A Critical Growth Nonlinear Bi-harmonic Problem in $R^4$

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Abstract This article concerns with the problem

$$-\Delta^2 u = \mu \frac{u}{|x|^{4n-2|v|}} + f(x, u), \quad x \in \Omega;$$

$$u = 0 \quad x \in \partial \Omega$$

There $f$ has critical growth at both $+\infty$ and $-\infty$ with the same $\alpha_0$, through a Hardy Inequality of [4], We prove the existence of a nontrivial solution of above problem by using Mountain Pass Theorem.

Keywords bi-harmonic equation; critical growth; Mountain Pass Theorem

0 Introduction

When $N > p$, the article [1] had discussed the nonlinear harmonic equation involving critical potential. But as $N = p = 2$, the corresponding question hasn't been studied. Then in 1995, in the article [2], D.G.de Figueiredo, Miyagaki and Ruf proved the existence of multiply solutions of nonlinear elliptic problem in $R^2$, where $f$ has subcritical growth and critical growth. After this article mainly, in 2004, Shen, Yao and Chen [3] have studied the existence of nontrivial solutions for quasi-linear elliptic equation involving critical potential:

$$\begin{cases}
-\Delta u - \mu \frac{u}{|x|^{2n-2|v|}} = \lambda u, & x \in \Omega \\
u = 0, & x \in \partial \Omega
\end{cases}$$

where $\Omega$ is a bounded domain in $R^2$, $0 \in \Omega \subset B_R(0), B_R(0)$ is a small ball centering origin with radius $R$ in $R^2$, and in this article, $f$ has subcritical. Then in 2005, Chen, Shen and Yao[4] have studied the existence of nontrivial solutions for nonlinear biharmonic equation involving critical potential:

$$\begin{cases}
\Delta^2 u - \mu \frac{u^2}{|x|^{4n-2|v|}} + f(x, u), & x \in \Omega \\
u = \frac{\partial u}{\partial \nu} = 0, & x \in \partial \Omega
\end{cases} \quad (0.1)$$

where $\Omega \subset B_R(0) \subset R^4$ is a bounded domain including the origin, $\mu \in R$, $v$ is the unit outer normal vector, and $f$ has subcritical growth(see[2]). According the article [2], we think what will happen if $f$ has critical growth in the problem (0.1). So in this paper, we have discussed the existence of nontrivial solutions for nonlinear bi-harmonic equation...
involving critical potential (0.1), but in here \( f \) has critical growth at \( +\infty \) (see[2]), it means if there exists \( \alpha_0 > 0 \), such that for all \( \alpha > \alpha_0 \)

\[
\lim_{t \to +\infty} \frac{|f(x, t)|}{e^{\alpha t^{1/3}}} = 0
\]  

(0.2)

and for all \( \alpha < \alpha_0 \)

\[
\lim_{t \to +\infty} \frac{|f(x, t)|}{e^{\alpha t^2}} = +\infty
\]

For easy reference we state new conditions on \( f \) that will be assumed below:

**\( H_1 \)** \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is continuous, \( f(x, 0) = 0 \)

**\( H_2 \)** \( \exists t_0 > 0, \exists M > 0, \) such that

\[
0 < F(x, t) = \int_0^t f(x, s)ds \leq M|f(x, t)|
\]

**\( H_3 \)** \( 0 < F(x, t) \leq \frac{1}{2} f(x, t) t, \forall t \in \mathbb{R} - \{0\}, \forall x \in \Omega \)

**\( H_4 \)** \( \lim_{t \to +\infty} \sup_{x \in \Omega} \frac{2F(x, t)}{t^2} < \lambda_1, \text{ uniformly in } (x, t) \)

Now we state the results which will be proved here. By “solution” in the theorems below we mean weak solution \( u \in H_0^2(\Omega) \).

**Theorem 0.1** Assume \( (H_1), (H_2), (H_3), (H_4), \mu < 1 \) and \( f \) has critical growth at both \( +\infty \) and \( -\infty \). Furthermore assume

\[
(H_5) \quad \lim_{t \to +\infty} f(x, t) e^{-\alpha_0 t^2} \geq \beta, \quad \beta > \frac{16(4 - \mu)}{\alpha_0 (1 + M) R^4}
\]

Then, problem (0.1) has a nontrivial solution.

In this paper, we define \( \|u\|^2 = \int_{\Omega} |\Delta u|^2, |u|_p = (\int |u|^p)^{1/p}. \)

1 The proof of lemmas

We know the functional of equation (0.1) is

\[
\Phi(u) = \frac{1}{2} \int_{\Omega} |\Delta u|^2 dx - \frac{\mu}{2} \int_{\Omega} \frac{u^2}{|x|^4 \ln^2 R/|x|} dx - \int_{\Omega} F(x, u) dx
\]

We assume \( (H_1), (H_2) \) and the existence of positive constance \( C \)

And \( \alpha_0 > 0, \text{ when } \alpha > \alpha_0, \left| f(x, t) \right| \leq C e^{\alpha_0 t} \quad \forall x \in \Omega, t \in \mathbb{R} \)  

(1.1)

It follows easily from \( (H_1) \) and \( (H_2) \) that

(1) there is a constant \( C > 0, \) such that

\[
F(x, t) \geq C e^{\frac{\alpha_0}{2}|t|}, \quad \forall |t| \geq t_0
\]  

(1.2)
Lemma 1.1 (see [4]) Assume \( u \in H^2_0(\Omega) \), then

\[
\int_\Omega \frac{u^2}{|x|^4 \ln^2 R/|x|} \, dx \leq \int_\Omega |\Delta u|^2 \, dx
\]  

(1.4)

where the constant 1 is optimal.

Lemma 1.2 (see [2]) \( f(x,u_n) \to f(x,u) \) in \( L^1(\Omega) \). where \( \{u_n\} \) is a (PS) sequence.

Set

Lemma 1.3 Assume \((H_1),(H_2)\) and \((H_3)\), if \( f \) has critical growth at both \(+\infty\) and \(-\infty\) with the same \( \alpha_0 \), then \( \Phi \) satisfies \((PS)\) for all \( c \in (-\infty, 8(1-\mu)\sqrt{\pi/3\alpha_0}) \).

Proof: Let \( \{u_n\} \subset H^1_0(\Omega) \) be a Palais-Smale sequence, i.e.

(1.5)

\[
\frac{1}{2} \int_\Omega |\Delta u_n|^2 \, dx - \frac{\mu}{2} \int_\Omega \frac{u_n^2}{|x|^4 \ln^2 R/|x|} \, dx - \int_\Omega F(x,u_n) \, dx \to c
\]

(1.6)

\[
\int_\Omega \Delta u_n \Delta v \, dx - \mu \int_\Omega \frac{u_n v}{|x|^4 \ln^2 R/|x|} \, dx - \int_\Omega f(x,u_n) v \, dx = o(1)||v||
\]

For \( \forall v \in H^1_0(\Omega) \).

From (1.3) and (1.5), for any \( \epsilon > 0 \), we have

\[
\frac{1}{2} ||u_n||^2 - \frac{\mu}{2} \int_\Omega \frac{|u_n|^2}{|x|^4 \ln^2 R/|x|} \, dx \leq C + \int_\Omega F(x,u_n) \, dx
\]

\[
\leq C + \epsilon \int_\Omega f(x,u_n) u_n \, dx
\]

(1.7)

We assume \( v = u_n \) in (1.6), can obtain

\[
||u_n||^2 - \mu \int_\Omega \frac{|u_n|^2}{|x|^4 \ln R/|x|} \, dx = \int_\Omega f(x,u_n) u_n \, dx + o(1)||u_n||
\]

(1.8)

Substitute (1.8) to (1.7), we have

\[
\frac{1}{2} ||u_n||^2 - \frac{\mu}{2} \int_\Omega \frac{|u_n|^2}{|x|^4 \ln R/|x|} \, dx \leq C + \epsilon(||u_n||^2 - \mu \int_\Omega \frac{|u_n|^2}{|x|^4 \ln R/|x|} \, dx + o(1)||u_n||)
\]

Set \( \epsilon = \frac{1}{4} \) from Lemma 1.1, we know there is a constant \( C \), such that

\[
||u_n||^2 \leq C
\]

Now we take a subsequence of \( \{u_n\} \) denoted again by \( \{u_n\} \), such that, for some \( u \in H^2_0 \), we have

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From Lemma 1.2, when \( n \to \infty \), (1.6) become
\[
\int _{\Omega } \Delta u \Delta v dx - \mu \int _{\Omega } \frac{uv}{|x|^4 \ln^2 R/|x|} dx - \frac{1}{2} \int _{\Omega } F(x, u) dx = 0 \tag{1.9}
\]
Let \( v = u \) in the (1.9), and using (1.3) then
\[
2\Phi (u) = \int _{\Omega } |\Delta u|^2 dx - \mu \int _{\Omega } \frac{u^2}{|x|^4 \ln^2 R/|x|} dx - 2 \int _{\Omega } F(x, u) dx \geq \left( \frac{1}{\epsilon} - 2 \right) \int _{\Omega } F(x, u) dx
\]
So, \( \Phi (u) \geq 0 \). Now we separate the proof into three cases:

Case 1 \( c = 0 \).

From Lemma 1.2, using Lebesgue dominated convergence theorem, we can get
\[
F(x, u_n) \to F(x, u) \text{ in } L^1(\Omega).
\]
So, from (1.5) and (1.6), set \( v = u_n \) we obtain
\[
\lim _{n \to \infty } \left( \frac{1}{2} \int _{\Omega } f(x, u_n) u_n dx - \int _{\Omega } F(x, u_n) dx \right) = c = 0
\]
then, from (1.8), we have
\[
0 \leq \Phi (u) \leq \liminf \Phi (u_n) = \frac{1}{2} \int _{\Omega } f(x, u_n) u_n dx - \int _{\Omega } F(x, u_n) dx = 0
\]
So, \( ||u_n|| \to ||u|| \) and then \( u_n \to u \) in \( H_0^2 \), The proof is finished in this case.

Case 2 \( c = 0, u = 0 \). In this case, we will show that it cannot happen for a (Ps) sequence. First we claim that, for some \( q > 1 \), we have
\[
\int _{\Omega } |f(x, u_n)|^q dx \leq \text{const}
\tag{1.10}
\]
From (1.1), set a fixed \( q > 1 \), then
\[
\int _{\Omega } |f(x, u_n)|^q dx \leq C \int _{\Omega } e^{\alpha q ||u_n||^{4/3}} dx = C \int _{\Omega } e^{\alpha q ||u_n||^{4/3}(\frac{\omega_n}{\omega})^{4/3}}
\]
Using Moser-Trudinger Inequality (N=4)(see[5]): for any \( u \in H^{1,4}_0(\Omega) \)
\[
\sup ||u||_{H^{1,4}_0} \leq \int _{\Omega } e^{\gamma ||u||^{1/3}} dx \leq C ||\Omega||, \quad \forall \gamma \leq 4\omega_3^{1/3}
\]
where \( ||u||_{H^{1,4}_0} = ||u||_4 + ||Du||_4, \omega_3 = 4\pi/3 \) is the volume of unit ball when \( n = 3 \), \( ||\Omega|| \) is the lebesgue measure of \( \Omega \).

Then we can deduce the integral (1.10) is bounded independently of \( n \), if
\[
\alpha q ||u_n||^{4/3} \leq \gamma \leq 4\omega_3^{1/3}
\]
From (1.5), Lemma 1.1, (H1) and \( u = 0 \), we have
\[
\lim _{n \to \infty } ||u_n||^2 = \frac{2(c + \epsilon)}{1 - \mu}
\]
Then it will be indeed the case for $c < 8(1 - \mu)\sqrt{\frac{\pi}{3\alpha^0}}$ if we choose $q > 1$ sufficiently close to $1$, $\alpha$ sufficiently close to $\alpha_0$ and sufficiently small.

Let (1.6) subtract (1.9), and assume $v = u_n - u$, then we have

$$\int_{\Omega} |\Delta(u_n - u)|^2 dx - \mu \int_{\Omega} \frac{|u_n - u|^2}{|x|^4 \ln^2 R/|x|} dx - \int_{\Omega} (f(x, u_n) - f(x, u))(u_n - u) dx = o(1)||u_n - u||$$

We estimate the third integral above using Holder inequality and $|u_n - u|_{L^q} \to 0$, then we have

$$\int_{\Omega} |(f(x, u_n) - f(x, u))(u_n - u)| dx \to 0$$

So, through Lemma 1.1, we know $||u_n|| \to 0$. But, from (1.5), which implies $||u_n||^2 \to \frac{2c}{1 - 4\mu} \neq 0$. It is contradiction.

Case 3 $c \neq 0, u \neq 0$.

Like case 2, we can proof (1.10). Because $||u_n||^2 \leq C$, so it means when $aqC^{2/3} \leq 4\left(\frac{4\pi}{3}\right)^{1/3}$, (1.10) is true. At the same time, we can know $u_n \to u$ in $H^2_0$. Then the lemma is proved.

In the case 3 of above, we actually can obtain

$$\Phi(u) = c$$

and $c < 8(1 - \mu)\sqrt{\frac{\pi}{3\alpha^0}}$.

Lemma 1.4 Assume $(H_1), (H_2), (H_3)$ and (1.1), then existence $a > 0, \rho > 0$, such that $\Phi(u) \geq a$, if $||u|| = \rho$.

Proof: From $(H_3)$, we know there are $\lambda_0 < \lambda_1, \delta > 0$, such that

$$F(x, t) \leq \frac{1}{2}\lambda_0 t^2, \quad |t| \leq \delta$$

In other way, from (1.1), to $q > 2$

$$F(x, t) \leq Ce^{\alpha|t|^{4/3}}|t|^q \leq \delta$$

Putting these two estimates together we obtain

$$F(x, t) \leq \frac{1}{2}\lambda_0 t^2 + Ce^{\alpha|t|^{4/3}}|t|^q \quad \forall t \in \mathbb{R}$$

(1.11)

From (1.11), and using Holder inequality, for $p > 1$, we have

$$\Phi(u) = \frac{1}{2} \int_{\Omega} |\Delta u|^2 - \frac{\mu}{2} \int_{\Omega} \frac{u^2}{|x|^4 \ln^2 R/|x|} - \int_{\Omega} F(x, u)$$

$$\geq \frac{1}{2}(1 - \mu)||u||^2 - \frac{1}{2}\lambda_0 \int_{\Omega} u^2 - C \int_{\Omega} e^{\alpha|u|^{4/3}}|u|^\eta$$

$$\geq \frac{1}{2}(1 - \frac{\lambda_0}{\lambda_1})(1 - \mu)||u||^2 - C(\int_{\Omega} e^{\alpha|u|^{4/3}}|u|^\eta)^{1/p}(\int_{\Omega} |u|^\eta')^{1/p'}$$

$$\geq \frac{1}{2}(1 - \frac{\lambda_0}{\lambda_1})(1 - \mu)||u||^2 - C||u||^q$$

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Now choose $\rho > 0$, as the point where the function $g(s) = \frac{1}{2}(1 - \frac{\lambda_0}{\lambda_1})(1 - \mu)s^2 - Cs^0$ assumes its maximum. Take $a = g(\rho)$. Then the proof is complete.

Remarks on the conditions above, we easily to prove there is $e \in H_0^2$, $||e|| > \rho$, such that $\Phi(e) \leq 0$.

2 The proof of Theorem 0.1

It follows from the assumptions that $\Phi$ satisfies $(PS)$ for all $c < 8(1 - \mu) \frac{\pi}{3\alpha_0}$, see lemma 1.3. At the same time, through lemma 1.4 and $(H_4)$, we can know that $\Phi$ has a local minimum at 0. To conclude via the Mountain Pass Theorem it suffices to show that there is a $\omega \in H_0^2$, $||\omega|| = 1$, such that $\max \{\Phi(t\omega) : t \geq 0\} < c$. For that matter we start by introducing the following functions

$$\omega_n(x) = \frac{1}{2\sqrt{2\pi}} \begin{cases} 
\frac{(\ln n)^{1/2}}{\ln |x|^{1/2}}, & 0 \leq |x| \leq \frac{R}{n} \\
\frac{\ln |x|^{1/2}}{\ln (n^{1/2})}, & \frac{R}{n} \leq |x| \leq R \\
0, & |x| \geq R
\end{cases}$$

which indicate that $\omega_n(x) \in H_0^2(B_R(0))$ and $||\omega_n|| = 1$ for all $n = 1, 2, ...$.

We claim that there exists $n$ such that

$$\max \{\Phi(t\omega_n) : t \geq 0\} < 8(1 - \mu) \sqrt{\frac{\pi}{3\alpha_0}}$$

Assume by contradiction that this is not the case. So, for all $n$, this maximum is large or equal to $8(1 - \mu) \sqrt{\frac{\pi}{3\alpha_0}}$. Set $t_n > 0$, such that

$$\max \{\Phi(t\omega_n) : t \geq 0\} = \Phi(t_n\omega_n) \geq 8(1 - \mu) \sqrt{\frac{\pi}{3\alpha_0}} \tag{2.1}$$

it is to say, from (2.1) and $(H_3)$

$$8(1 - \mu) \sqrt{\frac{\pi}{3\alpha_0}} \leq \frac{1}{2} \int_{B_R(0)} |\Delta t_n\omega_n|^2 - \frac{\mu}{2} \int_{B_R(0)} \frac{|t_n\omega_n|^2}{|x|^4 \ln^2 R/|x|} - \int_{B_R(0)} F(x, t_n\omega_n)$$

$$\leq \frac{1}{2} t_n^2 - \frac{\mu}{2} (2\pi^2) \int_{R/\sqrt{n}}^R \frac{r^3 \ln^2 R/r}{8\pi^2 r^4 \ln n \ln^2 R/r}$$

$$\leq \frac{1}{2} t_n^2 - \frac{\mu}{8} t_n^2$$

$$= \frac{1}{2} (1 - \frac{1}{4\mu}) t_n^2$$

so it means

$$t_n^2 \geq \frac{16(1 - \mu)}{1 - 1/4\mu} \sqrt{\frac{\pi}{3\alpha_0}} \tag{2.2}$$

At the same time, we know $\frac{d\Phi(t_n\omega_n)}{dt_n} = 0$, then
From (H5), for given $\varepsilon > 0$, there exists $s_\varepsilon > s_\alpha$ such that

$$f(x, s) \geq (\beta - \varepsilon)e^{\alpha_0 s^2}, \quad \forall s > s_\varepsilon$$

so

$$(1 - \frac{1}{4}\mu)t_n^2 \geq (\beta - \varepsilon)\int_{B_R(0)} e^{\alpha_0 t_n^2 s_n^2} \geq (\beta - \varepsilon)(2\pi^2) \int_0^{R/n} r^3 e^{\alpha_0 t_n^2 \ln n \frac{\ln n}{s_n^2}} dr \geq (\beta - \varepsilon)\pi^2 \frac{R^4}{2n^4} e^{\alpha_0 t_n^2 \ln n \frac{\ln n}{s_n^2}} = \frac{1}{2}(\beta - \varepsilon)\pi^2 R^4 e^{\ln n \left(\frac{\alpha_0 t_n^2}{s_n^2} - 4\right)}$$

which implies readily that $t_n$ is bounded. And moreover (2.2) together with (2.4), we can deduce that $t_n^2 \to \frac{32\pi^2}{\alpha_0}$.

Then let us estimate (2.3) more precisely.

$$(1 - \frac{1}{4}\mu)t_n^2 \geq (\beta - \varepsilon)\int_{B_R(0)} e^{\alpha_0 t_n^2 s_n^2} dx$$

Passing to the limit in above and assume $t = \frac{\ln R}{\ln n}$, then we can obtain

$$\frac{8(4 - \mu)\pi^2}{\alpha_0} \geq 2\pi^2(\beta - \varepsilon)\left[\int_0^{R/n} e^{32\pi^2 \ln n \frac{\ln n}{s_n^2}} r^3 dr + \int_{R/n}^R e^{32\pi^2 \ln n \frac{\ln n}{s_n^2}} r^3 dr\right]$$

$$= 2\pi^2(\beta - \varepsilon)[\frac{1}{4} R^4 + R^4 \ln n \int_0^1 e^{4t^2 \ln n - 4t \ln n} dt]$$

$$= \frac{1}{2}\pi^2(\beta - \varepsilon)R^4[1 + 4 \ln n \int_0^1 e^{4\ln n(t^2 - t)} dt]$$

which implies $\beta \leq \frac{16(4-\mu)}{\alpha_0(1+M)R^4}$, if we let $M = 4 \ln n \int_0^1 e^{4\ln n(t^2 - t)} dt[M \text{ see[2]}$, then it is contradiction to (H5).

So, the theorem is proved.

References


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