Numerical Solutions of Fractional Order Eigen-Value Problems using Bernstein Operational Matrices

Osama H. Mohammed\(^{(a)}\), Bashaer M. Abdali\(^{(b)}\)

\(^{a,b}\)Al-Nahrain University, college of science, Department of Mathematics, Baghdad, Iraq
Corresponding Author: Osama H. Mohammed

Abstract: In this article, we offer an easy and active computational technique for finding the solution of the fractional order Sturm-Liouville problems (FOSLPs) with variable coefficients using operational matrices of (BPs). The fractional order derivatives (FODs) are characterized in the Caputo sense. The proposed technique transform the fractional order differential equations (FDEs) into a linear system of algebraic equations, then the eigenvalues can be computed by finding the roots of the characteristics polynomials. Some tested problems are given in order to illustrate the effectiveness and efficiency of the method.

Keywords: Fractional order eigen-value problems, Sturm-Liouville problems, Bernstein polynomials.

I. Introduction

(FDEs) are generalized of classical integer order ones which are gained by exchanging integer order derivatives by fractional ones.

Most (FDEs) do not own analytic solutions, therefore approximate and numerical techniques must to use [1].

various numerical and approximate technique to solve (FDEs) have been given such as Adomian decomposition method [2], Variational iteration method [3], Homotopy analysis method [4], Homotopy perturbation method [5], Collocation method [6], wavelet method [7], finite element method [8] and spectral tau method [9].

The concept of operational matrices (OM) recently were acclimated for solving various types of (FDEs). Using the numerical techniques in combination with (OM) in some orthogonal polynomials, for the solution of (FDEs) on finite and infinite intervals, gives very accurate solutions for (FDEs) [9].

This article looks for finding the approximate values of the eigenvalues of some classes of the (FOSLPs). Some beforehand works have been published about the (FOSLPs) such as [10]-[19]. The essential feature of this work is to find the eigenvalues of the (FOSLPs) using the (OM) of the (BPs). BPs. play an eminent part in various areas of mathematics, these polynomials have extremely been used in the solution of integral equations, differential equations and approximation theory [20], [21].

The (OM) for (BPs) are introduced in order to solve different types of differential equations among them [20] used the (OM) for (BPs) for solving High Even-Order differentials equations, [21] have been used (OM) of (BPs) for solving Volterra integral equations, [22] investigated the solution of the nonlinear Volterra-Fredholm-Hammerstein integral equations using the (OM) of (BPs), while, [23] try to solve the physiology problems by the aid of the (OM) of (BPs), [24] have been used (OM) of (BPs) for solving high order delay differential equations. The (OM) of (BPs) have been used for solving multiterm variable order (FDEs) by [25].

The reminder of the article is ordered as follow. A concise review of the definitions of the (FODs) and integrations are offered in part 2. (BPs) have been given in part 3. The proposed method will be presented in part 4. Some numerical results are shown in part 5, at last a conclusion have been towed.

II. Fractional Order Derivatives and Integrals

In this part we shall give some basic definitions and properties of the (FODs) and integrals [26].

Def. (1):
The Riemann –Liouville (R-L) fractional integral of order $\alpha > 0$ is defined as follows:

$$I^\alpha_x f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad x > 0, \alpha \in \mathbb{R}^+.$$

Def. (2):
The Caputo fractional derivative of order $\alpha > 0$ is defined as follows:
\[ cD^\alpha_x f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x f^{(m)}(\tau) \frac{\tau^{\alpha-1}}{(x-\tau)^{\alpha+m-1}} d\tau, \quad m - 1 < \alpha < m; \]
\[ \frac{d^m}{dx^m} f(x), \quad \alpha = m. \]

For \( \alpha > 0 \), we have [1]:

1. \( cD^\alpha_x (t_\alpha f(x)) = f(x) \).
2. \( cD^\alpha_x \left( cD^\alpha_x f(x) \right) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!}. \)
3. \( cD^\alpha_x (c_1 f(x) + c_2 g(x)) = c_1 \ cD^\alpha_x (f(x)) + c_2 \ cD^\alpha_x (g(x)). \)

Where \( c_1 \) and \( c_2 \) are constants.

### III. Bernstein Polynomials (BPs)

The well known (BPs) of degree \([27]\) are given on \([0, 1]\) as:
\[ b^n_i(x) = \binom{n}{i} x^i (1 - x)^{n-i}, \quad i = 0, 1, \ldots, n \quad (1) \]

Eq. (1) can be also written as:
\[ b^n_i(x) = \sum_{i=0}^{n} (-1)^{i-1} \binom{n}{i} x^i, i = 0, \ldots, n. \quad (2) \]

The Bernstein vector \( B(x) = [b^n_0(x), b^n_1(x), \ldots, b^n_n(x)] \) [3] can be prescribed in the form:
\[ B(x) = AT_n(x) \quad (4) \]

Where
\[
A = \begin{bmatrix} (-1)^0 \binom{n}{0} & (-1)^1 \binom{n}{1} & \cdots & (-1)^{n-0} \binom{n}{n-0} \\ 0 & (-1)^0 \binom{n}{0} & \cdots & (-1)^{n-0} \binom{n}{n-0} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (-1)^0 \binom{n}{n} \end{bmatrix}_{(n+1)(n+1)} \quad (5)
\]

and
\[ T_n(x) = \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{bmatrix} \quad (6) \]

### III.I. Function Approximation:

Any \( f(x) \in L_2(0,1) \), can be decomposed in terms of the (BPs). Hence if \( f(x) \) can be decomposed as:
\[ f(x) = \sum_{i=0}^{n} c_i b^n_i(x) = c^T B(x) \quad (7) \]

Where \( c = [c_0, c_1, \ldots, c_n]^T \) B(x) = \( [b^n_0(x), b^n_1(x), \ldots, b^n_n(x)]^T \). Then
\[ c_i = \int_0^1 f(x) b^n_i(x) dx, i = 0, 1, \ldots, n. \]

Where \( d^n_i(x) \) are called the dual basis function to the Bernstein basis of degree \( n \) which has been derived in [27] in explicit representation as:
\[ d^n_i(x) = \sum_{k=0}^{n} \lambda_{jk} b^n_k(x), \quad j = 0, 1, \ldots, n. \]

Where
\[ \lambda_{jk} = \frac{(-1)^{i+k} \min(i,k)}{\binom{n}{i} \binom{n}{k}} \sum_{i=0}^{\min(i,k)} (2 i + 1) \binom{n+i+1}{n-j} \binom{n-i}{n-j} \binom{n+i+1}{n-k} \binom{n-i}{n-k} \quad (8) \]

For \( j,k = 0, 1, \ldots, n. \)

### III.II. The (OM) of the derivatives:

The derivative of the vector \( B(x) \) can be written as:
\[ \frac{d}{dx} B(x) = D^{(1)} B(x) \quad \quad (9) \]

Where \( D^{(1)} \) is handled by:
\[ D^{(1)} = AVB^* \quad \quad (10) \]

Where
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$$V = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{pmatrix}_{(n+1) \times n}$$ and

$$B^* = \begin{pmatrix} A_{11}^* \\ \vdots \\ A_{k1}^* \\ \vdots \\ A_{n1}^* \end{pmatrix}_{n \times (n+1)}$$

Where $A_{ik}^*$ is the $k^{th}$ row of $A^{-1}$ for $k = 1, 2, \ldots, n.$

Employ Eq. (9), it is obvious that

$$\frac{d^n B(x)}{dx^n} = (D^{(1)})^n B(x), \ n \in \mathbb{N} \quad \text{(11)}$$

and

$$D^{(n)} = (D^{(1)})^n, \ n = 1, 2, \ldots, \ldots \quad \text{(12)}$$

In the following theorem the (OM) of the (FODs) of the (BPs) will be offered.

**Theorem (1)[27]:** Let $B(x)$ be a Bernstein vector and $\alpha > 0$ then

$$D^n B(x) \approx D^{[\alpha]} B(x) \quad \text{(13)}$$

Where

$$D^{[\alpha]} = \begin{pmatrix} \sum_{i=0}^{n} \alpha_{0i,0} & \sum_{i=0}^{n} \alpha_{0i,1} & \cdots & \sum_{i=0}^{n} \alpha_{0i,n} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^{n} \alpha_{n,0} & \sum_{i=0}^{n} \alpha_{n,1} & \cdots & \sum_{i=0}^{n} \alpha_{n,n} \end{pmatrix}_{n \times n}$$

Here $\omega_{i,j}$ is given by

$$\omega_{i,j} = (-1)^{i-j} \binom{n}{j} \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} \sum_{k=0}^{n} \lambda_{ik} \mu_{kj} \quad \text{(15)}$$

Where $\lambda_{ik}$ is given in Eq. (8) and $\mu_{kj}$ is represented by

$$\mu_{kj} = \sum_{s=k}^{n} (-1)^{s-k} \binom{n}{s-k} \frac{1}{s-\alpha} \quad \text{(16)}$$

**IV. The Proposed Method**

Given the following fractional order Sturm-Liouville problem:

$$D_{x}^{\alpha} y(x) + q(x) y^{(1)}(x) = \lambda H(x) y(x), a \leq x \leq b, m - 1 < \alpha \leq m. \quad \text{(17)}$$

With the (BCs):

$$d_{11} y(a) + d_{12} y^{(1)}(a) = 0 \quad \text{(18)}$$
$$d_{21} y(b) + d_{22} y^{(1)}(b) = 0 \quad \text{(19)}$$

Where $d_{11}, d_{12}, d_{21}$ and $d_{22}$ are constants, and $q(x), H(x)$ are given functions. We suppose that the solution $y(x)$ can be approximated by using the (BPs) as follows:

$$y(x) \approx \sum_{n=0}^{n} c_{i} b_{i}^\top(x) = c_{i}^\top B(x) \quad \text{(20)}$$

Where $c=[c_{0}, c_{1}, \ldots, c_{n}]^\top$ and $B(x)=[b_{0}^\top(x), b_{1}^\top(x), \ldots, b_{n}^\top(x)]^\top$

The basic thought is to basically get a homogenous system of equations in the unknowns $c_{i}, \ 0 \leq i \leq n,$ the roots of whose characteristic equation comprise the eigenvalues of the problem. Using Eq.(13) and Eq.(20) we have:

$$D_{x}^{\alpha} y(x) \approx c_{i}^\top D^{(n)} B(x) \quad \text{(21)}$$

To begin we apply the (BCs) Eq.(18) and Eq.(19) using Eqs. (20) and (9), so we obtain the following two equations for the unknown coefficients $c_{i}$:

$$\begin{align*}
\{d_{11} c_{i} T B(a) + d_{12} c_{i} T D^{(1)} B(a) &= 0 \\
\{d_{21} c_{i} T B(b) + d_{22} c_{i} T D^{(1)} B(b) &= 0
\end{align*} \quad \text{(22)}$$

Or

$$\begin{align*}
c_{i} \Gamma_{a} & = \{d_{11} c_{i} T B(a) + d_{12} c_{i} T D^{(1)} B(a) \\
c_{i} \Gamma_{b} & \quad \text{(23)}
\end{align*}$$

Where

$$\begin{align*}
\Gamma_{a} & = d_{11} B(a) + d_{12} D^{(1)} B(a) \\
\Gamma_{b} & = d_{21} B(b) + d_{22} D^{(1)} B(b)
\end{align*}$$

Next, substituting Eqs.(9),(20)and(21) into Eq.(17) and by using the linearity property, we obtain:

$$c_{i} T D^{(0)} B(x) + c_{i} T q(x) D^{(1)} B(x) = \lambda H(x) c_{i} T B(x) \quad \text{(24)}$$

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Recall that we have \( n+1 \) unknown coefficients \( c_i \), \( i=0,1,\ldots,n \), and only two equations from the (BCs) so we require an additional \( n-1 \) equations. To get like equations we collocate Eq.(24) at \( n-1 \) points and for appropriate collocation points we use

\[
x_i = \left( \frac{1}{2} \right) \left( \cos \left( \frac{i\pi}{n} \right) + 1 \right) \quad i=1,\ldots,n-1 \quad \ldots(25)
\]

So we have:

\[
c^T D^{(0)} B(x_i) + c^T q(x_i) D^{(1)} B(x_i) = \lambda H(x_i) c^T B(x_i), \quad i=1,\ldots,n-1 \quad \ldots(26)
\]

Or

\[
c^T \Gamma_{1x} + c^T q(x_i) \Gamma_{1x} - \lambda H(x_i) c^T \Gamma_{10} = 0 \quad \ldots(27)
\]

Where

\[
\Gamma_{ij} = D^{(0)} B(x_i)\text{and } \Gamma_{ij} = D^{(1)} B(x_i), \quad j=0,1.
\]

Combining Eqs.(23) and (27), we get a complete system of \( n+1 \) equations which can be written after some simplifications as

\[
(A- \lambda K)c=0, \quad \ldots \quad (28)
\]

Where

\[
A = \begin{bmatrix}
(\Gamma_{1x} + q(x_i) \Gamma_{1x})^T & 0 \\
\Gamma_a^T & 0
\end{bmatrix}_{(n+1) \times (n+1)} \quad \text{and} \quad K = \begin{bmatrix}
(H(x_i) \Gamma_{10})^T & 0 \\
0 & 0
\end{bmatrix}_{(n+1) \times (n+1)}
\]

At last, for getting a nontrivial solution Eq.(28) should possess a nonzero solution which means that

\[
\det (A- \lambda K) = 0 \quad \ldots \quad (29)
\]

Where \( \det (A- \lambda K) \) is a polynomial of degree \( n-2 \) in \( \lambda \), the eigenvalues of problem (17)–(19) should be those that satisfy Eq.(29).

### V. Numerical Results

In this section some Sturm-Liouville problems have been considered in order to find the approximate results of its eigenvalues using the proposed approach given in section 4.

**Example 1:** Consider the following fractional order eigenvalue problem.

\[
^D D_x^{(1)} [y(x)] + \lambda y(x) = 0, \quad y(0) = 0, \quad y(1) = 0, \quad x \in (0,1) \quad \ldots \quad (30)
\]

S.t.: \( y^{(1)}(0) = 0, \quad y(1) = 0 \), where \( 1 < \alpha \leq 2 \).

\[
\ldots \quad (31)
\]

Following tables 1 and 2 represent the approximate values of the 1st three eigenvalues of problem (30)–(31) compared with the results obtained by [17] when \( \alpha = 2 \).

**Table 1:** The approximate values of the 1st three eigenvalues of the problem (30)–(31) compared with the results obtained by [17] for distinct values of \( \alpha \).

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \alpha = 1.85, n=8 )</th>
<th>( \alpha = 1.85, n=20 )</th>
<th>( \alpha = 1.95, n=8 )</th>
<th>( \alpha = 1.95, n=20 )</th>
<th>( \alpha = 2 )</th>
<th>( \alpha = 2, n=20 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 )</td>
<td>3.686884</td>
<td>3.681348</td>
<td>3.631415</td>
<td>3.628756</td>
<td>3.623089</td>
<td>3.623089</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>20.476917</td>
<td>20.429199</td>
<td>20.429199</td>
<td>20.429199</td>
<td>23.441949</td>
<td>23.441949</td>
</tr>
<tr>
<td>( \lambda_3 )</td>
<td>49.231315</td>
<td>49.108741</td>
<td>57.699948</td>
<td>57.581008</td>
<td>62.960266</td>
<td>62.960266</td>
</tr>
</tbody>
</table>

**Table 2:** The approximate values of the 1st three eigenvalues of the problem (30)–(31) for distinct values of \( n \) with \( \alpha = 1.75 \).

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \alpha = 1.75 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 6 )</td>
<td>( \lambda_1 )</td>
</tr>
<tr>
<td></td>
<td>3.8153558</td>
</tr>
<tr>
<td></td>
<td>3.8039381</td>
</tr>
<tr>
<td></td>
<td>3.800443</td>
</tr>
<tr>
<td></td>
<td>3.797901</td>
</tr>
<tr>
<td></td>
<td>3.797257</td>
</tr>
<tr>
<td>( n = 10 )</td>
<td>( \lambda_2 )</td>
</tr>
<tr>
<td></td>
<td>19.721111</td>
</tr>
<tr>
<td></td>
<td>19.563688</td>
</tr>
<tr>
<td></td>
<td>19.553688</td>
</tr>
<tr>
<td></td>
<td>19.550035</td>
</tr>
<tr>
<td>( n = 12 )</td>
<td>( \lambda_3 )</td>
</tr>
<tr>
<td></td>
<td>41.760745</td>
</tr>
<tr>
<td></td>
<td>42.966502</td>
</tr>
<tr>
<td></td>
<td>42.974093</td>
</tr>
<tr>
<td>( n = 14 )</td>
<td></td>
</tr>
<tr>
<td>( n = 16 )</td>
<td></td>
</tr>
</tbody>
</table>

**Example 2:** Consider the singular fractional eigenvalue problem.

\[
^D D_x^{(1)} [y(x)] + \left( \frac{1}{\alpha} \lambda \right) y(x) = 0, \quad x \in (0,1) \quad \ldots \quad (32)
\]

S.t.: \( y(0) = 0, \quad y^{(1)}(1) = 0 \).

\[
\ldots \quad (33)
\]

Following table 3 represent the approximate values of the 1st three eigenvalues of problem (32)–(33) compared with the results obtained by [11] when \( \alpha = 2 \).
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Table 3: The approximate values of the 1st three eigenvalues of the problem (32)–(33) for distinct values of α.

<table>
<thead>
<tr>
<th>λ_1</th>
<th>n=8</th>
<th>λ_2</th>
<th>n=20, [11]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>α=1.75</td>
<td>α=1.85</td>
<td>α=1.9</td>
</tr>
<tr>
<td>λ_1</td>
<td>1.00229</td>
<td>0.86995</td>
<td>0.82720</td>
</tr>
<tr>
<td>λ_3</td>
<td>35.19086</td>
<td>42.74263</td>
<td>47.29187</td>
</tr>
</tbody>
</table>

Example 3: Consider the following fractional order eigenvalue problem.

\[ D^\alpha y(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(1) = 0, \quad \lambda > 0 \]  

Following table 4 represent the approximate values of the 1st three eigenvalues of problem (34)–(35) for distinct values of α.

Table 4: The approximate values of the 1st three eigenvalues of the problem (34)–(35) for distinct values of α.

<table>
<thead>
<tr>
<th>λ_1</th>
<th>n=8</th>
<th>λ_2</th>
<th>n=20, [11]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>α=1.5</td>
<td>α=1.75</td>
<td>α=1.85</td>
</tr>
<tr>
<td>λ_1</td>
<td>2.12365</td>
<td>2.19259</td>
<td>2.27997</td>
</tr>
<tr>
<td>λ_2</td>
<td>13.83558</td>
<td>15.96978</td>
<td>17.97726</td>
</tr>
<tr>
<td>λ_3</td>
<td>24.28174</td>
<td>38.45971</td>
<td>46.04442</td>
</tr>
</tbody>
</table>

VI. Conclusions

This article presents powerful technique for calculating the eigenvalues of the fractional order Sturm–Liouville problems. The proposed technique is easy in that it is based on using the operational matrices of the fractional order fractional differential equations (FDEs) into a linear system of algebraic equations, then the eigenvalue may be computed by finding the roots of the characteristic polynomials. The proposed method is simple to execute, efficient and yields precise outcomes when it is compared with the existing methods.

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References


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