Research on three different Portfolio Models with singular Covariance Matrix

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Abstract: In this paper, we will solve three portfolio models with singular covariance matrix. These portfolio models include Mean-Variance Portfolio Model, Value-at-Risk portfolio Model, and Conditional Value-at-Risk portfolio Model. By studying and calculating, we fond: the effective boundary of these three types of portfolio models must be the effective boundary of their maximal linearly independent groups or their maximal linearly independent groups and risk-free assets.

Key words: Singular Covariance Matrix; Maximum Linear Unrelated Group; Effective borders

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I. Introduction

Markowitz first proposed in 1952 that the expected yield of risk assets and the use of variance to quantitatively replace returns and risks, so that abstract risk data can measure and estimate risks. However, both theory and practice show that variance is not an effective risk measure because it treats up and down deviations equally. The risk measure of variance is defined as follows:

\[ P(\xi_p) = \sigma^2 = \sum_{i=1}^{N} \sum_{j \neq i} \sigma_i \sigma_j \]

VaR: Under normal market conditions, the worst expected loss within a holding period of a given confidence interval. It represents the fractional \( \alpha \) of the profit and loss distribution of an investment instrument or portfolio. Compared to variance, VaR considers the investor's risk aversion. However, the verification found that the effect of VaR's resulting hand rate distribution and given confidence level only considered the probability of adverse events, and did not consider the degree of loss at the time of adverse events. Therefore, VaR is non-secondary and its non-convex. Therefore, it is not a consistency risk measure. VaR's risk measurement is defined as follows:

\[ P(\xi_p) = \text{VaR} = \min \{ \alpha \in R : \psi(\chi, \alpha) \geq \beta \} \]

CVaR: Proposed by Rockferer and Uryasev. CVaR refers to a loss that exceeds VaR's conditional expectation. CVaR not only retains VaR's point, but also overcomes VaR's limitations. It is not only a consistent risk measure, but also a convex risk measure, and the measurement of risk is more accurate. CVaR's risk measurement is defined as follows:

\[ P(\xi_p) = \text{CVaR} = (1 - \beta)^{-1} \int_{f(x,y) \geq \alpha \beta(x)} f(x,y)p(y)dy \]

Create function by literature:

\[ F_\beta(x, \alpha) = (1 - \beta)^{-1} \int_{f(x,y) \geq \alpha \beta(x)} [f(x,y) - \alpha \beta(x)]^+ p(y)dy \]

And

\[ [t]^+ = \begin{cases} t, t > 0 \\ 0, t \leq 0 \end{cases} \]

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By transforming:
\[ \text{VaR} = \mu(x) + c_1(\beta)\sigma(x), \quad \text{and} \quad c_1(\beta) = \sqrt{2} \text{erf}^{-1}(2\beta - 1) \]
\[ \text{CVaR} = \mu(x) + c_2(\beta)\sigma(x), \quad \text{and} \quad c_2(\beta) = \left(\sqrt{2} \exp\{ \text{erf}^{-1}(2\beta - 1)^2(1 - \beta)\}\right)^{-1} \]
then define
\[ \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \, dt \]
\[ \text{VaR} = -R + c_1(\beta)\sigma(x) \]
\[ \text{CVaR} = -R + c_2(\beta)\sigma(x) \]

**II. The discussion of Mean-Variance Portfolio Model with singular covariance matrix**

**Theorem 2.1:** First set \( \xi_1, \xi_2, \ldots, \xi_r \) (r<n) is a very linearly independent group of N risk assets. When \( V \) is a singular Matrix, which means \( |V| = 0 \), the optimal selection of the risk asset portfolio in N gets the optimal result, or the best result for the optimal selection of the R risk asset portfolio. Either the optimal result is obtained for the optimal choice of this R type of risk asset and a risk-free asset portfolio.

**Lemma 2.1:** If there is a real number that is not all zero \( k_1, k_2, \ldots, k_n \), so that \( k_1 \xi_1 + k_2 \xi_2 + \ldots + k_n \xi_n = a \) (a is a constant), then indicate \( \xi_1, \xi_2, \ldots, \xi_n \) have the relationship of Linear correlation.

**Lemma 2.2:** If there are real numbers \( k_0, k_1, k_2, \ldots, k_n \), so that \( \xi = k_0 + k_1 \xi_1 + k_2 \xi_2 + \ldots + k_n \xi_n \), then we can say that \( \xi \) can be linearly replaced by \( \xi_1, \xi_2, \ldots, \xi_n \).

**Lemma 2.3:** If \( \xi_1, \xi_2, \ldots, \xi_r \) is \( \xi_1, \xi_2, \ldots, \xi_n \), whose \( r \) maximal linearly independent vectors are rearranged to the first \( r \) positions, then \( \xi_1, \xi_2, \ldots, \xi_r \) is the maximal linear independent group of \( \xi_1, \xi_2, \ldots, \xi_n \), and \( \xi_1, \xi_2, \ldots, \xi_n \) can be linearly replaced by \( \xi_1, \xi_2, \ldots, \xi_r \).

**Proof:**
Since \( V \) is a singular matrix, then \( \xi_1, \xi_2, \ldots, \xi_n \) must be linearly related. From Lemma 2.1, there are real numbers \( k_1, k_2, \ldots, k_n \) (\( k_1, k_2, \ldots, k_n \) is not all zero), then \( k_1 \xi_1 + k_2 \xi_2 + \ldots + k_n \xi_n = a \) (a is a constant), which means \( P \left( \sum_{i=1}^{n} k_i \xi_i = a \right) = 1 \).

From Lemma 3, let \( \xi_1, \xi_2, \ldots, \xi_n \) is the maximally linearly independent group of \( \xi_1, \xi_2, \ldots, \xi_n \), and \( \xi_1, \xi_2, \ldots, \xi_n \) can be represented by linear representation of \( \xi_1, \xi_2, \ldots, \xi_n \). Also from Lemma 2.2 there are real numbers \( k_{0i}, k_{1i}, k_{2i}, \ldots, k_{ni} \), so that \( \xi_{fi} = k_{0i} + k_{1i} \xi_1 + k_{2i} \xi_2 + \ldots + k_{ni} \xi_n \). Then set \( \eta_i = \xi_{fi} - \sum_{i=1}^{n} k_{0i} \xi_{fi} = k_{0i}, \quad D (\eta_i) = 0 \), which means \( \eta_i \) is equivalent to \( (1-\sum_{i=1}^{r} k_{0i})r_f, \) \( t=r+1, r+2, \ldots, n \), \( r_f \) is risk-free interest rate. Let the investment ratio vector of \( n \) kinds of assets be \( (W_1, W_2, \ldots, W_n) \in \mathbb{R}^n \), then the total portfolio returns rate \( \xi_p = \sum_{i=1}^{n} W_i \xi_i \), \( R = (r_1, r_2, \ldots, r_n) \) and \( V \in \mathbb{R}^{n \times n} \) are the yield expectation vector and the singular covariance matrix of the asset \( \xi_1, \xi_2, \ldots, \xi_n \), respectively. Let the investment ratio vector of \( r \) assets be \( \bar{W} = (W^{r_1}, W^{r_2}, \ldots, W^{r_r}) \in \mathbb{R}^r \), and records
If for all \( t \), there is \( 1 - \sum_{i=1}^{r} k_i = 0 \), that is, \( k_{0i} = E \eta_i = (1 - \sum_{i=1}^{r} k_i) r_f = 0 \) \((t=r+1, r+2,...,n)\),

so \( \xi^{\tau}_i = k_{0i} + k_{1i} \xi^{\tau}_1 + k_{2i} \xi^{\tau}_2 + ... + k_{ni} \xi^{\tau}_n = k_{1i} \xi^{\tau}_1 + k_{2i} \xi^{\tau}_2 + ... + k_{ni} \xi^{\tau}_n \),

that is, the \( t \)th asset is \( \xi^{\tau}_1, \xi^{\tau}_2, \ldots, \xi^{\tau}_n \) in proportion to \( k_{1i}, k_{2i}, \ldots, k_{ni} \).

\[
\xi_p = \sum_{i=1}^{r} w_i \xi^{\tau}_i = \sum_{i=1}^{r} w_i \xi^{\tau}_1 + \sum_{i=r+1}^{n} w_i \xi^{\tau}_i = \sum_{i=1}^{r} w_i \xi^{\tau}_1 + \sum_{i=r+1}^{n} w_i (k_{0i} + \sum_{i=1}^{r} k_i \xi^{\tau}_i) = \sum_{i=1}^{r} w_i \xi^{\tau}_1 + \sum_{i=r+1}^{n} w_i \sum_{i=1}^{r} k_i \xi^{\tau}_i
\]

Then \( \bar{w}^{\tau}_i = w_i + \sum_{i=r+1}^{n} w_i k_{0i} \). It is easy to prove \( \sum_{i=1}^{r} \bar{w}^{\tau}_i = 1 = \sum_{i=1}^{n} w_i \)

\[
u = E \eta_i = \sum_{i=1}^{r} w_i \bar{r}^{\tau}_i = \bar{w}^{\tau} \bar{R}
\]

\[
\sigma^2 = Cov(\xi_p, \xi_p) = Cov(\sum_{i=1}^{r} \bar{w}^{\tau}_i \xi^{\tau}_1, \sum_{i=1}^{r} \bar{w}^{\tau}_i \xi^{\tau}_1) = \bar{w}^{\tau} \bar{W} \bar{W}^{\tau}
\]

Case 2: If \( t \) exists, let \( 1 - \sum_{i=1}^{r} k_i \neq 0 \), that is \( k_{(t)} \) \((t=r+1, r+2,...,n)\) not all be 0.

\[
\xi_p = \sum_{i=1}^{n} w_i \xi^{\tau}_i = \sum_{i=1}^{r} w_i \xi^{\tau}_1 + \sum_{i=r+1}^{n} w_i \xi^{\tau}_i = \sum_{i=1}^{r} w_i \xi^{\tau}_1 + \sum_{i=r+1}^{n} w_i (k_{0i} + \sum_{i=1}^{r} k_i \xi^{\tau}_i)
\]

\[
= \sum_{i=1}^{r} (w_i + \sum_{i=r+1}^{n} w_i k_{0i}) \xi^{\tau}_1 + \sum_{i=r+1}^{n} w_i \xi_{10} + \sum_{i=r+1}^{n} w_i k_{0i}
\]

So \( \nu = E \eta_i = \sum_{i=1}^{r} w_i \bar{r}^{\tau}_i + \sum_{i=r+1}^{n} w_i k_{0i} \)

\[
\sigma^2 = Cov(\xi_p, \xi_p) = Cov(\sum_{i=1}^{r} \bar{w}^{\tau}_i \xi^{\tau}_1, \sum_{i=1}^{r} \bar{w}^{\tau}_i \xi^{\tau}_1, \sum_{i=1}^{r} \bar{w}^{\tau}_i \xi^{\tau}_1 + \sum_{i=r+1}^{n} w_i k_{0i})
\]

\[
= Cov(\sum_{i=1}^{r} \bar{w}^{\tau}_i \xi^{\tau}_1, \sum_{i=1}^{r} \bar{w}^{\tau}_i \xi^{\tau}_1) = \bar{w}^{\tau} \bar{W} \bar{W}^{\tau}
\]

\[
\sum_{i=1}^{n} w_i = \sum_{i=1}^{r} w_i + \sum_{i=r+1}^{n} w_i = \sum_{i=1}^{r} w_i (\bar{r}^{\tau}_1 - \sum_{i=r+1}^{n} w_i k_{0i}) + \sum_{i=r+1}^{n} w_i = \sum_{i=1}^{r} \bar{w}^{\tau}_i - \sum_{i=1}^{n} w_i k_{0i}
\]

Because

\[
k_{0i} = E \eta_i = (1 - \sum_{i=1}^{r} k_i) r_f = 0 \) \((t=r+1, r+2,...,n)\),

\( r_f \) is risk free rate, \( r_f > 0 \), so

\[
\sum_{i=1}^{n} w_i = \sum_{i=1}^{r} \bar{w}^{\tau}_i + \frac{1}{r_f} \sum_{i=r+1}^{n} k_{0i} \bar{r}^{\tau}_i
\]

III. Optimal solution and its effective frontier of mean - risk model when V is Singular matrix

(1) Mean - variance model

Case 1: If for all \( t \), there is \( 1 - \sum_{i=1}^{r} k_i = 0 \). It can be converted to
Research on three different Portfolio Models with singular Covariance Matrix

\[
\begin{aligned}
\min \sigma^2 &= W^T \Sigma W \\
\text{s.t.} \sum_{i=1}^n w_i r_i &= \sum_{i=1}^r \tilde{w}_i r_i^* = u \\
\sum_{i=1}^n w_i = \sum_{i=1}^r \tilde{w}_i^2 &= 1 
\end{aligned}
\]

This is solution of the optimal solution and its effective boundary of mean - variance portfolio without risk-free assets under non-singular matrix.

By reference, we can get:

Optimal solution is \( \tilde{W}^* = \frac{(uc - b)\tilde{\Sigma}^{-1}I}{\Delta} \)

Effective boundary is \( \tilde{\sigma}^2_p = \frac{1}{\Delta} - \frac{(u - b)^2 c}{\Delta c^2} = 1 \)

And let

\( a = \tilde{R}^T \tilde{\Sigma} \tilde{R} \)  \( b = \tilde{I}^T \tilde{\Sigma}^{-1} \tilde{R} \)  \( c = \tilde{I}^T \tilde{\Sigma}^{-1} \tilde{I} \)  \( \Delta = ac - b \) \((a>0,c>0)\)

Case 2: If for all \( t \), there is \( 1 - \sum_{i=1}^t k_{i0} \neq 0 \), so that \( k_{i0} \) \((r+1,r+2,\ldots,n)\) is not all zero.

\[
\begin{aligned}
\min \sigma^2 &= \tilde{W}^T \tilde{\Sigma} \tilde{W} \\
\text{s.t.} \sum_{i=1}^n w_i r_i &= \sum_{i=1}^r \tilde{w}_i r_i^* + \sum_{t=r+1}^n w_i^* k_{i0} = u \\
\sum_{i=1}^n w_i &= \sum_{i=1}^r \tilde{w}_i^2 + \frac{1}{r_f} \sum_{t=r+1}^n k_{i0} \tilde{w}_i = 1 
\end{aligned}
\]

Optimal solution is \( \tilde{W}^* = \frac{(u - r_f)}{H}(\tilde{\Sigma}^{-1} \tilde{R} - r_f \tilde{\Sigma}^{-1} \tilde{I}) \)

Effective boundary is \( \tilde{\sigma}^2_p = \frac{(u - r_f)^2}{H} \)

And let

\( a = \tilde{R}^T \tilde{\Sigma} \tilde{R} \)  \( b = \tilde{I}^T \tilde{\Sigma}^{-1} \tilde{R} \)  \( c = \tilde{I}^T \tilde{\Sigma}^{-1} \tilde{I} \)  \( \Delta = ac - b \) \((a>0,c>0)\)

2) Mean—VaR model

\[
\begin{aligned}
\min VaR &= c_\gamma(\beta)\sigma(x) - u \\
\text{s.t.} \sum_{i=1}^n w_i r_i &= u \\
\sum_{i=1}^n w_i &= 1 
\end{aligned}
\]
From the reference, it is strictly greater than 0 at a given confidence level ($\alpha > 50\%$).
In addition, it can be proved that the inequality constraint is tight at the optimal solution, so the mean-VaR model's solution is the same as the standard mean-variance model, with the same boundary combination. So the optimal solution and effective boundary are solved in the following two cases:

Case 1: If for all $t$, there is $1 - \sum_{i=1}^{r} k_i = 0$, it can be converted to

$$\min VaR = c_1(\beta) \sqrt{\mathbf{w}^T \mathbf{V} \mathbf{w}} - u$$

Subject to

$$\sum_{i=1}^{n} w_i r_i = \sum_{i=1}^{r} w_i^t r_i^t = u$$

$$\sum_{i=1}^{n} w_i = \sum_{i=1}^{n} w_i^t = 1$$

This is solution of the optimal solution and its effective boundary of mean-VaR portfolio without risk-free assets under non-singular matrix.

By reference, we can get:

Optimal solution is

$$\mathbf{w}^* = \frac{(uc - b)\mathbf{V}^{-1}\mathbf{I}}{\Delta}$$

Effective boundary is

$$\frac{([VaR + u] / c_1(\beta))^2}{1} - \frac{(u - \frac{b}{c})^2}{\Delta^2} = 1$$

And let

$$a = \mathbf{R}^T \mathbf{V} \mathbf{R} \quad b = \mathbf{I}^T \mathbf{V}^{-1} \mathbf{R} \quad c = \mathbf{I}^T \mathbf{V}^{-1} \mathbf{I} \quad \Delta = ac - b \quad (a > 0, c > 0)$$

Case 2: If for all $t$, there is $1 - \sum_{i=1}^{r} k_i \neq 0$, so that $k_{t_0}$ ($t = r+1, r+2, ..., n$) is not all zero.

$$\min VaR = c_1(\beta) \sqrt{\mathbf{w}^T \mathbf{V} \mathbf{w}} - u$$

Subject to

$$\sum_{i=1}^{n} w_i r_i = \sum_{i=1}^{r} w_i^t r_i^t + \sum_{i=r+1}^{n} w_i^t k_{t_0} = u$$

$$\sum_{i=1}^{n} w_i = \sum_{i=1}^{r} w_i^t + \frac{1}{r_f} \sum_{i=r+1}^{n} k_{t_0} w_i = 1$$

Optimal solution is

$$\mathbf{w}^* = \frac{(u - r_f)}{H} (\mathbf{V}^{-1} \mathbf{R} - r_f \mathbf{V}^{-1} \mathbf{I})$$

Effective boundary is

$$\frac{VaR + u}{c_1(\beta)} = \frac{u - r_f}{\sqrt{H}}$$

And let

$$a = \mathbf{R}^T \mathbf{V} \mathbf{R} \quad b = \mathbf{I}^T \mathbf{V}^{-1} \mathbf{R} \quad c = \mathbf{I}^T \mathbf{V}^{-1} \mathbf{I} \quad \Delta = ac - b \quad (a > 0, c > 0)$$

(3) Mean-CVaR model
Research on three different Portfolio Models with singular Covariance Matrix

\[
\min_{\mathcal{W}} CVaR = c_2(\beta)\sigma(x) - u
\]
\[
\text{s.t.} \sum_{i=1}^{n} w_i r_i = u
\]
\[
\sum_{i=1}^{n} w_i = 1
\]

The solution of mean-CVaR model is the same as the solution of standard mean-variance, with the same boundary combination. So in both cases the optimal solution and the effective boundary are solved as follows:

Case 1: If for all t, there is \(1 - \sum_{i=1}^{t} k_{it} = 0\), it can be converted to

\[
\min_{\mathcal{W}} CVaR = c_2(\beta)\sqrt{W^T\Sigma W} - u
\]
\[
\text{s.t.} \sum_{i=1}^{r} w_i r_i = u
\]
\[
\sum_{i=1}^{n} w_i = 1
\]

This is solution of the optimal solution and its effective boundary of mean-CVaR portfolio without risk-free assets under non-singular matrix.

By reference, we can get:
Optimal solution is
\[
\mathcal{W}^* = \frac{(uc - b)\sqrt{V}^{-1}I}{\Delta}
\]
Effective boundary is
\[
\frac{[(CVaR + u)/c_2(\beta)]^2}{\frac{(u-b)^2}{\Delta} - \frac{1}{c}} = 1
\]

And let
\[
a = \mathcal{R}^T\sqrt{\mathcal{V}} \quad b = \mathbf{I}^T\mathbf{V}^{-1}\mathbf{R} \quad c = \mathbf{I}^T\mathbf{V}^{-1}\mathbf{I} \quad \Delta = ac - b \quad (a>0,c>0)
\]

Case 2: If for all t, there is \(1 - \sum_{i=1}^{t} k_0 = 0\), so that \(k_{i0} = 0\) (\(t=r+1,r+2,...,n\)) is not all zero.

\[
\min_{\mathcal{W}} CVaR = c_2(\beta)\sqrt{W^T\Sigma W} - u
\]
\[
\text{s.t.} \sum_{i=1}^{n} w_i r_i = \sum_{i=1}^{r} w_i r_i + \sum_{t=r+1}^{n} w_{i}\cdot k_{i0} = u
\]
\[
\sum_{i=1}^{n} w_i = \sum_{i=1}^{r} w_i + \frac{1}{r_f}\sum_{t=r+1}^{n} k_{i0} w_i = 1
\]

Optimal solution is
\[
\mathcal{W}^* = \frac{(u-r_f)}{H} (\sqrt{\mathcal{V}^{-1}\mathcal{R} - r_f\sqrt{\mathcal{V}^{-1}I})
\]
Effective boundary is
\[
\frac{CVaR + u}{c_2(\beta)} = \frac{u-r_f}{\sqrt{H}}
\]

And let

DOI: 10.9790/5728-1405023339 www.iosrjournals.org 38 | Page
Research on three different Portfolio Models with singular Covariance Matrix

$\mathbf{a} = \mathbf{R}^T \mathbf{\bar{V}} \mathbf{R} \quad \mathbf{b} = \mathbf{I}^T \mathbf{\bar{V}}^{-1} \mathbf{R} \\
\mathbf{c} = \mathbf{I}^T \mathbf{\bar{V}}^{-1} \mathbf{\bar{I}} \quad \Delta = \mathbf{ac} - \mathbf{b} \quad (a>0, c>0)$

IV. Conclusion

(1) If the $\mathbf{V}$ of these three kinds of mean-risk model is a singular covariance matrix, first determine whether there is $\sum_{i=1}^{r} k_i = 0$ for all $t$. If yes, it is case 1, the equivalent is the optimization problem of the risk portfolio composed of the maximally linear irrelevant group; if not all 0, the equivalent is the risk portfolio of the extremely linear irrelevant group and a risk-free asset. Optimization problem.

(2) If $\mathbf{A}$ of these three kinds of mean-risk models is a singular covariance matrix, the effective boundary of the $n$ kinds of risk assets is $F$, and the effective boundary of the risky asset portfolio composed of the extremely linear irrelevant group is the risky asset composed of the extremely linear independent group. The effective boundary of the combination and a risk-free asset is $F_2$, either $F = F_1$, or $F = F_2$.

References