The conformal theory of two-dimensional Finsler space has been studied by M. Matsumoto and based on the above work, B. N. Prasad and D. K. Diwedi discussed the theory of conformal change of three-dimensional Finsler space.

Nabil L. Youssef, S. H. Abed and A. Soleiman investigated intrinsically conformal changes in Finsler geometry. Also they studied conformal change of Barthel connection and its curvature tensor, the conformal changes of Cartan and Berwald connections as well as their curvature tensors. Shun-ichi Hojo, M. Matsumoto and K. Okubo discussed the theory of conformally Berwald Finsler spaces and its applications to $(\alpha, \beta)$ - metrics.

The notion of Douglas space has been introduced by M. Matsumoto and S. Bacso as a generalization of Berwald space from the view point of geodesic equations. It is remarkable that a Finsler space is a Douglas space or is of Douglas type if and only if the Douglas tensor vanishes identically.

M. Matsumoto studied on Finsler spaces with $(\alpha, \beta)$ - metric of Douglas type. Hong-Suh Park and Eun-See Choi explained Finsler spaces with an approximate Matsumoto metric of Douglas type. The authors S. Bacso and I. papp studied on a generalized Douglas space. Also the team of authors Benling Li, Yibing Shen and Zhongmin Shen studied on a class of Douglas metrics.

The first part of the present paper is devoted to study the condition for the Finsler space with special $(\alpha, \beta)$ - metric to be a Berwald type (Theorem 3.1). The second part is devoted to find the condition for the Finsler space with special $(\alpha, \beta)$ - metric to be Berwald space (Theorem 4.1) and finally, we apply the conformal change of Finsler space with the special $(\alpha, \beta)$ - metric of Douglas type (Theorem 5.1). We study conformal change of Berwald space with special $(\alpha, \beta)$ - metric $L = \alpha - \frac{\beta^2}{\alpha}$ and also we obtain the conditions for Finsler space with an $(\alpha, \beta)$ - metric $L^2 = 2\alpha\beta$ to be conformally Berwald.

II. Preliminaries

To find the condition for the Finsler space with special $(\alpha, \beta)$ - metric to be Berwald space (Theorem 4.1) and finally, we apply the conformal change of Finsler space with the special $(\alpha, \beta)$ - metric of Douglas type (Theorem 5.1). We study conformal change of Berwald space with special $(\alpha, \beta)$ - metric $L = \alpha - \frac{\beta^2}{\alpha}$ and also we obtain the conditions for Finsler space with an $(\alpha, \beta)$ - metric $L^2 = 2\alpha\beta$ to be conformally Berwald.

The geodesics of a Finsler space $F^n = (M^n, L)$ are given by the differential equations:

$$\frac{d^2 x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0,$$

Where $2G^i(x, y) = \gamma^i_{jk}(x, y)y^j_y^k$ and $\gamma^i_{jk}(x, y)$ are Christoffel symbols constructed from $g_{ij}(x, y)$ with respect to $x^i$.

A Finsler space $F^n$ is said to be of Douglas type if

$$D^i_j \equiv G^i_j(x, y)y^j - G^i(x, y)y^j$$

are homogeneous polynomials in $(y^i)$ of degree three.
Let $hp(r)$ denote the homogeneous polynomial in $y^i$ of degree $r$.

We use the following definition in future.

**Definition 2.1:** The Finsler space $F^n$ is of Douglas type if and only if the Douglas tensor

$$D^h_{ij} = G^h_{ij} = \frac{1}{n+1} (G_{ijk} y^h + G_{ij} \delta^h_k + G_{jk} \delta^h_i + G_{ki} \delta^h_j)$$

vanishes identically, where $G^h_{ij} = \delta^h_k G^h_{ij}$ is the curvature tensor of the Berwald connection.

The covariant differentiation with respect to the Levi-Civita connection $\{\gamma_{ik}^j(x)\}$ of $R^n$ is denoted by $(\gamma)$. We use the symbols as follows:

$$r_{ij} = \frac{1}{2} (b_{ij} + b_{ji}), \quad s_{ij} = \frac{1}{2} (b_{ij} + b_{ji})$$

The functions $G^i(x, y)$ of $F^n$ with $(\alpha, \beta)$-metric are written in the form

$$2G^i = \left\{ \begin{array}{c} i \\ 0 \end{array} \right\} + 2B^i,$$

$$B^i = \frac{\alpha L_{ij}}{L_\alpha} s^j_0 + C^* \left[ \frac{\beta L_{ij}}{\alpha L_\alpha} y^i - \frac{\alpha L_{ij}}{L_\alpha} \left( \frac{1}{\alpha} y^i - \frac{\beta}{\alpha} b^i \right) \right].$$

Where $L_\alpha = \partial L/\partial \alpha$, $L_\beta = \partial L/\partial \beta$, $L_{ij} = \partial L/\partial \alpha \partial \beta$, the subscript 0 means contraction by $y^i$ and we put

$$C^* = \frac{2(\beta L_{00} - 2\alpha x_{0ij})}{\beta L_{ij}},$$

Where $y^2 = b^i e^{\alpha}$, $b^i = b_i$, and $b^2 = a^i b_i$. Since $\{\begin{array}{c} i \\ 0 \end{array}\}$ are functions of $h$, $F^n$ is a Berwald space as a factor, then the dimension $n$ is equal to 2 and $b^2$ vanishes. In this case, we have $d_i (x, y)$ satisfying $a^2 = \beta^2$ and $d_i b^i = 2$.

### III. Berwald Space With Special Metric

$L = \alpha - \frac{\beta^2}{a}$ to be a Berwald space.

For the special $(\alpha, \beta)$-metric $L = \alpha - \frac{\beta^2}{a}$, the above equation is written as

$$(a^2 + \beta^2)B_{ijkl} y^i y^k + a^2 (\alpha - 2 \beta) (B_{ijkl} b^k - b_{ij}) y^j = 0$$

Where $B_{ijkl} = a_{ij} B_{kl}$. We suppose that $F^n$ is a Berwald space. Then $B_{ijkl}$ and $b_{ij}$ are functions of position alone. Therefore, we separate (3.1) as the rational and irrational terms in $y^i$ as:

$$(a^2 + \beta^2)B_{ijkl} y^i y^k - 2a^2 \beta (B_{ijkl} b^k - b_{ij}) y^j + a^2 (B_{ijkl} b^k - b_{ij}) y^j = 0,$$

Which yields two equations

$$a^2 (B_{ijkl} b^k - b_{ij}) y^j = 0,$$

$$a^2 (B_{ijkl} b^k - b_{ij}) y^j = 0.$$ 

Substituting (3.3) in (3.2), we have

$$B_{ijkl} y^i y^k = 0$$

and hence $B_{ijkl} + B_{kij} = 0$. Since $B_{ijkl}$ is symmetric in $(j, i)$, we get $B_{ijkl} = 0$ and from (3.3), we have $b_{ij} = 0$.

Conversely, if $b_{ij} = 0$, then $B_{ijkl} = 0$ are uniquely determined from (3.1). Hence, we conclude the following

**Theorem 3.1:** A Finsler space with a special $(\alpha, \beta)$-metric $L = \alpha - \frac{\beta^2}{a}$ is Berwald if and only if $b_{ij} = 0$.

### IV. Conformally Berwald Finsler Space

In this section, we find the conditions for Finsler space with a $(\alpha, \beta)$-metric $L = \alpha - \frac{\beta^2}{a}$ to be conformally Berwald.

**Definition 4.1:** A Finsler space $F^n = (M^n, L)$ is called conformally Berwald if there exists a conformal change $L \rightarrow \tilde{L} = e^{2\alpha} L$ such that the changed space $\tilde{F}^n = (\tilde{M}^n, \tilde{L})$ is a Berwald space, that is, $\tilde{G}^i_\chi$ are functions of position alone.
V. Conclusion

M. Matsumoto and S. Bacso introduced the notion of Douglas space as a generalization of Berwald space from the viewpoint of geodesic equations. Douglas metrics can be viewed as generalized Berwald metrics. The study on Douglas metrics will enhance our understanding on the geometrical meaning of non-Riemannian quantities. In this paper, we found the conditions for a Finsler space with special \((\alpha, \beta)\) – metric \(L = \alpha - \frac{\beta^2}{n}\) to be a Douglas space and also to be a Berwald space. Further, we found the conditions for a conformally transformed Douglas space with the above mentioned special \((\alpha, \beta)\) – metric to be a Douglas space.

References

[7]. M. Matsumoto, On Finsler spaces with \((\alpha, \beta)\) – metric of Douglas type, Tensor, N.S., 60(1998).
[14]. S.K. Narasimhamurthy and G.N. Latha Kumari, On a hypersurface of a special Finsler space with a metric \(L = \alpha + \frac{\beta^2}{n} + \frac{\beta^3}{n^2} + \beta\), ADJM, 9(1)(2010), 36-44.