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# On Transfinite Cardinal Numbers

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***Abstract:*** *For transfinite cardinal number $\alpha$, using Zorn's lemma we have given a simple proof which is understandable to undergraduate students, of the result $\alpha + \alpha = \alpha\alpha = \alpha$, that is, idempotency for addition and multiplication. Moreover for a cardinal number $\beta$ with $2 \le \beta < \alpha$ we obtain easily $\alpha + \beta = \alpha\beta = \alpha$, $\alpha^{\beta} = \alpha^{\alpha} = 2^{\alpha}$, $\alpha^{\beta} < \beta^{\alpha}$. Using these results we get many results directly as*
$\aleph_0 + C = \aleph_0 C = C + C = CC = C$, $\aleph_0^{\aleph_0} = C = C^{\aleph_0} = 2^{\aleph_0}$, $C^C = \aleph_0^{C} = 2^{C}$ *where $\aleph_0$ = card **N**, C = card **R**.*

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## I. Introduction

Let A and B be any two nonempty sets. A *relation* ρ from A to B is defined as a subset of A x B. Thus ρ is a relation from A to B if and only if (abbreviated by *iff*) ρ ⊆ A x B. In particular subset ρ of A x A is called a *relation* (or a *binary relation*)on A. We denote xρy for (x, y) ∈ ρ.

A binary relation ρ on a set A is said to be

(i) *reflexive* if $(a, a) \in \rho\ \forall\ a \in A$.

(ii) *symmetric* if $(a, b) \in \rho \Rightarrow (b, a) \in \rho$.

(iii) *anti-symmetric* if $(a, b) \in \rho$ and $(b, a) \in \rho \Rightarrow a = b$.

(iv) *transitive* if $(a, b) \in \rho$ and $(b, c) \in \rho \Rightarrow (a, c) \in \rho$.

A relation ρ on a set A is called an *equivalence relation* on A if it is simultaneously reflexive, symmetric and transitive on A.

An equivalence relation on a set is usually denoted by ∼ (wiggle or tilde).

Let ∼ be an equivalence relation of a set X and let $a \in X$. The *equivalence class* of $a$, denoted by $[a]$ or $\bar{a}$ is defined as $[a] = \{x \in X \mid x \sim a\}$ = the set of those elements of X which related to $a$ under ∼.

Let X be a nonempty set. A collection of nonempty mutually disjoint subsets of X whose union is X is called a *partition* of the set X.

**1.1 Theorem.** Let ∼ be an equivalence relation in a nonempty set X and $a$ and b be arbitrary elements in X. Then

(i) $a \in [a]$(ii) $b \in [a]$ iff $[a] = [b]$ iff $a \sim b$.

(iii) Any two equivalence classes are either disjoint or identical.

**1.2 Theorem.** (*Fundamental Theorem on Equivalence Relation*): An equivalence relation ∼ on a set X partitions the set X and conversely every partition of X induces an equivalence relation on X.

**Alternative statement:** The distinct equivalence classes of an equivalence relation on X provides us with a decomposition of X as a union of mutually disjoint subsets. Conversely, given a decomposition of X as a union of mutually disjoint, nonempty subsets, we can define an equivalence relation on X for which these subsets are the distinct equivalence classes.

A binary relation ρ on a set X is called a *partial order relation* if it is reflexive,
anti-symmetric and transitive. Generally partial order relation is denoted by ≤ and a set with partial order elation is termed a *partial ordered set* (poset).

Let (P, ≤) be a poset such that any two elements in it are comparable, i. e. $x \le y$ or $y \le x\ \forall\ x, y \in P$. Then '≤' is called linear (or total) order and (P, ≤) is called a *totally* (or *linearly*) *ordered set* or a *chain*.

Let A and B be two sets. It is natural to ask whether both sets contain same number of elements or not. If A and B are finite sets having same number of elements then we say that A and B are equivalent sets and in this case we write A ∼ B.

**Definition:**

We say that the sets A and B are *equivalent* (*equipotent*) if there is a one – one onto function f: A → B. In this case we write A ∼ B.

'∼ ' is an equivalence relation on a family of sets.

A set S is said to be a *finite set* if S = $\boldsymbol{\phi}$, an empty set or S ∼ {1, 2, 3, . . . , n} for some n ∈ N.

A set X is said to be *countable* (*atmost countable*) if either X is finite or X ∼ N = {1, 2, 3, . . .}.

An infinite countable set = enumerable, denumerable.
A set X is *uncountable* means X is not countable (i. e. neither finite nor denumerable).
**1.3 Theorem.** 'Is equivalent to ' is an equivalence relation on a collection of sets.
**Cardinal numbers (*Cardinality*):** The relation 'is equivalent to' is an equivalence relation on a family of sets. Hence by the fundamental theorem of equivalence relations, all sets are partitioned into mutually disjoint distinct classes of equivalent sets.
Above theorems with proofs, definitions are standard and they are also given in books given in references [1] to [5]. Concepts of countable and uncountable sets were introduced by Cantor G. and most of the results on the concepts due to himself.
Let A be any set and let α denote the family of sets which are equivalent to A.
Then α is called a *cardinal number* or simply *cardinal* of A. It is written as α = car (A) or
α = |A| or α = ⋕(A). α = |A| means α represents all sets which are equivalent to A.
Cardinal number is associated with 'measure of size'. There are other definitions given by Frege (1884) and B. Russell (1902) identified the cardinal number |A| of the set A with the set of all sets equivalent to A and J. von Neumann (1988) suggested the selection of a fixed set C from the set of all sets equivalent to A. With any one of these definition one obtains what is essential that an object associated in common with those and only those sets which are equivalent to each other. Cardinality of sets under study are identified sets (equivalent sets) and arranged in order also.
The cardinal number of sets $\phi$, {1}, {1, 2}, {1, 2, 3}, . . . , {1, 2, . . . , n} are denoted by
0, 1, 2, 3, . . . , n respectively and each is a finite cardinal.
The cardinal number of **N** is denoted by $\aleph_0$ (aleph naught) or ɑ. Thus cardinality of any denumerable set is ɑ. So |**Z**| = |**Q**| = |**N**x**N**| = ɑ, since **N**x**N** ~ **N**.
The cardinality of an infinite set is called an *infinite cardinal* or *transfinite cardinal*.
Let a set X be equivalent to the interval [0, 1]. Then X is said to have cardinality C and said to have the power of continuum. For any a < b, each of the interval [a, b], [a, b), (a, b], (a, b) has cardinality C.
The set R has cardinality C and |**R** – **Q**| = C = |[a, ∞)| = |$(-\infty, a)$|, for any a ∈ **R**.

**Result 1:** [3] Every infinite set contains a denumerable set.
Let X be an infinite set.
So it has a X – {$a_1$} as an infinite set, so ∃ $a_2$ ∈ X – {$a_1$} and X – {$a_1$, $a_2$} is an infinite set,
so ∃ $a_3$ ∈ X – {$a_1$, $a_2$}. Continuing in this way we get a denumerable set D = {$a_1$, $a_2$, $a_3$, . . .} ⊆ X.
**Cardinal Arithmetic:** We define addition and multiplication of cardinal numbers α, β.
Let A, B be sets such that α = |A| and β = |B|. We define αβ = |AxB|.
For A∩B = $\phi$, we define α + β = |A∪B|.
These definitions are well defined in the sense that if A ~ A', B ~ B', A∩B = ∅ = A'∩B' gives ∃ bijections f: A ⟶ A', g: B ⟶ B'.
Then $h_1$: A∪B ⟶ A'∪B' given by $h_1(x) = \begin{cases} f(x), x \in A \\ g(x), x \in B \end{cases}$ ∀ x ∈ A∪B is one – one and onto.
∴ A∪B ~ A'∪B', i. e. |A∪B| = |A'∪B'|.
Now $h_2$: AxB ⟶ A'xB' given by $h_2$(x, y) = (f(x), g(y)) ∀ (x, y) ∈ AxB is one – one and onto, and so
AxB ~ A'xB'. Hence |AxB| = |A'xB'|.
Let A, B be sets and α = |A|, β = |B|.Then A ~ Ax{1}, B ~ Bx{2}, and Ax{1}∩Bx{2} = ∅.
Moreover |Ax{1}| = α, |Bx{2}| = β and |Ax{1}∪Bx{2}| = α + β and |AxB| = α β.
**Exponents:** We now define exponents in cardinal numbers. Let A and B be sets and let
α = |A| and β = |B|. Let $B^A$ be the family (set) of all functions from A (exponent) into B.
Then we define $\beta^\alpha = |B^A|$.
**Result 2:** For sets A, B, C with B ⊆ C, we have $B^A \subseteq C^A \Rightarrow |B^A| \le |C^A|$, i. e. $|B|^{|A|} \le |C|^{|A|}$.
Thus if α, β, $\gamma$ are cardinal numbers with β ≤ $\gamma$ then $\beta^\alpha \le \gamma^\alpha$.
Ex. Let A = {*a*, *b*, *c*}, B = {1, 2}. Then |A| = 3, |B| = 2. $B^A$ = family of all functions from A into B = {{(*a*, *1*), (*b*, *1*), (*c*, *1*)}, {(*a*, *1*), (*b*, *1*), (*c*, *2*)}, {(*a*, *1*), (*b*, *2*), (*c*, *1*)}, {(*a*, *2*), (*b*, *1*), (*c*, *1*)}, {(*a*, *1*), (*b*, *2*), (*c*, *2*)}, {(*a*, *2*), (*b*, *1*), (*c*, *2*)}, {(*a*, *2*), (*b*, *2*), (*c*, *1*)}, {(*a*, *2*), (*b*, *2*), (*c*, *2*)}}. $|B^A| = 8 = 2^3 = \beta^\alpha$ where $\beta$ = 2 = |B|, α = 3 = |A|.
**1.4 Theorem**. [2]The operation of addition and multiplication of cardinal numbers are associative and commutative; and multiplication is distributive with respect to addition, and laws of indices also hold,
i. e. if α, β, ϒ are any cardinal numbers then
(a)(i) α + β = β + α, (ii) αβ = βα, (iii) α + (β + ϒ), (iv) α(βϒ) = (αβ)ϒ,
(v) α(β + ϒ) = αβ + αϒ, (b)(i) $\alpha^\beta\alpha^\Upsilon = \alpha^{\beta+\Upsilon}$, (ii) $(\alpha^\beta)^\Upsilon = \alpha^{\beta\Upsilon}$, (iii) $(\alpha\beta)^\Upsilon = \alpha^\Upsilon\beta^\Upsilon$.
**Definition:** [2]Let A and B be sets and α = |A| and β = |B|. We say that α ≤ β iff A is equivalent to a subset of B, that is there is a one – one function f: A ⟶ B. α < β means α ≤ β and α ≠ β.

[$\beta \geq \alpha$ means $\alpha \leq \beta$ and $\beta > \alpha$ means $\alpha < \beta$.]
(i) Let A and B be finite sets with n = |A|, m = |B|. Then n ≤ m as cardinal numbers iff n ≤ m as nonnegative integers. If A is equivalent to a proper subset of B, then n < m.
(ii) For any sets A, B with $A \subseteq B$, we have $|A| \leq |B|$, since f: A ⟶ B given by f(x) = x, x ∈ A, is 1-1.
Now $\mathbf{N} \subseteq \mathbf{R} \Rightarrow \mathfrak{a} \leq \mathfrak{C}$. Since **R** is not denumerable, $\mathfrak{a} \neq \mathfrak{C}$. Therefore $\mathfrak{a} < \mathfrak{C}$.
If A is any infinite set, then A contains a denumerable subset D, i. e. $D \subseteq A$. So $|D| \leq |A|$,
i. e. $\mathfrak{a} \leq |A|$ for any infinite set A.
(iii) For any set A with $|A| = \alpha$, the identity mapping I: A ⟶ A (given by I(x) = x ∀ x ∈ A)
is one – one. Therefore $|A| \leq |A|$, i. e. $\alpha \leq \alpha$ for each cardinal number α. (≤ is reflexive)
(iv) Let A, B, C be sets such that $\alpha = |A|$, $\beta = |B|$, $\gamma = |C|$. $\alpha \leq \beta$ and $\beta \leq \gamma$ ⇒
∃ functions f: A ⟶ B, g: B ⟶ C which are one – one. ⇒ g∘f: A ⟶ C is also one – one. Hence $|A| \leq |C|$, i. e. $\alpha \leq \gamma$. (≤ is transitive)
(v) If α, β, γ are cardinal numbers and $\alpha \leq \beta$ then $\alpha + \gamma \leq \beta + \gamma$ and $\alpha\gamma \leq \beta\gamma$.
Proof are trivial and available in books mentioned in references or in any standard books on Set Theory.
**Continuum Hypothesis:** There is no cardinal number which lies between $\mathfrak{a}$ and $\mathfrak{C}$ ($\mathfrak{a} < \mathfrak{C}$).
In other words, there is no cardinal number β such that $\mathfrak{a} < \beta < \mathfrak{C}$.
In 1963 it was shown that the continuum hypothesis is independent of our axioms of set theory.
The generalized continuum hypothesis: There is no cardinal numbers strictly between α and $2^{\alpha}$ for any transfinite cardinal number α. $\mathfrak{C}^{\mathfrak{a}} = (2^{\mathfrak{a}})^{\mathfrak{a}} = 2^{\mathfrak{a}^2} = 2^{\mathfrak{a}} = \mathfrak{C}$.
$0 < 1 < 2 < 3 < \ldots < \mathfrak{a} < \mathfrak{C} < 2^{\mathfrak{C}} < 2^{2^{\mathfrak{C}}} < \ldots$ in which there are infinitely many cardinal numbers. It is evident that there is only one kind of countable infinity, symbolized by $\mathfrak{a}$ or $\aleph_0$, and beyond this there is an infinite hierarchy of uncountable infinities which are all distinct from one another.
**Note**: We know that if *a* and b are real numbers such that $a \leq b$ and $b \leq a$ then $a = b$. According to Shroeder-Bernstein theorem, this property also holds in case of transfinite cardinal numbers and also for cardinal numbers. Hence a set of cardinal numbers is linearly ordered (a chain).
Following theorem given in one seminar by Cantor as an open problem, and 19 year old Felix Bernstein has settled it, so it referred also as the Cantor–Bernstein theorem.
**Shroeder-Bernstein Theorem**:[2, 3,5] If A and B are sets such that $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$. In other words, if α, β are cardinal numbers such that $\alpha \leq \beta$ and $\beta \leq \alpha$ then $\alpha = \beta$.
['≤' is a partial order relation on a set of cardinals.]
This can be stated in equivalent form as follows: Let X, Y, $X_1$ be sets such that $X \supseteq Y \supseteq X_1$ and $X \sim X_1$ then $X \sim Y$.
[$X \supseteq Y \supseteq X_1$ and $|Y| \leq |X|$ and $|X_1| \leq |Y|$ and $X \sim X_1$, then hypothesis is $|Y| \leq |X|$ and $|X| \leq |Y|$].
***Law of Trichotomy***: If α, β are any two cardinal numbers, then either $\alpha < \beta$ or $\alpha = \beta$ or $\alpha > \beta$.
**Cantor's Theorem:** (i) Let X be a set and $|X| = \alpha$. Then $|P(X)| = 2^{\alpha}$, i. e. $|P(X)| = 2^{|X|}$.
Power set of X is $P(X) = \{S : S \subseteq X\}$ also denoted by $2^X$.
(ii) Cantor's theorem also stated as "$\alpha < 2^{\alpha}$" for any cardinal number α.
**Result**. [2] For $\mathfrak{a} = |N|$, $\mathfrak{C} = |[0, 1]| = |R|$, we can prove:
(i) $\mathfrak{a}\mathfrak{C} = \mathfrak{C}$ (ii) $1 + \mathfrak{C} = \mathfrak{C}$ (iii) $1 + \mathfrak{a} = 1\,\mathfrak{a} = \mathfrak{a} = \mathfrak{a}\mathfrak{a}$ (iv) $\mathfrak{a} + \mathfrak{C} = \mathfrak{C}$ (v) $\mathfrak{C} + \mathfrak{C} = \mathfrak{C}$
(vi) $\mathfrak{a} + \beta = \beta$ for any infinite cardinal number β. (vii) $\mathfrak{C} = 2^{\mathfrak{a}}$ .(viii) $\mathfrak{C}\mathfrak{C} = \mathfrak{C}$.
**Maximal and minimal elements:** Let (P, ≤) be a poset. An element b ∈ P is called a *maximal element* of P iff there is no element in P which strictly dominates b,
i. e. for x ∈ P with $x \geq b \Rightarrow x = b$, i. e. $b \leq x \Rightarrow x = b$. An element *a* ∈ P is called *minimal element* of P iff there is no element in P which is strictly preceeds '*a*' i. e. for x ∈ P, $x \leq a \Rightarrow x = a$.
**Zorn's Lemma:** Let X be a nonempty poset such that every totally ordered subset of X has an upper bound in X. Then X contains a maximal element.
By a partial odering on a family $\mathcal{F}$ of sets, means $(\mathcal{F}, \subseteq)$ is a poset. A chain of sets (*nest* / *tower*) is a family $\mathcal{B}$ of sets such that $A \subseteq B$ or $B \subseteq A$ ∀ A, B ∈ $\mathcal{B}$. M ∈ $\mathcal{F}$ is a maximal element means M is not proper subset of any member of $\mathcal{F}$.

## II. Main Theorem

**2.1 Theorem.** If α is any infinite cardinal, then $\alpha + \alpha = \alpha\alpha = \alpha$ and for any cardinal number β,
$0 < \beta \leq \alpha$, $\alpha + \beta = \alpha\beta = \alpha$.
**Proof:** (i) Let A be any set with infinite cardinal number α, i. e. $\alpha = |A|$.Then
$|Ax\{0\}| = |Ax\{1\}| = \alpha$, so $|Ax\{0, 1\}| = |(Ax\{0\}) \cup (Ax\{1\})| = |Ax\{0\}| + |Ax\{1\}| = \alpha + \alpha$,
($\because (Ax\{0\}) \cap (Ax\{1\}) = \phi$)
Let $\mathcal{F} = \{(B, f) : B \subseteq A$ and f: B ⟶ Bx{0, 1} is a bijection}. Now A is an infinite set, so A has a countable

subset C, say and we have C ~ C x{0, 1} (∵ C x{0, 1} is countable).
Thus ∃ a bijection g: C ⟶ C x{0, 1} and so (C, g) ∈ $\mathcal{F}$. ⇒ $\mathcal{F}$ is a nonempty set (family).
Then ≤ defined for (X, f), (Y, g) ∈ $\mathcal{F}$ by (X, f) ≤ (Y, g) iff g is an extension of f on Y,
i. e. f(x) = g(x) ∀ x ∈ X ⊆ Y. Clearly ≤ is a partial order relation on $\mathcal{F}$, i. e. ($\mathcal{F}$, ≤) is a poset.
Consider any chain $\mathcal{C} = \{(B_j, f_j) \in \mathcal{F} \mid j \in \Delta\}$ in $\mathcal{F}$, where Δ is an ordered index set such that $(B_i, f_i) \le (B_j, f_j)$ if i ≤ j in Δ. Let $D = \bigcup_{j\in\Delta} B_j$. Define h for any x ∈ D by
$h(x) = f_j(x)$ if $x \in B_j$ for some j ∈ Δ and $f_j(x) \in B_j x\{0, 1\} \subseteq \bigcup_{j\in\Delta} B_j x\{0, 1\} = Dx\{0, 1\}$.
As $\mathcal{C}$ is a chain, so h: D ⟶ Dx{0, 1} is a well defined function.
Function h *is one – one*, since x ≠ y in D ⇒ x ≠ y in some $B_j$, j ∈ Δ ⇒ $f_j(x) \ne f_j(y)$, i. e. h(x) ≠ h(y).
To show that h *is onto*: For any (x, n) ∈ Dx{0, 1}, we have x ∈ D, so x ∈ $B_j$ for some j ∈ Δ and
n ∈ {0, 1}, i. e. (x, n) ∈ $B_j$x{0, 1} = codomain of $f_j$ ⇒ x' ∈ $B_j$ such that $f_j$(x') = (x, n), i. e. x' ∈ D
and h(x') = $f_j$(x') = (x, n). So h is onto.
Thus h: D ⟶ Dx{0, 1} is a one – one onto function and D ⊆ A,
showing (D, h) ∈ $\mathcal{F}$and (B, f) ≤ (D, h) ∀ (B, f) ∈ $\mathcal{C}$.
∴ Every chain $\mathcal{C}$ in $\mathcal{F}$ has an upper bound in $\mathcal{F}$. By Zorn's lemma $\mathcal{F}$ has a maximal element (E, k) with k: E ⟶ Ex{0, 1} is a bijection and E ⊆ A. Suppose the subset A – E of A is not finite.
Then A – E has an infinite countable set G and G ~ Gx{0, 1}, since Gx{0, 1} is also an infinite countable set.
So there is a bijection, say q: G ⟶ Gx{0, 1}.
As A∩G = $\phi$, the function p: E∪G ⟶ (E∪G)x{0, 1} defined by $p(x) = \begin{cases} k(x), \text{if } x \in E \\ q(x), \text{if } x \in G, \end{cases}$ is a bijection.
This proves (E, k) ≤ (E∪G, p) ∈ $\mathcal{F}$, E ⊊ E∪G, a contradiction to (E, k) as a maximal element in $\mathcal{F}$.
So supposition 'set A – E is not finite' is wrong. Hence A – E is a finite subset of A where E is an infinite subset of A, and hence
α = |A| = |E|= |Ex{0, 1}| = |(Ex{0})∪(Ex{1})| = |Ex{0}| + |Ex{1}| = α + α.
(ii) Let cardinal number β ≤ α. Then we have α ≤ α + β ≤ α + α = α ⇒ α + β = α.
(iii) Let A be any set with infinite cardinal number α, i. e. α = |A|. Then |AxA| = |A|.|A| = αα = $\alpha^2$.
Let $\mathcal{F}$ = {(B, f) : B ⊆ A and f: B ⟶ BxB is a bijection}. Now A is an infinite set, so A has a countable subset C, say and we have C ~ CxC (∵ CxC is countable). Thus ∃ a bijection g: C ⟶ CxC and so
(C, g) ∈ $\mathcal{F}$. ⇒ $\mathcal{F}$ is a nonempty set (family). Then ≤ defined for (X, f), (Y, g) ∈ $\mathcal{F}$ by
(X, f) ≤ (Y, g) iff g is an extension of f on Y, i. e. f(x) = g(x) ∀ x ∈ X ⊆ Y.
Clearly ≤ is a partial order relation on $\mathcal{F}$, i. e. ($\mathcal{F}$, ≤) is a poset.
Consider any chain $\mathcal{C} = \{(B_j, f_j) \in \mathcal{F} \mid j \in \Delta\}$ in $\mathcal{F}$, where Δ is an ordered index set such that
$(B_i, f_i) \le (B_j, f_j)$ if i ≤ j in Δ. Let $D = \bigcup_{j\in\Delta} B_j$. Define h for any x ∈ D by
$h(x) = f_j(x)$ if $x \in B_j$ for some j ∈ Δ and $f_j(x) \in B_j x\{0, 1\} \subseteq \bigcup_{j\in\Delta} B_j x B_j = DxD$, since $\{B_j x B_j\}_{j\in\Delta}$ is an increasing sequence of sets and $\bigcup_{j\in\Delta} B_j = D$.
As $\mathcal{C}$ is a chain, so h: D ⟶ DxD is a well defined function.
Function h *is one – one*, since x ≠ y in D ⇒ x ≠ y in some $B_j$, j ∈ Δ ⇒ $f_j(x) \ne f_j(y)$, i. e. h(x) ≠ h(y).
To show that h *is onto*: For any (x, y) ∈ DxD, we have x, y ∈ D, so x ∈ $B_i$, y ∈ $B_j$ for some i, j ∈ Δ and
$B_i$ and $B_j$ are comparable sets. Consider $B_i \subseteq B_j$, so x, y ∈ $B_j$ and $f_j$(z) = (x, y) for some z ∈ $B_j$,
since $f_j$: $B_j$ ⟶ $B_j$x$B_j$ is an onto function. Then z ∈ D with h(z) = (x, y) and hence h is onto.
Thus h: D ⟶ DxD is a bijection and D ⊆ A, showing (D, h) ∈ $\mathcal{F}$and (B, f) ≤ (D, h) ∀ (B, f) ∈ $\mathcal{C}$.
∴ Every chain $\mathcal{C}$ in $\mathcal{F}$ has an upper bound in $\mathcal{F}$. By Zorn's lemma $\mathcal{F}$ has a maximal element (E, k)
with k: E ⟶ ExE is a bijection and E ⊆ A and so |E| = |ExE| = |E||E|.
We now prove cardinality of A – E is |A – E| < |E| = β, say (and β = ββ = $\beta^2$).
Suppose |A – E| ≥ β. Then ∃ a subset G of A – E of cardinality β.
Now A∩G = $\phi$ and |(ExG)∪(GxE)∪(GxG)| = |ExG| + |GxE| + |GxG| = $\beta^2 + \beta^2 + \beta^2$ = β + β + β = β by (i),
i. e. G ~ (ExG)∪(GxE)∪(GxG), so ∃ a bijection, say q: G ⟶ (ExG)∪(GxE)∪(GxG) and
(E∪G)x(E∪G) = (ExE)∪ (ExG)∪(GxE)∪(GxG). As A∩G = $\phi$, the sets ExG, GxE, GxG are pairwise disjoint
and the function p: E∪G ⟶ (E∪G)x(E∪G) defined by $p(x) = \begin{cases} k(x), \text{if } x \in E \\ q(x), \text{if } x \in G, \end{cases}$ is a bijection.
This proves (E, k) ≤ (E∪G, p) ∈ $\mathcal{F}$, E ⊊ E∪G, a contradiction to (E, k) as a maximal element in $\mathcal{F}$.
So supposition '|A – E| ≥ β' is wrong. Hence |A – E| < β.
Now β = |E| ≤ α =|A| = |E∪(A – E)| = |E| + |A – E| ≤ β + β = β,
since E ⊆ A, |A – E| ≤ |E| = β.
⇒ α = β, and as ββ = $\beta^2$ = β, we have αα = α for any transfinite cardinal number.
Here it is followed that for any infinite set A, we have A and AxA have same cardinality.
(iv) Let α be a cardinality of an infinite set and β is a cardinal number such that

$1 \leq \beta \leq \alpha$. Then $\alpha = \alpha 1 \leq \alpha\beta \leq \alpha\alpha = \alpha \Rightarrow \alpha\beta = \alpha$.

**2.2 Corollary.** If $\alpha$, $\beta$ are nonzero cardinal numbers and one of them is transfinite then
$\alpha + \beta = \alpha\beta = \max\{\alpha, \beta\}$ and $1\aleph_0 = n\aleph_0 = \aleph_0\aleph_0 = \aleph_0$ and $1\mathfrak{C} = n\mathfrak{C} = \aleph_0\mathfrak{C} = \mathfrak{C}\mathfrak{C} = \mathfrak{C}$.

**Proof:** Any two cardinal numbers are comparable, so we assume $1 \leq \beta \leq \alpha$, and hence $\max\{\alpha, \beta\} = \alpha$, and by theorem 2.1 $\alpha + \beta = \alpha\beta = \alpha = \max\{\alpha, \beta\}$.
Then $1 \leq n < \aleph_0 < \mathfrak{C}$ for any $n \in \mathbf{N}$, gives $1\aleph_0 = n\aleph_0 = \aleph_0\aleph_0 = \aleph_0$ and $1\mathfrak{C} = n\mathfrak{C} = \aleph_0\mathfrak{C} = \mathfrak{C}\mathfrak{C} = \mathfrak{C}$.
Note that $\aleph_0 + \aleph_0 + \aleph_0 + \ldots = \aleph_0$, $\mathfrak{C} + \mathfrak{C} + \mathfrak{C} + \ldots = \mathfrak{C}$ and for any cardinal number $\alpha$;
we have $\alpha^1 = \alpha$, $1^\alpha = 1$.

**2.3 Corollary.** Let $\alpha$, $\beta$ be cardinal numbers with $n \in \mathbf{N}$, $2 \leq n < \beta < \alpha$ and $\alpha$ is transfinite.
Then $\alpha^\beta = \alpha^n = \alpha^2 = \alpha$, $\alpha^\alpha = \beta^\alpha = n^\alpha = 2^\alpha$ and $\alpha^\beta < \beta^\alpha$.

**Proof:** Let $\alpha = \aleph_0$, $2 \leq n \leq \alpha$. Now $2 < 3 < 4 < 5 < 6 < 7 < 8 < \ldots$
$\Rightarrow \mathfrak{C} = 2^{\aleph_0} \leq 3^{\aleph_0} \leq 2^{2\aleph_0} \leq 5^{\aleph_0} \leq 6^{\aleph_0} \leq 7^{\aleph_0} \leq 2^{3\aleph_0} \leq \ldots$ and $n\aleph_0 = \aleph_0 \ \forall\ n \in \mathbf{N}$, $2^{\aleph_0} = \mathfrak{C}$.
Hence $n^\alpha = 2^\alpha = \mathfrak{C}\ \forall$ integer $n \geq 2$, where $\alpha = \aleph_0$. Also $2 < \aleph_0 < \mathfrak{C}$
$\Rightarrow \mathfrak{C} = 2^{\aleph_0} \leq {\aleph_0}^{\aleph_0} \leq \mathfrak{C}^{\aleph_0} = 2^{\aleph_0\aleph_0} = 2^{\aleph_0} = \mathfrak{C}$. Thus ${\aleph_0}^{\aleph_0} = \mathfrak{C}$.
Next consider $\alpha > \aleph_0$, i. e. $\alpha \geq \mathfrak{C}$ and $2 \leq n \leq \beta < \alpha$.
Then $\exists$ infinite cardinal $\gamma$ such that, $\aleph_0 \leq \gamma < \alpha$, $\beta \leq \gamma$ and $\alpha = 2^\gamma$, $2\gamma = n\gamma = \beta\gamma = \gamma$, $\alpha\gamma = \alpha$.
Hence $\alpha^\beta = 2^{\beta\gamma} = 2^\gamma = \alpha$, $\alpha^n = 2^{n\gamma} = 2^\gamma = \alpha = \alpha^2$, by theorem 2.1. Now $\alpha^\alpha = 2^{\alpha\gamma} = 2^\alpha$. As $2 \leq n < \beta < \alpha$
$\Rightarrow 2^\alpha \leq n^\alpha \leq \beta^\alpha \leq \alpha^\alpha = 2^\alpha$. Hence the results $\alpha^\alpha = \beta^\alpha = n^\alpha = 2^\alpha$ and $\alpha^\beta < \beta^\alpha$.
Note that from above corollary: $\aleph_0\aleph_0\aleph_0 \ldots = \mathfrak{C} = 2^{\aleph_0}$, $\mathfrak{C}\mathfrak{C}\mathfrak{C} \ldots = 2^{\mathfrak{C}}$

## III. Conclusions

1. On a set of transfinite cardinals 'addition' and 'multiplication' are commutative, associative, hold laws of indices, and multiplication is distributive over addition.
2. Transfinite cardinals do not satisfy cancellation laws for addition and multiplication, since $\aleph_0 + \aleph_0 = \aleph_0\aleph_0 = 1\aleph_0 = 1 + \aleph_0$, $0 \neq \aleph_0 \neq 1$.
For the positive integers $a, b, c$ we have $b < c \Rightarrow ab < ac$, $b^a < c^a$, but these strict inequalities do not hold for transfinite numbers, since $\aleph_0 < \mathfrak{C}$ but $\aleph_0\mathfrak{C} = \mathfrak{C}\mathfrak{C} = \mathfrak{C}$, ${\aleph_0}^{\mathfrak{C}} = \mathfrak{C}^{\mathfrak{C}} = 2^{\mathfrak{C}}$.
For natural numbers 2, 3, 4 we have $2 < 3 < 4$ and $2^3 < 3^2$, $2^4 = 4^2$; but such results do not hold for transfinite numbers, since we have (by corollary 2.3) for any transfinite numbers $\alpha$, $\beta$ with
$\beta < \alpha \Rightarrow 2^\alpha = \beta^\alpha > \alpha^\beta = \alpha$.
3. For transfinite number $\alpha$ and $n \in \mathbf{N}$, we have $\sum_{i=1}^{n} \alpha = \prod_{k=1}^{n} \alpha = \alpha$ and
$\alpha + \alpha + \alpha + \ldots = \alpha$, $\alpha\alpha\alpha \ldots = 2^\alpha > \alpha$, which is not true in case of any positive number,
since for any $n \in \mathbf{N}$; $n + n + n + \ldots = \aleph_0 > n$ and $n.n.n \ldots = \begin{cases} 1 < 2^1, \text{if } n = 1 \\ \mathfrak{C} > 2^n, \text{if } n \geq 2. \end{cases}$
4. $|\mathbf{N}| = \aleph_0$ is called the countable infinity and any $\alpha > \aleph_0$ is called an uncountable infinity.
For example $\mathfrak{C} = |\mathbf{R}| = |P(\mathbf{N})|$ is an uncountable infinity.
By Cantor's theorem $\aleph_0 = |P(\mathbf{N})| < \mathfrak{C} = |P(P(\mathbf{N}))| < |P(P(\mathbf{R}))| < \ldots$, so there are infinitely many distinct infinities and also there are infinitely many distinct uncountable infinities.
5. Equation $\mathfrak{C} + \alpha = \mathfrak{C}$ has solutions of any finite cardinal number n, $\aleph_0$ and $\mathfrak{C}$.
Equation $\mathfrak{C}\alpha = \mathfrak{C}$ has solutions of finite positive cardinal number n, $\aleph_0$ and $\mathfrak{C}$. These linear equations over cardinal numbers in $\alpha$ have infinitely many solutions in the set of cardinals.
$\aleph_0 + \alpha = \mathfrak{C}$ and $\aleph_0\alpha = \mathfrak{C}$ both have unique solution $\alpha = \mathfrak{C}$ and the equation $\mathfrak{C}\alpha = \aleph_0$ has no solution in the set of cardinals. $\mathfrak{C}^\alpha = \mathfrak{C}$ has solutions as finite positive cardinal numbers and $\aleph_0$.

## References


[1]. Carothers N. L., *Real Analysis*, Cambridge University Press (2000).
[2]. Lipschutz Samour, *Theory and Problems of Set Theory and Related Topics*, Schaum Publishing Co., New York.
[3]. Pervin W. J., *Foundation of General Topology*, Academic Press.
[4]. Sharma J. N., *Topology*, 31st Edition, Krishna Prakashan Media (P) Ltd, Meerut, (2006).
[5]. Stoll R. R., *Set Theory and Logic*, 2nd Indian Reprint, Eureka Publishing House, New Delhi (1976).


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