IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 15, Issue 5 Ser. I (sep – Oct 2019), PP 09-13
www.iosrjournals.org

The Movement Techniques of a Bishop on a Chess Board Generates a Finite Sum of Two Terms

M. Laisin[1] andW. Uwandu[2]
[1]*Department of Mathematics Chukwuemeka Odumegwu Ojukwu University, Uli, Anambra State, Nigeria*
[2]*Department of Mathematics, Abia State Polytechnic, Aba, Abia State, Nigeria*

Abstract: *In this paper the authors carried some self-standing studies to examinethe game of chess and its counting techniques associatedwith non-attacking bishop positions. The authors studied the chess boardfor bishop placement with forbidden positions and varied the diagonal movement to the direction$\theta_r = 45^0$(to the right) and $\theta_l = 135^0$(to the left). We discussed the general movement of a bishopas generating function of two diagonal sums. Furthermore, weconstructed the movement techniques of a bishop placements in the game of chess that generates a finite sum of terms $\mathcal{X}_{\theta_r}\mu\beta^u$ and $\mathcal{X}_{\theta_l}\,\varphi\delta^v$ respectively. Finally, we applied it to combinatorial problems that generates' the diagonal movement of a bishop to give the sum of two algebraic expansion.*

Key words*: Chess movements; Laurent series;Permutation;Puiseux expansion; Rings*

Date of Submission: 26-08-2019 Date of Acceptance: 10-09-2019

I. Introduction

Bishop polynomial have a significant role to play in the theory of permutations with forbidden positions. Studies have shown thatpolynomial of either the bishop or rook on any board can be generated recursively(Laisin M. , 2018; Abigail, 2004; RIORDAN, 1980) using acell decomposition technique of Riordan. The counting techniques generatedby non-attacking bishop placements in the game of chess and its diagonal movement tocapture piecesto the right (direction 45^o) orto the left (direction 135^o) in the same direction with the bishop generates a polynomial of finite sum (Laisin M. , 2018).However, many authors have researched in this area such as(Butler, 1985; Farrell & Whitehead, 1991; Goldman, Joichi, & White, 1977; Goldman., Joichi, Reiner, & White, 1976; Haglund, 1996; Michaels, 2013; Abigail, 2004)which has extended this area of studies to more advanced combinatorial approaches. Most freshly,(LAISIN, 2018; Laisin M. , 2018; Skoch, 2015; Laisin, Okoli, & Okaa-onwuogu, 2019) and(Michaels, 2013; Ono, Haglund, & Sze, 1998; Shanaz , 1999)carried outself-standing studies in various connections to either bishop or rook polynomials. Furthermore, enumeration over finite fields,hypergeometric series and group representation theory have enjoyed new developments in the area of combinatorics.

However, thebishop polynomialof a board B is denoted as $\mathfrak{B}(x, B)$ for the case of finding the number of ways in which$n$ non-attacking bishops can be placed. It is clear that different pieces in the game of chess have unique movement laws, nevertheless, our point of interest is on the potency of the bishop movement.

Furthermore, authors have carried out studies to demonstrate thatpolynomial of either the bishop or rook of any board can be computed recursively.We shall present a construction for the movement techniques of a bishop placements in the game of chess that generates a finite sum of terms $\mathcal{X}_{\theta_r}\mu\beta^u$ *and* $\mathcal{X}_{\theta_l}\,\varphi\delta^v$.

II. Preliminaries

Definition 2.1.1

A ring R is a set with two laws of composition $+$and $\times$ called addition and multiplication, which satisfy these axions;

a. With the composition $+, R$is an abelian group, with identity denoted by 0. This abelian group is denoted by R^+

b. Multiplication is associative and has an identity denoted by 1.

c. Distributive law for all $a, b, c \in R, \Rightarrow (a + b)c = ac + bc \;\&\; c(a + b) = ca + cb$(Artin, 1991)

A chess board B of a ring is a chess board which is closed under the operations of addition subtraction, and multiplication and which contains the first placement ($b_0(B) = 1$). A bishop polynomial with forbidden positions is denoted as $\mathfrak{B}(x, B)$, given by

$$\mathfrak{B}(x,B) = \sum_{i=1}^{k} b_i(B)x^i,$$

where $\mathfrak{B}(x,B)$ has coefficients $b_i(B)$ representing the number of ways ofbishop's placements on B. Furthermore, on m ×n board B, we have $b_0(B) = 1$ and thecoefficientsare determined by

$$\mathfrak{B}(x,B) = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k!\, x^k = \sum_{k=0}^{\min(m,n)} \frac{n!\, m!}{k!\,(n-k)!\,(m-k)!} x^k.$$

(Vilenkin, 1969; LAISIN, 2018).

Definition 2.1.2

Let B be an $n \times n$ board and its diagonal denoted by $\mathfrak{D}^\theta$ and suppose that;

$$F(x_1, x_2, \dots, x_k) = \sum f(n_1, n_2, \dots, n_k) x_1^{n_1} x_2^{n_2} \dots x_k^{n_k} \in K[[(x_1, x_2, \dots, x_n)]]$$

Then, the $\mathfrak{D}^\theta$ is the power series in a single variable x defined by

$$\mathfrak{D}^\theta = \mathfrak{D}^\theta(x) = \sum_n f(n, n, \dots, n) x^n$$

Standard Basis 2.2

Let $\mathfrak{D}^\theta = F^n$ be the space of diagonal vectors and let e_i denote the diagonal vector with $b_0(B) = 1$ in the i^{th} position and zeros elsewhere. Then, the n vectors e_i from a basis for F^n . That is every vector $X = (x_1, x_2, \dots, x_k)$ has the unique expression;

$$XE = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

as the linear combination of $E = (e_1, e_2, \dots, e_n)$(Artin, 1991)

Lemma 2.2.1.

Let S be a set of vectors of V and let W be a subspace of V. If $S \subset W$ then span $S \subset W$(Artin, 1991)

Theorem 2.2.2.

If $m \times n$ board isunrestricted,then number of ways to place k rooks is;

$$\binom{m}{k}\binom{n}{k} k!$$

where $0 \le k \le \min\{m,n\}$.(Skoch, 2015)

Theorem 2.2.3 (The Fundamental Right Diagonal Theorem)

Let B be a chessboard containing a right diagonal ($\theta = 45^0$)with k cells. Let $B - \theta_{r,l}$be the board obtained from B byremoving right diagonal r and left diagonal l (i.e. one of the k left diagonals containing acell in the right diagonal r). we have;

$\mathfrak{B}(x,B) = \mathfrak{B}(x, B-\theta_r) + x\sum_{r=0}^{n} \mathfrak{B}(x, B-\theta_{r,l})$(Shanaz , 1999)

III. Main Results

Lemma3.1

If orderismaintained, then thenumberofwaystoarrangen-Bishopsamongm-positions ($m \ge n$) is given by;

$$\frac{1}{C_{(m,\ n)}} \sum_{i=0}^{n} (-1)^i b_i(B) C_{(m-i,\ n-i)}, \quad if\ m \ne n$$

or

$$\sum_{i=0}^{n} (-1)^i b_i(B), \qquad if\ \ n = n$$

Proof

Case i. when $m \ne n$

$$\mathfrak{B}(x,B) = 1 - \frac{b_1(B)C_{(m-1,\ n-1)}}{C_{(m,\ n)}} + \frac{b_2(B)C_{(m-2,\ n-2)}}{C_{(m,\ n)}} - \cdots (-1)^i \frac{b_i(B)C_{(m-n,\ n-n)}}{C_{(m,\ n)}}$$

$$= \frac{1}{C_{(m,\ n)}} \sum_{i=0}^{n} (-1)^i b_i(B) C_{(m-i,\ n-i)}$$

Case ii.when $n = n$.

$$\mathfrak{B}(x,B) = 1 - b_1(B)C_{(n-1,\ n-1)} + b_2(B)C_{(n-2,\ n-2)} - \cdots (-1)^i b_i(B)C_{(n-n,\ n-n)}$$

$$= \sum_{i=0}^{n} (-1)^i b_i(B)$$

Theorem 3.1
Let B be an $m \times n$board and suppose $\mathfrak{B}$ is a power series in s and t that represent a rational function with$\mathfrak{B}(x, B) \epsilon K[[s,t]] \cap K(s,t)$. Then, the bishop movement is algebraic with $\theta_r = 45^0 \ or\ \theta_l = 135^0$.
Proof
Suppose that$n \times n$ board B is a subset of the ring that contains $K\left[\left[s, \frac{x}{s}\right]\right]$,then, thebishop movementthrough a direction of $\theta = 45^0 \ or\ 135^0$is a diagonal and the bishop polynomial is given as follows;

$$\mathfrak{B}(x, B) = \sum_{k=0}^{n} b_k^{\theta}(B)x^k = 1 + b_1^{\theta}(B)x + b_2^{\theta}(B)x^2 + \cdots + b_n^{\theta}(B)x^n$$

Now, assume that K is closed algebraically and also that char $K = 0$, holds for any $K = p > 0$. Let
$L(x,s) = \mathfrak{B}\left(s, \frac{x}{s}\right) \in K\left[\left[s, \frac{x}{s}\right]\right]$…(1)
where$\frac{x}{s} \epsilon B$, $L(x,s)$ is a Laurent series in s and x. If $x^i s^j$ appears in $L(x,s)$, then $i \geq 0 \ and\ j \geq -i$ .It is clear that

$$\mathfrak{B}(x^*, B) = \mathfrak{D}^{\theta} = [s^0]L(x,s),$$

with $\mathfrak{D}^{\theta}$ as the diagonal of B and $L(x,s)$ as the constant term of the Laurent series in s whose coefficients are the power series in x. Since the bishop movement($\mathfrak{B}(x,B)$) is rational, it implies that$L(x,s) = \frac{P}{Q}$, where $P, Q \in K[s,x]$. Given that $L(x,s)$ is a rational function of s whose coefficients lie in the field K[x], that decomposes into a partial fraction;

$$L(x,s) = \sum_{j=0}^{h} \frac{N_j(s)}{(s - \xi_j)^{\alpha_j}} \qquad \ldots (2)$$

where;(i)$\alpha_j \in P$, (ii)$\xi_1, \xi_2, \ldots, \xi_h$ are the distinct zeros of Q(s) and
(iii) $N_j(s) \in K(\xi_1, \xi_2, \ldots, \xi_h)[s]$
Furthermore, by expanding each term of (2) such that each expansion lies inB and the ring in which the expansion (1) of the Laurent series lies. Thus, all our bishop movements in the direction of$\theta = 45^0 \ or\ 135^0$in the direction to capture a piece are taking place inside a single ring. Since the coefficients of Q(s) are polynomials in x and by Puiseux's theorem, we can let$\xi_j \in K((\sqrt[r]{x}))$ for some $r \in B$.Then $N_j(s) \in K((\sqrt[r]{x}))$[s]. Replacing $\xi_1, \xi_2, \ldots, \xi_h$ by $\beta_1, \beta_2, \ldots, \beta_m$ and $\delta_1, \delta_2, \ldots, \delta_n$ we have

$$\beta_i \in \sqrt[r]{x} K\left[\left[\sqrt[r]{x}\right]\right] \qquad \forall\, i = 1,2, \ldots, m;$$
$$\delta_j \notin \sqrt[r]{x} K\left[\left[\sqrt[r]{x}\right]\right] \qquad \forall\, j = 1,2, \ldots, n$$

It follows that the Puiseuxexpansion of $\beta_i$gives only positive exponents and that of$\delta_j$gives some negative exponent. Similarly, $\delta_j^{-1} \in K\left[\left[\sqrt[r]{x}\right]\right]$.
Thus,

$$L(x,s) = \sum_{j=0}^{m} \frac{N_j(s)}{(1 - s^{-1}\beta_i)^{\tau_j}} + \sum_{j=0}^{n} \frac{N_j(s)}{(s - s\delta_j^{-1})^{\omega_j}}$$
$$= \sum_{j=0}^{m} p_i(s) \sum_{k \geq 0} \binom{\tau_j}{k} s^{-1}\beta_i^k + \sum_{j=0}^{n} q_j(s) \sum_{k \geq 0} \binom{\omega_j}{k} s^k \delta_j^{-k} \qquad \ldots (3)$$

For some $\tau_j, \omega_j \ \in B$ and $p_i(s), q_j(s) \in K((\sqrt[r]{x}))[s, s^{-1}]$. Thus, theexpansion of (3) for $L(x,s)$ is such that;

$$L(x,s) \in K\left[\left[\sqrt[r]{x}, \sqrt[r]{\frac{x}{s}}\right]\right] [\sqrt[r]{s}^{-1}] \supset K\left[\left[s, \frac{x}{s}\right]\right]$$

Since $L(x,s) \in K\left[\left[s, \frac{x}{s}\right]\right] \subset K[[s,t]]$ the expansion of (3) agrees with the series
$L(x,s) = \mathfrak{B}\left(s, \frac{x}{s}\right) \in \mathfrak{B}(x,B) \subset K[[s,t]] \cap K(s,t)$.

Multiplying both sides of (3) by s^0, we have; $\mathfrak{D}^\theta = s^0 L(x,s)$, hence, this gives us a finite sum of terms to the right-hand side of (3).We have that;

$$\mathfrak{D}^\theta = s^0 L(x,s) = \mathcal{X}_{\theta_r}\mu\beta^u \ and \ \mathcal{X}_{\theta_l}\,\varphi\delta^v$$

where $\theta_r, \theta_l \in \mathfrak{D}^\theta, \quad u, v \in \mathbb{Z} \ and \ \beta, \ \delta, \ \mu, \ \varphi$ are algebraic. Hence $\mathfrak{D}^\theta$ is algebraic, thus, the bishop movement along $\mathfrak{D}^\theta$ is also algebraic. ■

IV. NUMERICAL APPLICATIONS

Example 4.1

Let B be a $2x2$ board and $x_{1=}$s, $x_{2=}$t. Then $\mathfrak{B}(s,t)$=$\frac{1}{1-s-t}$=$\sum_{i,j}\binom{i+j}{i}s^i t^j$

$$\mathfrak{D}^\theta = \sum_n \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$$

Example 4.2

Suppose the rational function for lemma 1 is generating the diagonal movement of a bishop. Show that the sum of two algebraic expansion is the bishop movement at $n = 0$.

Solution

Let$t = \frac{x}{s} \in B$, then $L(x,s) = \mathfrak{B}\left(s, \frac{x}{s}\right) = \frac{1}{1-s-\frac{x}{s}}$

Implies $L(x,s) = \frac{-s}{s^2-s+x}$

Let the roots of$s^2 - s + x$ be $\alpha \ and \ \beta$ then we have

$$s^2 - s + x = s^2 - (\alpha+\beta)s + \alpha\beta = (s-\alpha)(s-\beta)$$

Implies, the sum of roots $\alpha + \beta = 1$ and product of roots $\alpha\beta = x$. By solving for $\alpha \ and \ \beta$ we have;

$$\beta = \frac{1 + +\sqrt{1-4x}}{2} = 1 + \cdots$$

and$\alpha = \frac{1-\sqrt{1-4x}}{2} = x + \cdots$

Thus, the expansion of $L(x,s)$ as a partial fraction is

$$L(x,s) = \frac{1}{\beta-\alpha}\left(\frac{\alpha}{s-\alpha} - \frac{\beta}{s-\beta}\right)$$

As in Laurent series we have

$$L(x,s) = \sum_{i,j}\binom{i+j}{i} s^i \left(\frac{x}{s}\right)^j = \sum_{i,j} c_{(i+j,i)}\, s^i \left(\frac{x}{s}\right)^j$$

$$= \sum_{i,j} c_{(i+j,i)}\, x^i s^{i-j}$$

.

Clearly, we can see that $\frac{\alpha}{s-\alpha}$ is a rational fraction with two expansions in power of s. That is;

$$\frac{\frac{\alpha}{s}}{1-\frac{\alpha}{s}} = \sum_{n\geq 1} \alpha^n s^{-n}$$

and

$$\frac{-1}{1-\frac{s}{\alpha}} = -\sum_{n\geq 0} \alpha^{-n} s^n$$

.

The correct movement is the one on the board B that lies in the ring $k\left[\left[\sqrt[r]{x}, \sqrt[r]{\frac{x}{s}}\right]\right][\sqrt[r]{s}^{-1}]$ where $r = 1$.

However, the two possible expansions are;

$\alpha = x +$ higher order terms

and in a similar way, we have;

$\frac{\beta}{s-\beta} = \frac{1}{1-\frac{s}{\beta}}$ and $\beta - \alpha = \sqrt{1-4x}$

$$\therefore \qquad L(x,s) = \frac{1}{\sqrt{1-4x}}\left(\frac{\frac{\alpha}{s}}{1-\alpha s^{-1}} + \frac{1}{1-s\beta^{-1}}\right)$$

$$= \frac{1}{\sqrt{1-4x}}\left(\sum_{n\geq 1} \alpha^n s^{-n} + \sum_{n\geq 0} \beta^{-n} s^n\right)$$

This imply that the rook movement along the diagonal when $n = 0$ is;

$$\mathfrak{D}^{\theta} = [s^0]L(x,s) = \frac{1}{\sqrt{1-4x}}$$

V. Conclusion

Polynomials generated by bishop movements are not just interesting for their own sake. They have a variety of algorithms to generate combinatorial structures which will always be needed, simply because of the fundamental nature of the movement laws. One could need different kindsof combinatorial formulae for different kinds of pieces in the game of chess due to the movement laws. However, we constructed the movement techniques generated by bishop placements in the game of chess that generated a finite sum of terms $\mathcal{X}_{\theta_r}\mu\beta^u$ and $\mathcal{X}_{\theta_l}\,\varphi\delta^v$ for each diagonal. Finally,the result generated was applied to combinatorial problems with the diagonal movement such as that of a bishop to give us the sum of two algebraic expansions.

.

VI. Recommendations

In the result in this paper, a fundamental assumption is that non-attacking bishops with forbiddenpositions can be allowed in the placement process. However, there are a plenitude of real-life movement techniques generated by bishop movements where attacking bishops can be allowed. It would be interesting to develop precise formulaefor this more general situation.

References

[1]. Abigail, M. (2004). A block decomposition algorithm for computing rook polynomials,.

[2]. Artin, M. (1991). Algebra. Massachusetts Institute of Technology: Prentice Hall Upper Saddle River New Jersey 07458.

[3]. Barbeau, E. J. (2003). Polynomials. Springer, New York.

[4]. Berge, C. (1971). Principles of Combinatorics; vol. 72 in Mathematics in . New York: Science and Engineering a series of monographs and textbooks, Academic press, vol. 72.

[5]. Butler, F. (1985). Rook theory and cycle-counting permutation statistics. Advances in Applied Mathematics, 124-135.

[6]. Chung, F., & Graham, R. (1995). On the cover polynomial of a digraph. Journal of Combinatorial Theory, Series B., 65: 273–290.

[7]. Farrell, E. J., & Whitehead, E. (1991). Matching, rook and chromatic polynomials and chromatically vector equivalent graphs. Journal of Combinatorics Mathematics and Combinatorics Computing , 9, 107-118.

[8]. Goldman, J., Joichi, J. T., & White, D. (1977). Rook theory III. Rook polynomials and the chromatic structure of graphs. Jornal of Combinatorial Theory, Series B, 25, 135-142.

[9]. Goldman., J., Joichi, J. T., Reiner, D., & White, D. (1976). Rook theory II. Boards of binomial type. SIAM Journal on Applied Mathematics, 31, 617-633.

[10]. Haglund, J. (1996). Rook theory and hypergeometric series. Advances in Applied Math, 17:408–459.

[11]. Joichi, J. T., Goldman, J., & White, D. (1978). Rook theory IV. Orthogonal sequences of rook polynomials. Studies in Applied Math., 56, 267–272.

[12]. LAISIN, M. (2018). Construction of Rook Polynomials Using Generating Functions and m×n Arrays. International Journal of New Technology and Research (IJNTR), 4(2), 47-51.

[13]. Laisin, M. (2018). Enumerative Techniques for Bishop Polynomials Generated by a θ^o Board With an mxn array. International Journal of Scientific and Research Publications (IJSRP) 9(1), (ISSN: 2250-3153), DOI: http://dx.doi.org/10.29322/IJSRP.9.01.2019.p8599.

[14]. Laisin, M., Okoli, O. C., & Okaa-onwuogu, C. A. (2019). The bishop movement on a chess board is a noncommutative diagonal movement generated by a Laurent monomial. COOU Journal of Physical Sciences (CJPS), ISSN: 2465-6739, vol. 2, No. 2, Pg. 1- 6.

[15]. Michaels, J. G. (2013). Arrangements with Forbidden Positions. In Discrete Mathematics and its Applications (pp. 158-171).

[16]. Ono, K., Haglund, J., & Sze, L. (1998). Rock theory and t-cores. Journal of Combinatorial Theory, Series A(84), 9-37.

[17]. Riordan, J. (1958). An Introduction to Combinatorial Analysis. New York: Princeton University Press, John Wiley and Sons.

[18]. RIORDAN, J. (1980). An Introduction to Combinatorial Analysis. New York: Princeton University Press, (originally published by John Wiley and Sons) Chapman and Hall.

[19]. Shanaz , A. (1999). On the Distribution of Non-Attacking Bishops on a Chessboard C.

[20]. Skoch, S. R. (2015). "I Don't Play Chess: A Study of Chess Piece Generating Polynomials". Senior Independent Study Theses, (p. Paper 6559).

[21]. Vilenkin, N. Y. (1969). Combinatorics (Kombinatorika). Moscow (In Russian): Nauka Publishers.

[22]. White, D., Goldman, J., & Joichi, J. T. (1975). Rook theory I. Rook equivalence of Ferrers boards. AMS (pp. 485-492). America: AMS.

M. Laisin. " The Movement Techniques of a Bishop on a Chess Board Generates a Finite Sum of Two Terms." IOSR Journal of Mathematics (IOSR-JM) 15.5 (2019): 09-13.