A Necessary and Sufficient Condition for Local and Global Null – Controllability.

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Abstract: The problem of null-controllability for linear non-autonomous system of the form

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t), t \in [0, \infty) \]

(1.1)

to the origin in a finite time \( t > 0 \). We take \( A(t) \) and \( B(t) \) to be an \( n \times n \) and \( n \times m \) continuous bounded operator matrices; the state \( x(t) \in \mathbb{R}^n \) and the control function \( u(t) \in \mathbb{R}^m \). Usually, in controllability problems, the values of controls at each instant of time are free, but in this case, we insist that each of the control value at each instant of time lies in a prescribed set \( \Omega \) in \( \mathbb{R}^m \), which is not empty. Now, let us define a function \( \mu: \mathbb{R} \to \Omega, \) from \( \mathbb{R} \) into \( \Omega \) and then assume they are measurable on \([0, \infty)\). Then such control function \( u(t) \in \mu(\Omega) \) is said to be admissible. In this paper, we assume that \( X \) and \( \Omega \) are complex Banach spaces. According to Aniaku and Jackreece [2], a point \( x_0 \in X \) is said to be null-controllable if there is a finite time \( T = T(x_0) \) and an admissible control \( u(t) \in \Omega \) such that the corresponding trajectory \( x(t) \) of (1.1) initiated at \( x_0 \) satisfies \( x(T) = 0 \). System (1.1) is said to be locally controllable if then \( \Psi \) of all null-controllable points contains some neighborhood of the origin. I.e. \( \Psi = X \).

1.2 Notations and Definitions.

Here, we present some notations and definitions which we hope to encounter in this paper. Firstly we have notations.

1.2.1 Notations:

Let \( X \) and \( Y \) be complex Banach spaces, \( \| \cdot \| \) is the norm in the space under consideration.

\( X^* \) denotes the duel of \( X \).
\( \text{Int } M \) denotes the interior of \( M \) in \( X \).
\( \mathcal{M}(\Omega) \) denote the set of the function from \( \mathbb{R} \) into \( \Omega \) that are measurable on \([t_0, \infty)\).
\( \text{Span } \{ M \} \) denotes the span of \( M \).

For each \( k \in \mathbb{N} \), \( M^k \) denote the set of all vector of the form \( x^k = (x_1, x_2, \ldots, x_k) \) with \( x_j \in M \),

\( j = 1, 2, 3, \ldots, k \).

\( X^k \) stands for a Banach space endowed with the norm \( \| x^k \| = \| x_1 \| + \| x_2 \| + \cdots + \| x_k \| \).

\( \mathbb{C} \) will denote the set of all complex numbers. Complex conjugates of \( \lambda \) is denoted by \( \overline{\lambda} \).

\( D_r \) denotes the disc \( \{ \lambda \in \mathbb{C} : |\lambda| \leq r \} \).

\( \mathbb{N} \) denotes the set of counting numbers or natural numbers.

If \( A \in L(X, Y) \), then \( \sigma(A) \) denotes the spectrum of \( A \).

\( L(X, Y) \) denotes the Banach space of all bounded linear operators.

We now define the controllability which are prescribed to lie in \( \Omega \) which is very important in our paper. This we precisely call \( \Omega \)-null controllability. Nevertheless, we have the following definitions.
1.2.2 Definitions:

Definition 1.1: Ω-null controllability [1].
The linear system (1.1) is Ω-null controllable at \((x_0,0)\) if given the initial point \(x(0) = x_0\) there exists a control function \(u(t) \in \mu(\Omega)\) Such that the solution \(x(t)\) of (1.1) satisfies \(x(t) = 0\) for \(t \in [0,\infty)\).

Definition 1.2: Globally Ω-null controllability [2].
The linear system (1.1) is the globally Ω-null controllable at 0 if the system (1.1) is Ω-null controllable at \((x_0,0)\) for all \(x_0 \in \mathbb{R}^n\)

Definition 1.3: Locally Ω-null controllability [2].
The linear system (1.1) is locally Ω-null controllable at 0 if there exist an open set \(V \in \mathbb{R}^n\) containing the origin 0, such that (1.1) is null controllable at \((x_0,0)\) for all \(x_0 \in V\).

Definition 1.4: Baire Functions.
These are functions obtained from continuous functions by transfinite iteration of the operation of forming point wise limit of sequence of functions.

Definition 1.5: Support of Function.
The support of a real-valued function \(f\) is the subset of the domain containing those elements which are not mapped to zero.

Definition 1.6: Scalar function.
A scalar function \(T: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}\) is denoted as
\[
J(x_o, T, \lambda) = x_0^1 \phi^1(T, t_0) \lambda + \int_{t_0}^t H_\Omega(B^1(T)Q^1(T, \tau) \lambda) \, d\tau
\]

Definition 1.7: Positive Definite Function. [3]
A function \(F(x, y)\) is called positive definite in a domain \(D\) if \(F(0,0) = 0\) and \(F(x, y) > 0\) for all other points \((x, y)\) in \(D\).

II. Review Of Literature.

We note that different necessary and sufficient conditions for global Ω-Null controllability have been studied by different researchers. In the equation (1.1), if \(A\) and \(B\) are constants, Kalman [4] showed that if \(\Omega = \mathbb{R}^m\) then a necessary and sufficient condition for global \(\mathbb{R}^m\)-null controllability is that the rank \(P\) = \(n\), where
\[
P = [B AB A^2 B \ldots A^{n-1} B]
\]
We have to point out that Lee and Markus [5] considered constraint set \(\Omega \subseteq R^m\) which has the origin as its interior, and showed that the rank\((P) = n\) is necessary and sufficient condition for system (1.1) to be locally Ω-null controllable. He also showed that if each eigenvalue \(\lambda\) of \(A\) satisfies \(R_e(\lambda) < 0\), then system (1.1) is globally Ω-null controllable.

Saperstone and Yorke [6] eliminated the assumption that the origin must be an interior point of \(\Omega\) for global Ω-null controllability using \(m = 1\) and \(\Omega = [0,1]\). In their study, they found that system (1.1) is locally Ω-null controllable if and only if the rank \(P\) = \(n\) and \(A\) has no real eigenvalues.

Brammer [7] in his own study, found a more general constrained set where he showed that if there exists a control function \(u \in \Omega\) satisfying \(Bu = 0\), and the convex hall of \(\Omega\) has a non empty interior, then the necessary and sufficient conditions for a local Ω-null controllability are rank\((P) = n\) and the non-existence of a real eigenvectors \(V\) of \(A^1\) satisfy \(V^1 Bu = 0\) for all \(u \in \Omega\). He found that if in addition, no eigenvalues of \(A\) has a positive real part, then problem (1.1) becomes one in which global Ω-null controllability applies.

Now, for non-autonomous system, Kalman [4] showed controllability result when \(\Omega = \mathbb{R}^m\). He showed that system (1.1) is \(\mathbb{R}^m\)-null controllable if and only if the gramain \(W(t_0, t_1)\) is positive definite for some \(t_1 \in [t_0, \infty)\) Where
\[
W(t_0, t_1) = \int_{t_0}^{t_1} \varphi(t, \tau) Bu(t, \tau) B^1(t)Q^1(t, \tau) d\tau
\]
and \(\varphi(t, \tau)\) is the fundamental matrix for the system (1.1). The problem Ω-null controllability was also studied by Dauer [8] and also Chukwu and Gronski [9]. Chukwu and Slliman [10] and Granthanand Vincent [11] considered the problem of steering a nonlinear system to a target point. They formulated a way of determining the difference between the set of state which can be steered to a required target point and those which cannot be
steered to it. We have to state at Pachter and Jacobson [12] developed the condition for Ω-null controllability for the case where $A(t)$ and $B(t)$ are time invariant, and $Ω$ is a closed convex cone which contains the origin.

Now, we want to study the problem of Ω-null controllability when $A(t)$ and $B(t)$ are time-varying. Our result for global Ω-null controllability is for a constraint set $Ω$ which are compact and contains the origin, which may not be an interior point. This result for Ω-null controllability at $(x_0, t_0)$ has a wider application since it does not require $0 \in Ω$. It does not even require the existence of a control function $u \in Ω$ such that $Bu = 0$ as in Brammer [7]. So, we can check controllability of system with, for example $m = 1$ and $Ω = [1,2]$, where we know, many of the presently Ω-null controllability result do not apply. We shall show as example an autonomous system (1.1) which is neither globally Ω-null controllable, nor locally Ω-null controllable at some point $(x_0, t_0)$. Note that this our theorem can be used to divide the state space into two sets, viz: initial state which can be steered to the origin and those which cannot be driven to the origin by any admissible control function.

## III. The Main Theorem

Before we state our main result, we have to deal with the following proposition.

**Proposition 3.1:** let $Ω$ be a compact set and $G$ a subset of $\mathbb{R}^n$ which has the origin as interior point. Then system (1.1) is Ω-null controllable at $(x_0, t_0)$ if and only if

$$\min\{f(x_0, T, λ); λ \in G\} = 0$$

(3.1)

For some $T \in [t_0, \infty)

**Proof:**

Let $A(x_0)$ be the set of states which can be reached from $x_0$ at finite time $T$

$$A(x_0) = \left\{\varphi(t, o)x_0 + \int_0^T \varphi(T, r)B(r)u(r)\,dr : u(\cdot) \in \operatorname{dom}(Ω)\right\}$$

(3.2)

We note that the set $A(x_0)$ is convex and compact. It then follows that $x_0$ can be steered to the origin in a finite time $T$ if and only if $0 \notin A(x_0)$. So,

$$0 \leq \sup\{λ^1 : a \in A(x_0)\}$$

(3.3)

For all vectors $λ \in \mathbb{R}^n$.

Then (3.2) and (3.3) give

$$λ^1\varphi(T, 0)x_0 + \sup\left\{\int_0^T λ^1\varphi(T, r)B(r)u(r)\,dr : u(\cdot) \in \mathcal{M}(Ω) \geq 0\right\}$$

(3.4)

For all $λ \in \mathbb{R}^n$.

Note that in (3.4), we can commute the supremum and integral operation to get

$$0 \leq λ^1\varphi(T, 0)x_0 + \int_0^T H_t(B^1(T)^1r)\lambda \,dr = \int(x_0, T, λ).$$

(3.5)

For all $λ \in \mathbb{R}^n$ since $f(x_0, T, λ)$ is positively homogeneous in $λ$ we can restrict $λ$ to be in $G$. this then completes the proof.

We are now ready to state our main result.

**Theorem 3.1:**

Suppose $Ω$ is a compact set which contains the origin. Then system (1.1) is globally Ω-null controllable at $t_0$ if and only if

$$\int_{t_0}^T H_{t0}(B^1(r)Z(r)) \,dr = +\infty$$

(3.6)

for all non-zero solution $Z(\cdot)$ of the adjoint system

$$\dot{Z}(t) = -A^1(t)Z(t), t \in [t, \infty)$$

(3.7)

**Proof:**

Necessary condition:

Let us suppose that the system (1.1) is globally Ω-controllable at $t_0 = 0$, without loss of generality, let $Z(\cdot)$ be any non-zero solution of adjoint system (1.1). We then show that

$$\int_{0}^{\infty} H_{t0}(B^1(r)Z(r)) \,dr = +\infty$$

(3.8)

We show this by contradiction. Suppose that there is a non-zero solution $Z(\cdot)$ such that
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\[ \int_0^\infty H_\Omega(B^1(\tau)\tilde{Z}(\tau))d\tau < \infty \]

Then there is a positive constant \( \alpha < \infty \) such that
\[ \int_0^\infty H_\Omega(B^1(\tau)\tilde{Z}(\tau))d\tau < \alpha \]

If we define
\[ x_0^+ = \frac{-2\alpha\tilde{Z}(0)}{\tilde{Z}^1(0)\tilde{Z}(0)}, x_0^+ \neq 0 \]

We claim that \( x_0^+ \) cannot be steered to 0 by any admissible control \( u(.) \in M(\Omega) \). We prove this as follows: for each \( t \in [0, \infty) \) let us define
\[ \lambda_t = \varphi^1(0,t)\tilde{Z}(0), \lambda_t \neq 0 \]

So, given any \( t \in [0, \infty) \)
\[ f(x_0^+, t, \lambda_t) = x_0^+ \varphi^1(t, 0)\lambda_t + \int_0^t H_\Omega(B^1(\tau)\varphi^1(t, \tau)\lambda_t)d\tau \]
\[ = x_0^+ + \tilde{Z}(0) + \int_0^t H_\Omega(B^1(t)\tilde{Z}(\tau))d\tau \]
\[ = -2\beta + \beta < 0 \]

If we take \( G = \mathbb{R}^n \) in the proposition 3.1 above, we see that
\[ \min \{f(x_0^+, t, \lambda) : \lambda \in G \} \leq f(x_0^+, \lambda) < 0 \]

For all \( t \in [0, \infty) \) then, by proposition 3.1 above, the system (1.1) is not \( \Omega \)-null controllable at \( (x_0^+, 0) \).

**Sufficiency:**
We now assume that (3.8) holds. Then we also go by contradiction. Suppose that the system (1.1) is not globally \( \Omega \)-null controllable at \( t_0 = 0 \). Then, there exists an initial point \( x_0^+ \neq 0 \) which can not be steered to zero in finite time. If we can find a sequence of times \( \{t_k\}_{k=1}^\infty \) and a sequence of vectors \( \{\lambda_k\}_{k=1}^\infty \) having the properties
\[ (c_1) \lim_{k \to \infty} t_k = +\infty \]

\[ (c_2) f(x_0^+, t, \lambda_k) < 0 \quad \text{for} \quad k = 1, 2, 3, \ldots \]

Then, we construct an initial point \( \tilde{Z}_0 \neq 0 \) for (1.1) which makes the integral in (3.8) finite.

Let
\[ z_k = \frac{\varphi^1(t_k, 0)\lambda_k}{\varphi^1(t_k, 0)\lambda_k} \quad k = 1, 2, \ldots \]

We see that \( z_k \neq 0 \text{ since } \lambda_k \neq 0 \) and \( \varphi^1(t_k, 0) \) is invertible. So \( \{z_k\}_{k=1}^\infty \) is a sequence in \( \mathbb{R}^n \) which belongs to the set
\[ S = \{z \in \mathbb{R}^n : \|z\| = 1 \} \]

Since this set \( S \) is compact, we can find a subsequence \( \{z_{k_j}\}_{j=1}^\infty \) which converges to some vector \( \tilde{Z}_0 \in S \), we show that \( \tilde{Z}_0 \) is the initial point which does the job.

Let \( \tilde{Z}(.) \) be a trajectory of \( S^1 \) which is obtained by \( z(0) = \tilde{Z}_0 \)

Let \( \{t_{k_j}\}_{j=1}^\infty \) be a subsequence of times corresponding to \( \{z_{k_j}\}_{j=1}^\infty \)

By \( c_1 \) we have
\[ \lim_{j \to \infty} t_{k_j} = +\infty \]

and by \( c_2 \) it follows that
\[ x_0^+ \varphi^1(t_1, 0)\lambda_{k_j} + \int_0^{t_{k_j}} H_\Omega(B^1(\tau)\varphi(t_{k_j}, \tau)\lambda_{k_j})d\tau < 0 \quad j = 1, 2, 3, \ldots \]

Dividing by \( \|\varphi^1(t_{k_j}, 0)\lambda_{k_j}\| \) and noting that \( H_\Omega \) is positively homogeneous, we get
\[ \int_0^{t_{k_j}} H_\Omega(B^1(\tau)\varphi^1(0, \tau)z_{k_j})d\tau \leq \|x_0^+\| \|z_{k_j}\|, \quad j = 1, 2, 3, \ldots \]
\[ \leq \|x_0^+\| \quad j = 1, 2, 3, \ldots \]

\[ \Rightarrow \int_0^\infty H_\Omega(B^1(\tau)\tilde{Z}(\tau))d\tau \leq \|x_0^+\| < \infty. \]

This is a contradiction which we are looking for. So, the proof is completed.
3.2 Examples:

Example 1.
Consider the control system described by
\[ \dot{x}(t) = x(t) + u(t), t \in [0, \infty) \] (3.9)

where \( x(t) \) and \( u(t) \) are scalars.

We know that this system is \( R^1 \)-null controllable. Suppose we take \( \Omega = [0, 1] \). Then, this system is not globally \( \Omega \)-null controllable at \( t_0 = 0 \). This conclusion is obtained from Theorem 3.1, since for \( z_0 < 0, \int_{t_0}^{\infty} H_\Omega(B^1(t)z(t)) \, dt = 0 \) and thus
\[ \int_{t_0}^{\infty} H_\Omega(B^1(t)z(t)) \, dt < +\infty \]

Example 2.
Consider the time-varying two dimensional system defined by
\[ \begin{align*}
\dot{x}_1(t) &= u(t) \sin t \\
\dot{x}_2(t) &= \frac{1}{(t+1)^2} x_1(t) + u(t) t \sin t, \quad t \in [0, \infty)
\end{align*} \] (3.10)

Let us take the control constraint set to be \( \Omega = [0, 1] \). Then state fundamental matrix for the adjoint system is
\[ \varphi + (t,t_0) = \begin{pmatrix} 1 & \frac{t-t_0}{(t+1)(t_0+1)} \\ 0 & 1 \end{pmatrix} \]

So from Theorem 3.1, we see that (3.10) is globally \( \Omega \)-null controllable at \( t_0 = 0 \) if and only if
\[ \int_{0}^{\infty} \sup_{t \in [0,1]} t \sin t \cdot \tau \sin t \cdot \frac{1}{(t+1)(t_0+1)} \begin{pmatrix} z_0 \sin \tau \\ 1 \end{pmatrix} \, d\tau = \infty \] (3.11)

for all non-zero initial point \( z_0 = (z_{01}, z_{02})^T \).

Evaluating (3.11) we get
\[ \int_{0}^{\infty} I(\tau) \, d\tau = \int_{0}^{\infty} \max[0, z_{01} \sin \tau + z_{02} \tau \sin t(1 + \frac{1}{\tau + 1})] \, d\tau = +\infty \] (3.12)

for all \( z_0 \neq 0 \). Equation (3.2) is indeed true for both
(a) \( z_{01} \neq 0, z_{02} = 0 \).

and

(b) \( z_{01} = 0, z_{02} \neq 0 \).

For case (a), we have
\[ \int_{0}^{\infty} I(\tau) \, d\tau = \int_{0}^{\infty} \max[0, z_{01} \sin \tau] \, d\tau \]
\[ = \int_{\xi_1}^{\infty} z_{01} \sin \tau \, d\tau \]
where \( \xi_1 = \{ \tau \geq 0; z_{01} \sin \tau > 0 \} \) Since the range set \( \xi_1 \) of integration is the union of infinitely many intervals of length \( \pi \) it follows that
\[ \int_{0}^{\infty} I(\tau) \, d\tau = +\infty \]

For case (b), let \( T^+ = \frac{|z_{01}| + 1}{z_{02}} \)

Then it is enough to show that
\[ \int_{\xi_2}^{\infty} I(\tau) \, d\tau = +\infty \]

where \( \xi_2 = \{ \tau \geq T^+; z_{02} \sin \tau > 0 \} \). Since the range of integration is once again the union of infinitely many intervals of length \( \pi \) we conclude that (3.10) is globally \( \Omega \)-null controllable.

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