On Commutativity Property Of $Q_{k,m,n}$, $P_{k,m,n}$, $P_{k,m,\infty}$ and $Q_{k,m,\infty}$ Rings

1 G.Gopalakrishnamoorthy, H.Habeeb Rani2 and 3 V Thiripurasundari
1Advisor, Sri Krishnasamy arts and science college, Mettamalai, Sattur.
2Assistant Professor of Mathematics, H.K.R.H.College, Uthamapalayam, Theni district, Tamil nadu.
3Assistant Professor of Mathematics, Sri S.R.N.M.College, Sattur, Virudhunagar district, Tamil nadu

ABSTRACT: We study commutativity in Rings $R$ with the property that for fixed positive integers $k,m,n$, $x^kS^m = S^m x^k$ for all $x \in R$ and for all $n$-subsets $S$ of $R$.

I. Introduction

Recently G.Gopalakrishnamoorthy and S.Anitha have defined $Q_{k,n}$ -rings by the property that $x^kS = Sx^k$ for all $x \in R$ and for all $n$-subsets $S$ of $R$. They also have defined $Q_{k,n,\infty}$ -rings by the property that $x^kS = Sx^k$ for all $x \in R$ and for all infinite subsets $S$ of $R$, and defined $P_{k,n}$ - ring to be a ring $R$ with the property that $XY = YX$ for all $k$-subsets $X$ of $R$ and $n$-subsets $Y$ of $R$. Also they have defined $P_{k,\infty}$ - ring by the property that $XY = YX$ for all $k$-subsets $X$ of $R$ and all infinite subsets $Y$ of $R$. Obviously every $Q_{k,n}$ -ring is a $P_{k,n}$ -ring and every $P_{k,\infty}$ -ring is a $P_{k,n}$ -ring. It is proved that any $Q_{k,n}$ -ring with identity such that $|R| > n$ is commutative. If $n \leq 4$, $Q_{k,n}$ -rings are commutative. If $n \leq 8$, every $Q_{k,n}$ -ring with $1$ is commutative.

In this paper we define $Q_{k,m,n}$ -rings and $P_{k,m,n}$ -rings, thus generalizing the above concepts and discuss their commutativity.

II. Preliminaries

Let $R$ be an arbitrary ring not necessarily with identity. Let $D,N,Z$ and $C(R)$ denote the set of zero divisors, the set of nilpotents, the center and the commutator ideal of $R$ respectively. Let $|R|$ denote the cardinality of $R$. For any subset $Y$ of $R$, let $CR(Y), A1(Y), Ar(Y)$ and $A(Y)$ denote the centralizer of $Y$, the left, right and two sided annihilators of $Y$ respectively. For $x,y \in R$ the set $Lx, y, k$ is defined to be $\{w \in R \mid x^ky = wx^k\}$ where $k \geq 1$ is a fixed integer.

2.1 Definition

Let $k,m,n$ be three fixed positive integers. A ring $R$ is said to be $Q_{k,m,n}$-ring if $x^kS^m = S^m x^k$ for all $x \in R$ and for all $n$-subsets $S$ of $R$.

where $|R| > n$ and $S^m = \{s^m \mid s \in S\}$

2.2 Definition

Let $k,m,n$ be three fixed positive integers. A ring $R$ is said to be $P_{k,m,n}$-ring $X^mY = Y^mX^m$ for all $k$-subsets $X$ of $R$ and $n$-subsets $Y$ of $R$.

2.3 Definition

Let $k,m$ be two fixed positive integers. A ring $R$ is said to be $Q_{k,m,\infty}$-ring if $x^kS^m = S^m x^k$ for all $x \in R$ and for all infinite subsets $S$ of $R$.

where $|R| > n$ and $S^m = \{s^m \mid s \in S\}$

2.4 Definition

Let $k,m$ be two fixed positive integers. A ring $R$ is said to be $P_{k,m,\infty}$-ring $X^mY = Y^mX^m$ for all $k$-subsets $X$ of $R$ and for all infinite subsets $Y$ of $R$.

Taking $m = 1$ we note that $XY = YX$ for all $k$-subsets $X$ of $R$ and for all infinite subsets $Y$ of $R$.

We simply call $P_{k,1,\infty}$-ring as a $P_{k,\infty}$-ring.

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2.5 Note

i. Every \( Q_{k,m,n} \) ring is a \( Q_{k,m,\infty} \) ring

ii. Every \( Q_{k,m,n} \) ring is a \( P_{k,m,n} \) ring

iii. Every \( P_{k,m,n} \) ring is a \( P_{k,m,\infty} \) ring

iv. Every \( Q_{k,m,\infty} \) ring is a \( P_{k,m,\infty} \) ring

v. Every \( Q_{k,\infty} \) ring is a \( P_{k,\infty} \) ring

2.6 Definition

Let \( R \) be a ring and \( I \) be a subset of \( R \). Let \((k, m)\) be fixed positive integers. \( I \) is said to be a left \((k, m)\)-ideal of \( R \) if

i. \( x^m, y^m \in I \Rightarrow x^m y^m \in I \) and

ii. \( x^m \in I, r \in R \Rightarrow r x^m \in I \)

Similarly the right \((k, m)\)-Ideal and two sided \((k, m)\) ideal can be defined.

2.7 Lemma

Let \( R \) be a \( Q_{k,m,n} \) ring with \(|R| > n\). Then

i. for all \( x \in R, x^m \cdot R^m = R^m x^k \)

ii. If \( x^k \) is idempotent then \( x^k \) commutes with the \( m^m \) power of every \( a \in R \)

iii. \( N \) is a \((k, m)\) ideal of \( R \).

iv. \( |A_1(x^k)|^m = |A_1(x^k)|^m \)

v. If \( R \) is not commutative and \( (x^m)^a \in Z \) then \( R \cup A_1(x^k)^m \cup UC_0(x^k)^m \) and \( R \cup A_1(x^k)^m \cup UC_0(x^k)^m \) are non-empty.

Proof:

Let \( R \) be a \( Q_{k,m,n} \) ring with \(|R| > n\).

i. \( z \in R^m x^k \) if \( z = r x^k \) for some \( r \in R \)

- \( z \in S^m x^k \) for some \( n \)-subsets \( S \subset R \)
- \( z \in x^k S^m \) for some \( n \)-subsets \( S \subset R \)
- \( z = x^k (S^m) \) for some \( s \in S \subset R \)
- \( z \in x^k R^m \)

\( i.e., \) \( R^m x^k = x^k R^m \) for all \( x \in R \)

ii. Let \( x \in R \) be such that \( x^k \) is idempotent. Then for all \( a \in R \)

\[ x^k a^m = x^k a^m \] (since \( x^k \) is idempotent)

\[ = x^k (x^k a) \]

\[ = x^k (a^m, x^k) \] (since \( x^k R^m = R^m x^k \))

\[ = (x^k, a^m) x^k \]

\[ = (a^m, x^k) x^k \] (since \( x^k R^m = R^m x^k \))

\[ = a^m x^k \]

\[ = a^m x^k \] (since \( x^k \) is idempotent)

Hence \( x^k \) commutes with the \( m^m \) power of every \( a \in R \)

iii. Let \( x^m, y^m \in N \). Clearly \( x^m + y^m \in N \) (adopt the standard proof that \( N \) is an ideal in commutative rings)

Since \( x^m \in N \), \( (x^m)^n = 0 \) for some \( n \geq 1 \).

For all \( r \in R \)

\[ (r^k x^m)^n = (r^k x^m) \ldots (r^k x^m) \] \( n \) times

\[ = r^k (r^k x^m) \ldots (r^k x^m) x^m \]

\[ = r^{2k} (r^k x^m) \ldots (r^k x^m) x^{2m} \]

\[ = r^{3k} (r^k x^m) \ldots (r^k x^m) x^{3m} \]

\[ = r^{nk} (r^k x^m) \ldots (r^k x^m) x^{nk} \]

\[ = 0 \] (since \( x^m \) commutes, \( x^{nk} = x^{nm} = 0 \))

\( (r^k x^m)^n = 0 \)

Hence \( r^k x^m \in N \)

Thus \( x^m \in N, r \in R \Rightarrow r^k x^m \in N \)

So, \( N \) is a \((k, m)\) ideal of \( R \)
iv. Also,
\[ z^m \in A_1(x^k)^m \quad \text{iff} \quad z^m x^k = 0 \]
\[ \text{iff} \quad x^k z^m = 0 \quad \text{(using (i))} \]
\[ \text{iff} \quad z^m \in A_r(x^k)^m \]
Hence \[ |A_1(x^k)^m| = |A_r(x^k)^m| \]
v. Let \( R \) be a non-commutative ring and \( x^k \) does not belong to \( Z \). Then there exist \( y \in R \) such that \( x^k y^m \neq y^m x^k \). Consequently \( y \) does not belong to \( C_r(x^k)^m \). So \( C_r(x^k)^m \) is a proper subgroup of \( (R,+) \). Then from (i) and (iv) imply that \( A_1(x^k)^m \) and \( A_r(x^k)^m \) are also proper subgroups of \( (R,+) \). Since a group cannot be the union of two proper subgroups, (v) is proved.

2.8 Note
This generalizes lemma 2.8[4].

2.9 Lemma
If \( R \) is an infinite \( Q_{k,m,n} \) ring then \( R \) is commutative.

Proof
Let \( R \) be an infinite \( Q_{k,m,n} \) ring.
If \( R \) is commutative then there is nothing to prove.
Suppose \( R \) is non-commutative. Since all \( Q_{k,m,n} \) rings are commutative, \( k > 1, m > 1 \) and \( n > 1 \).
Assume that \( R \) is not a \( Q_{k,m,n} \) ring for any \( s < n \). Then there exist \( x \in R \) and an \( (n-1) \) subset \( H \) of \( R \) such that \( x^k H^m \neq H^m x^k \). Since \( R \) is infinite \( R \cap H \neq \emptyset \)
For any \( a \in R \), \( \forall x \in H \), \( x^k (H \{a\})^m = (H \{a\})^m x^k \)
So if we take \( h \in H \) for which \( x^k h^m \) does not belong to \( H^m x^k \)
We have \( x^k h^m = a^n x^k \quad (1) \)
Since (1) holds for all \( a \in R \setminus H \) it follows that for fixed \( b \in R \setminus H \), \( R \cap H^m \subseteq b^n + A_1(x^k)^m \quad (2) \)
Moreover
\[ b^n + c^m \notin H^m \]
That is \( b^n + c^m \notin b^n + A_1(x^k)^m \)
This implies \( b + c \notin H \)
Hence \( b^n + A_1(x^k)^m \subseteq R \setminus H^m \)
\[ R \setminus H^m = b^n + A_1(x^k)^m \]
Hence by (2) and (3) we have \( R \setminus H^m = b^n + A_1(x^k)^m \)
\[ |R| A_1(x^k)^m = |H^m| \]
Since \( A_1(x^k)^m \) is a proper subgroup of \( (R,+) \)
we have \( |R| A_1(x^k)^m = |H^m| \)
That is \( |H^m| \geq |A_1(x^k)^m| \)
The finiteness of \( H^m \) yields finiteness of \( R \), contradicting \( R \) is finite.
Hence \( R \) is commutative.

2.10 Note:
This generalizes lemma 2.10[4]

2.11 Lemma: (See[6])
If \( R \) is a finite ring with \( N \subseteq Z \), then \( R \) is commutative.

2.12 Remark
In view of lemma 2.11, we assume henceforth that \( R \) is finite.
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III. Commutativity of $Q_{k,m,n}$ Rings

3.1 Theorem If R is any $Q_{k,m,n}$ ring with identity such that $|R|>n$, then R is commutative.
Proof: If R is infinite, commutativity follows from Lemma 2.9. So, assume R is finite.
By Lemma 2.11, we need only to show that N $\subseteq$ Z.
Since u $\in$ N implies 1+u is invertible, it suffices to prove that invertible elements are central.
Let x $\in$ R be an invertible element. If x $\in$ Z, there is nothing to prove. Assume x $\notin$ Z,
then $x^m \notin Z$. Hence $C_k(x^m)$ is a proper subset of R. Choose y $\in$ R such that $y^m \notin C_k(x^m)$.
Then $y^m x^m \neq x^m y^m$. If H is any (n-1) subset R, which does not contain y, the condition
$(x^m y^m)^k y^m = z^m(x^m)^k (1)
Since x is invertible, there is unique z $\in$ R satisfying (1). Thus we have proved that every (n-1) subsets of R contains either y or z.
But $S = R \setminus \{y^m, z^m\}$ does not contain $y^m$ and $z^m$ and |S| $\geq$ n-1, a contradiction. This contradiction proves that non–central invertible elements cannot exist. This proves the theorem.

3.2 Remark
This theorem generalizes theorem 3.1[4]

3.3 Theorem
Let n $\geq$ 4 and let R be a $Q_{k,m,n}$ ring with identity such that $|R|>2n$ - 2 or if n is even and $|R|>2n$ - 4.
Then $(x^k)^m \in Z$ for all x $\in$ R.
Proof: If n $\geq$ 4 and R be a $Q_{k,m,n}$ ring.
We shall prove that if there exists x $\in$ R such that $(x^k)^m \notin Z$, then $|R|\leq 2n$ - 2 or $|R|\leq 2n$ - 4.
Since (n-1)$\leq 2n$ - 4, we may suppose that $|R|\geq n$. Suppose there exists x $\in$ R such that $(x^k)^m \notin Z$, by Lemma 2.8(v), there exists
$y \in R \setminus \{A_k(x^m) \cup C_k(x^k)^m\}.
If H is any (n-1) subset which does not contain y, we have
$(x^k)^m \setminus \{y^m\} \cup H = (\{y^m\} \cup H) \setminus (x^k)^m$.
Since $(x^k)^m \neq Z(x^m)^m$, there exists z $\in$ H such that $z(x^k)^m \neq z^m(x^k)^m$ is z $\in$ L$_{x,y,k}$.
So $H \cap L_{x,y,k} \neq \emptyset$.
Thus we have proved that any (n-1) subset of R must either contain y or intersect L$_{x,y,k}$
This condition cannot hold if $|R - L_{x,y,k}| \geq n$.
So, $|R - L_{x,y,k}| \leq (n-1)$
That is $|R| \leq |L_{x,y,k}| + (n-1)$.
Now, if w $\in$ L$_{x,y,k}$ then L$_{x,y,k} = w+ A_k(x^m)$
Hence $|L_{x,y,k}| = |A_k(x^m)|$.
Again by Lemma 2.8(v), $A_k(x^m) \neq R$.
So $|L_{x,y,k}| \leq \frac{|R|}{p}$ for some p $\geq 2$.
Substituting in (1), we get $|R| \leq \frac{|R|}{p} + (n-1)$
i.e., $|R| < 1 + \frac{1}{p} \leq (n-1)$
i.e., $|R| \leq \frac{p}{p-1} (n-1) \leq 2n$ - 2 (2)
Suppose that n is even, If $(A_k(x^m))$ has index atleast 3 in (R,+), the inequality (2) yields
$|R| \leq \frac{3(n-1)}{2} \leq 2n$ - 4
Thus we may assume that $|R| \geq \frac{3(n-1)}{2}$.
We shall show that $|R|\neq 2n$ - 2.
Suppose $|R| = 2n$ - 2, then $|A_k(x^m)| = \frac{2n-2}{2} = (n-1)$. So $|A_k(x^m)| = (n-1)$
We note that $A_k(x^m)$ is an (n-1) subset not intersecting L$_{x,y,k}$.
Hence y $\notin A_k(x^m)$.
Since y $\in R \setminus A_k(x^m) \cup C_k(x^m)$, we see that y $\notin A_k(x^m)$.
So, $A_k(x^m) \neq A_k(x^m) \cup C_k(x^m)$ and consequently $A_k(x^m) \cup C_k(x^m) \neq 0$.
Now, $x^m \setminus (y^m) \cup A_k(x^m) = (y^m) \cup A_k(x^m)$ and therefore $A_k(x^m) \cup C_k(x^m) \subseteq \{x^m, y^m, 0\}$.
Hence $A_k(x^m) \cup C_k(x^m) \neq \{0, x^m, y^m, 0\}$ is an additive subgroup of order 2.
Hence the map $\phi: A_k(x^m) \rightarrow A_k(x^m) \cup C_k(x^m)$ given by $\phi(w) = w(x^m)$ has kernel of index 2 in $A_k(x^m)$.
But $A_k(x^m)$ is odd and so we have a contradiction.
Hence $|R| \leq 2n$ - 4.
3.4 Lemma (Theorem 5[5])
Suppose the ring \( R \) is such that \( x^{m(n)} \in Z \), the centre of \( R \), for all \( x \in R \). Then if \( R \) has no non – zero nilideals, it must be commutative.

3.5 Theorem
Let \( n \geq 4 \) and \( R \) be a \( Q_{k,m,n} \) ring. If \( |R| > 2n - 2 \) or if \( n \) is even and \( |R| > 2n - 4 \), then \( R \) is commutative, provided \( R \) has no non- zero nilideals.

Proof: Follows from Theorem 3.3 and Lemma 3.4.

3.6 Theorem
Let \( n \geq 4 \) and let \( R \) be a \( Q_{k,m,n} \) ring with \( |R| > \frac{3}{2}(n-1) \). Then \( R \) is commutative, if one the following is satisfied
i. \( |R| \) is odd.
ii. \( (R,+)_0 \) is not the union of three proper subgroups.
iii. \( N \) is commutative.
iv. \( R^2 \neq \{0\} \).

Proof: (i) Assume \( |R| \) is odd.
Suppose that \( R \) is not commutative.

Since, \( |R| > \frac{3}{2}(n-1) > n \)
The arguments in the proof of theorem 3.3 gives
\( |A_1(x^m)| = |A_1(x^m)| = |R|/2 \)
This is impossible. So, \( R \) must be commutative.

(ii) Assume \( (R,+)_0 \) is not the union of three proper subgroups. Suppose that \( R \) is not commutative. Then by (i), \( |R| \) is even. By applying the first isomorphism theorem of groups:
\( |(x^k)^m| = |R(x^k)^m| = 2 \).
Hence for any \( u \in R \setminus A_1(x^m) \), \( (x^k)^m = \{0, (x^k)^mu\} \) and for any \( v \in R \setminus A_1(x^m) \)
\( R(x^k)^m = \{0,v(x^k)^m\} \)
By Lemma 2.8(i),
\( (x^k)^m R = R(x^k)^m \).
If \( y \in \mathbb{R} \setminus A_1(x^m) \cup A_1(x^m) \) then
\( \{0,(x^k)^m\} = (x^k)^m R = R(x^k)^m = \{0,y(x^k)^m\} \)
Hence \( y \in \mathbb{C}_R(x^k)^m \)
Thus \( R = A_1(x^m) \cup A_1(x^m) \cup \mathbb{C}_R(x^k)^m \) which is a contradiction to our assumption that \( (R,+)_0 \) is not the union of three proper subgroups.

(iii) Assume \( N \) is commutative.
Suppose \( R \) is not commutative. Then by theorem 3.1, if \( R \) is any \( Q_{k,m,n} \) ring with 1 such that
\( |R| > n \), then \( R \) is commutative.

Now, \( R \) does not have 1. Hence, \( R = D \) fro \( R \) is finite. If \( (x^k)^m \notin \mathbb{N} \), some power of \( (x^k)^m \) is an idempotent zero divisor \( e \neq 0 \).
Since, \( A_1(x^m) \subseteq A_1(x^m) \) and \( A_1(x^m) \neq R \),
We must have \( A_1(x^m) \subseteq A_1(x^m) \)
And similarly \( A_1(x^m) \subseteq A_1(x^m) \)
By lemma 2.8 (ii),\( e \) is central.

Hence, \( A_1(x^m) = A_1(x^m) = A_1(x^m) \subseteq \mathbb{C}_R(x^k)^m \).
Thus if \( y \notin A_1(x^m) \) then \( y \notin A_1(x^m) \) and \( y \notin A_1(x^m) \).
\( Y \notin A_1(x^m) \)
\( y \in R \setminus A_1(x^m) \)
\( \rightarrow \{0,(x^k)^m\} R = R(x^k)^m = \{0,y(x^k)^m\} \)
Hence \( y \notin \mathbb{C}_R(x^k)^m \) which is the contradiction to the assumption that \( (x^k)^m \notin \mathbb{Z} \).

Hence \( (x^k)^m \) is a non – central element.
If there exists two non – commutative elements, which is a contradiction to the assumption that \( N \) is commutative.

(iv) Assume \( R^2 \neq 0 \).
Suppose \( R \) is not commutative. Then there exists \( x \in R \) such that \( x \notin \mathbb{Z} \), the fact that \( (x^k)^m \notin \mathbb{N} \) yields
\( A_1(x^m) \supseteq A_1(x^m) \)
So, \( A_1(x^m) = R \)
Hence, \( (x^{k+1})^m R = R(x^{k+1})^n(0) \)
Choose \( y \in R \setminus A_1(x^m) \cup A_1(x^m) \) and \( w \in R \setminus A_1(x^m) \cup A_1(x^m) \)
Then \( y^{k+1} R = y^{k+1} \)
More over, \( \{0, (x^k)^m\} y = (x^k)^m R = R(x^k)^m = \{0,w(x^k)^m\} \) so that \( (x^k)^m y = w(x^k)^m \)
Thus \( (x^k)^m R^2 = (x^k)^m yR = w(x^k)^m R = \{ w(x^k)^m y,0 \} = \{ xy^{k+1},0 \} = (0) \)

If \( z \in Z \) then \( (x^k)^m + Z \notin Z \) so that \( (x+Z)R^2 = \{0\} \)

Hence \( R^3 = \{0\} \), which is a contradiction to the fact that \( R^3 \neq \{0\} \).

Hence \( R \) is commutative.

### IV. Further Results for Small \( n \)

**4.1 Theorem**

If \( n \leq 8 \), then every \( Q_{k,m,n} \) ring with 1 is commutative.

*Proof:* Let \( R \) be any \( Q_{k,m,n} \) ring with identity. Suppose \( R \) is not commutative.

We any assume that \( n=8 \).

Then by theorem 3.1, \( |R| \leq 8 \).

Since all rings with 1 having fewer than 8 elements are commutative, \( |R| = 8 \) and \( R \) is indecomposable. Since idempotents are central, we must have no idempotents except 0 and 1. Hence every element is either nilpotent or invertible. Since \( u \in N \), \( 1+u \) is invertible. If follows from lemma 2.12, there exists a pair \( x^m,y^m \) of non-commuting invertible elements. The groups of units is not commutative and has almost 7 elements, hence is isomorphic to \( S_3 \). Thus, there exists a unique non-zero nilpotent element \( w \) which by lemma 2.12 is not central. Hence there is an invertible element \( w \) such that \( u(w^k)^m \neq (w^k)^m u \). By Lemma 2.8(iii), \( (w^k)^m \) and \( u(w^k)^m \) are non-zero nilpotents. This gives a contradiction.

So, \( R \) is commutative.

### References


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