On Scalar Quasi weak m-power Commutative Algebras

G.Gopalakrishnamoorthy¹, M.Kamaraj² and S.Anitha³
¹Principal,Sri krishnasamy Arts and Science College ,Sattur – 626203, Tamilnadu.
²Dept. of Mathematics,Government Arts and Science College, Sivakasi – 626124.
³Lecturer, Raja Doraisingam Government Arts College, Sivagangai – 630 561, Tamil Nadu.
Corresponding Author: G.Gopalakrishnamoorthy

Abstract: A right near-ring N is called Quasi-weak commutative if xyz = yxz[3].A right near-ring N is called quasi weak m-power commutative if \(x^m y z = y^m xz\) for all \(x,y,z \in N\),where \(m \geq 1\) is a fixed integer [5].An algebra A over a commutative ring R is called scalar quasi-weak commutative if for every \(x,y,z \in A\) there exists \(\alpha = \alpha (x,y,z) \in R\) depending on \(x,y,z\) such that \(xyz = \alpha yxz\). In this paper we generalise the concept of scalar quasi-weak commutative as scalar quasi-weak m-power commutativity and prove many interesting results analogous to our own results[8].

I. Introduction:

Let A be an algebra (not necessarily associative) over a commutative ring R.A is called scalar commutative if for each \(x,y \in A\),there exists \(\alpha \in R\) depending on \(x,y\) such that \(xy = \alpha yx\).Rich[11] proved that if A is scalar commutative over a field F,then A is either commutative or anti-commutative.KOH,LUH and PUTCHA [9] proved that if A is scalar commutative with 1 and if R is a principal ideal domain ,then A is commutative. A near-ring N is said to be weak-commutative if \(xyz = xzy\) for all \(x,y,z \in N\)(Definition 9.4, p.289, Pliz[10]). An algebra A over a commutative ring R is called scalar quasi weak commutative, if for every \(x,y,z \in A\), there exists \(\alpha = \alpha (x,y,z) \in R\) depending on \(x,y,z\) such that \(xyz = \alpha yxz\)[3]. In this paper we define scalar quasi weak m-power commutativity and prove many interesting results analogous to our own results[8].

II. Preliminaries:

In this section we give some basic definitions and well known results which we use in the sequel.

2.1 Definition [10]:
Let N be a near-ring.N is said to be weak commutative if \(xyz = xzy\) for all \(x,y,z \in N\).

2.2 Definition:
Let N be a near-ring.N is said to be anti weak commutative if \(xyz = -xzy\) for all \(x,y,z \in N\).

2.3 Definition [2]:
Let A be an algebra (not necessarily associative) over a commutative ring R.A is called scalar commutative if for each \(x,y \in A\),there exists \(\alpha = \alpha (x,y) \in R\) depending on \(x,y\) such that \(xy = \alpha yx\).A is called scalar anti-commutative if \(xy = -yx\).

2.4 Lemma[5]:
Let N be a distributive near-ring.If \(xyz = \pm xzy\) for all \(x,y,z \in N\),then N is either weak commutative or weak anti-commutative.

3 Main Results:

3.1 Definition
Let A be an algebra (not necessarily associative) over a commutative ring R. A is called an scalar quasi-weak m-power commutative if for every \(x,y,z \in A\),ther exists scalar \(\alpha \in R\) depending on \(x,y,z\) such that \(x^m y z = \alpha y^m xz\).

3.2 Definition
Let A be an algebra (not necessarily associative) over a commutative ring R. A is called an scalar quasi-weak m-power anti-commutative if for every \(x,y,z \in A\),ther exists scalar \(\alpha \in R\) depending on \(x,y,z\) such that \(x^m y z = -\alpha y^m xz\).

3.3 Theorem:
Let A be an algebra (not necessarily associative) over a field F.Let \(m \in \mathbb{Z}^+\).
Let \((x+y)^m = x^m + y^m\) holds for all \(x,y \in A\).Assume \(a^m = \alpha \forall \alpha \in R\).If for each \(x,y,z \in A\),there exists a scalar \(\alpha \in F\) depending on \(x,y,z\) such that \(x^m y z = \alpha y^m x z\) then A is either quasi weak m-power commutative or quasi-weak m-power anti-commutative.
Proof:
Suppose $x^m y z = y^m x z$ for all $x, y, z \in A$, there is nothing to prove. Suppose not, we shall prove that $x^m y z = y^m x z$ for all $x, y, z \in A$.

First we shall prove that if $x^m y z \neq y^m x z$, then $x^{m+1} z = y^{m+1} z = 0$.

So, assume $x^m y z \neq y^m x z$.

Since $A$ is scalar quasi weak $m$-power commutative, there exists $\alpha = \alpha(x, y, z) \in F$ such that

$$x^m y z = \alpha y^m x z.$$  \hfill (1)

Also there exists a scalar $\gamma = \gamma(x, x + y, z) \in F$ such that

$$x^m (x + y) z = \gamma (x + y)^m x z.$$  \hfill (2)

(1) - (2) gives

$$x^m y z - x^{m+1} z = x^m y z - y^m x z = \alpha y^m x z - \gamma (x + y)^m x z.$$  \hfill (3)

Now if $y^m x z \neq 0$, if $y^m x z = 0$, then from (1) we get $x^m y z = 0$ and so $x^m y z = y^m x z$, contradicting our assumption that $x^m y z \neq y^m x z$.

Also $\alpha \neq 1$, if $\gamma = 1$, then from (3) we get $\alpha = \gamma = 1$. Then from (1) we get $x^m y z = y^m x z$, again a contradiction.

Now from (3) we get

$$x^{m+1} z = \frac{y^m - y^m - \alpha y^m x z}{1 - \alpha} y^m x z.$$  \hfill (4)

Similarly $y^{m+1} z = \delta y^m x z$ for some $\delta \in F$. \hfill (5)

Now corresponding to each choice of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in $F$, there is an $\eta \in F$ such that

$$\eta(x^m x + \alpha_1 x^m) = \eta(x^m x + \alpha_3 x^m) \quad \eta(x^m x + \alpha_2 y^m) = \eta(x^m x + \alpha_4 y^m).$$

Since $\alpha_1 = \alpha$ for all $\alpha \in F$, we get

$$\alpha_1 x^m z + \alpha_2 x^m yz + \alpha_3 y^m xz + \alpha_4 y^m x^m z = \eta(x^m x + \alpha x^m) \quad \eta(x^m x + \alpha x^m) = \eta(x^m x + \alpha x^m).$$

In (7) we choose $\alpha_2 = 0, \alpha_3 = \alpha_1 = 1, \alpha_4 = - \beta$.

The right hand side of (7) is zero where as the left hand side of (7) is

$$\beta (a^{-1} - \beta) x^m y z = 0$$

Since $x^m y z \neq 0$ and $\alpha \neq 1$, we get $\beta = 0$.

Hence from (4) we get $x^{m+1} z = 0$.

Also if in (7) we choose $\alpha_3 = 0, \alpha_4 = \alpha_2 = 1$ and $\alpha_1 = - \delta$ the right side of (7) is zero where as the left side of (7) is

$$(\delta + \alpha a^{-1}) x^m y z = 0$$

Since $x^m y z \neq 0$ and $\alpha \neq 1$, we get $\delta = 0$.

Hence from (5) we get $y^{m+1} z = 0$.

Then (6) becomes

$$\alpha_1 x^m z + \alpha_2 x^m yz + \alpha_3 y^m xz = \eta(x^m x + \alpha x^m) \quad \eta(x^m x + \alpha x^m) = \eta(x^m x + \alpha x^m).$$

This is true for any choice of $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in F$.

Choosing $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = - \alpha^{-1}$ we get

$$(1 - (a^{-1})^2) x^m y z = 0.$$  

Since $x^m y z \neq 0, 1 - (a^{-1})^2 \neq 0$.

Hence $(a^{-1})^2 = 1$ i.e., $\alpha = \pm 1$.

Since $\alpha \neq 1$, we get $\alpha = -1$.

i.e., $x^m y z = - y^m x z$ for all $x, y, z \in A$.

i.e., A is either quasi weak m power commutative or quasi-weak m power anti-commutative.
3.4 Note:
Taking m = 1, we get Theorem 3.2[8].

3.5 Lemma:
Let A be an algebra (not necessarily associative) over a commutative ring R. Let m ∈ Z^+.
Suppose A is scalar quasi weak m–power commutative. Then for all x, y, z ∈ A, α ∈ R, α x^m yz = 0 iff
α y^m xz = 0. Also x^m yz = 0 iff y^m xz = 0.

Proof:
Let x, y, z ∈ A and α ∈ R such that α x^m yz = 0. Since A is scalar quasi weak m–power commutative, there exists β = β(y, x, az) ∈ R such that y^m x(az) = β x^m y(az).

i.e., α y^m xz = β α x^m yz = 0.

Conversely assume α y^m xz = 0. Since A is scalar quasi weak m–power commutative, there exists γ = γ(x, y, az) ∈ R such that

\[ x^m y (az) = γ y^m x (az). \]

i.e., α x^m yz = γ α y^m xz = 0.

Thus α x^m yz = 0 iff α y^m xz = 0 \ ∀ α ∈ R.

Now assume x^m yz = 0. Since A is scalar quasi weak m–power commutative, there exists scalar δ(y, x, z) ∈ R such that y^m xz = δ x^m yz = 0.

Conversely assume y^m xz = 0. Then there exists scalar η = η(x, y, z) ∈ R such that x^m yz = η y^m xz = 0.

Then x^m yz = 0 iff y^m xz = 0.

3.6 Note:
Taking m = 1, we get Lemma 3.3[8].

3.7 Lemma:
Let A be an algebra (not necessarily associative) over a commutative ring R. Let m ∈ Z^+.
Suppose (x+y)^m = x^m + y^m for all x, y ∈ A and every element of R is m–potent (i.e., α^m = α \ ∀ α ∈ R).
Let x, y, z, u ∈ A, α, β ∈ R such that x^m u = u^m x, y^m xz = α x^m yz and (y+u)^m = β x^m (y+u)z, then

\[ (x–u – αx – β – x^m \alpha u) = 0. \]

Proof:
Given \( (y+u)^m xz = β x^m (y+u)z \)
\[ y^m xz = α x^m yz \quad \text{→ (1)} \]
and \[ x^m u = u^m x \quad \text{→ (2)} \]
From (1) we get
\[ (y^m + u^m)xz = β x^m (y+u)z \]
\[ (y^m xz + u^m xz) = β x^m (y+u)z \quad \text{→ (4)} \]
\[ \alpha x^m yz + u^m xz = β x^m yz + β x^m uz(\text{using (2)}) \]
\[ \alpha x^m yz + x^m uz = β x^m yz + β x^m uz(\text{using (3)}) \]
\[ x^m (\alpha u – β y – u) = 0 \]

By Lemma 3.5, we get
\[ (\alpha u – β y – u)^m xz = 0 \]
\[ (\alpha u + u^m) – (\beta y^m + (u)^m) xz = 0 \]
\[ (\alpha^m u + u^m) – (β y^m + (u)^m) xz = 0 \]
Since R is m–potent, we get
\[ (\alpha^m u + u^m) – β y^m xz = 0 \]
\[ (\alpha^m xz + u^m xz – β y^m xz) = 0 \]
\[ \alpha^m xz + u^m xz – β y^m xz = 0 \quad \text{→ (5)} \]
From (4) we get
\[ y^m xz – β y^m xz = β x^m uz - u^m xz \]
Multiply by α
\[ \alpha y^m xz – αβ x^m yz = αβ x^m uz - αu^m xz \quad \text{→ (6)} \]

From (5) and (6), we get
\[ αβ x^m uz - α u^m xz + u^m xz - β u^m xz = 0. \]
\[ (αβ x^m uz - α u^m xz + u^m xz - β u^m xz) = 0. \]
i.e., \( (u^m x - α u^m x - β u^m x + αβ x^m uz) = 0. \)
i.e., \( (x^m u - α x^m u - β x^m u + αβ x^m uz) = 0. \)

3.8 Corollary:
Taking \( u = x, \) we get
\[ (x^{m+1} - α x^{m+1} - β x^{m+1} + αβ x^{m+1}) = 0. \]
\[ (x – α x^m) (x – β x^m) = 0. \]
i.e., \( x^{m+1} (x – α x) (x – β x) = 0. \)
3.9 Note:
Taking m = 1, we get Lemma 3.4 [8] and corollary 3.5 [8].

3.10 Theorem:
Let A be an algebra (not necessarily associative) over a commutative ring R. Let m ∈ Z⁺.
Suppose (x+y)ᵐ = xᵐ + yᵐ for all x, y ∈ A and that A has no zero divisors. Assume every element of R is m-potent. If A is scalar quasi weak m-power commutative, then A is quasi weak m-power commutative.

Proof:
Let x, y, z ∈ A.
Since A is scalar quasi weak m-power commutative there exists scalars α = α(x, y, z) ∈ R and
β = β(y+x, x, z) ∈ R
such that

\[(y+x)^m xz = β x^m (y+x) z \quad \rightarrow (1)\]
\[y^m xz = α x^m yz \quad \rightarrow (2)\]

From (1) we get

\[(y+x)^m xz = β x^m yz + β x^{m+1} z\]

(ie)
\[y^m xz + x^{m+1} z = β x^m yz + β x^{m+1} z \quad \rightarrow (3)\]
\[α x^m yz + x^{m+1} z - β x^m yz - β x^{m+1} z = 0 \quad \text{(using (2))}\]
\[x^m (α y x - β y - β x) z = 0\]

By Lemma 3.3 we get
\[(α y x - β y - β x) y^m xz = 0\]
\[(α x^m y + x^{m+1} z - β x^m y - β x^{m+1} z) xz = 0\]
\[(α x^m y + x^{m+1} z - β x^m y - β x^{m+1} z) xz = 0 \quad \text{(since R is m potent)}\]
\[(i.e) \quad α x^m y z + x^{m+1} z - β x^m y z - β x^{m+1} z = 0\]
\[α x^m y z + x^{m+1} z - α β x^m y z = 0 \quad \rightarrow (4) \text{(using (2))}\]

Multiply (3) by α
\[α x^m y z - α β x^m y z + α x^{m+1} z - α β x^{m+1} z = 0 \quad \rightarrow (5)\]

From (4) and (5) we get,

\[x^{m+1} z - β x^{m+1} z - α x^{m+1} z + α β x^{m+1} z = 0\]
\[x^{m+1} (x^2 - α^2 x^2 - β x^2 + α β x^2) z = 0\]
\[x^{m+1} (x - α x)(x - β x) z = 0\]

Since A has no zero divisors, x = 0 or x-αx = 0 (or) x-βx = 0

If x = 0, then y^m xz = y^m xz
If x = αx, then from (2) we get
\[y^m xz = α x^m yz\]
\[α (y^m xz - x^m yz) = 0\]

Since α ≠ 0, y^m xz = x^m yz

If x = βx, then from (3) we get
\[y^m xz + x^{m+1} z = x^m yz + x^{m+1} z\]
\[y^m xz = x^m yz \quad \text{(since β = β^m)}\]

This A is quasi weak m-power commutative.

3.11 Note:
Taking m = 1, we get Lemma 3.6 [8]

3.12 Definition:
Let R be any ring. Let m > 1 be a fixed integer. An element a ∈ R is said to be m-potent if aᵐ = a.

3.13 Lemma:
Let A be an algebra with unity over a P.I.D R. Let m ∈ Z⁺. Assume (x + y)ᵐ = xᵐ + yᵐ
for all x, y ∈ A and that every element of R is m-potent. If A is scalar quasi weak m-power commutative, x ∈ A such that O(xᵐ) = 0, then xᵐ yz = yᵐ xz for all y, z ∈ A.

Proof:
Let x ∈ A such that O(xᵐ) = 0.
Let y, z ∈ A.
Then there exists scalars α = α(y, x, z) ∈ R and β = β(y + x, x, z) ∈ R such that
\[(y + x)^m xz = β x^m (y + x) z \quad \rightarrow (1)\]
\[y^m xz = α x^m yz \quad \rightarrow (2)\]

From (2) we get
\[(y + x)^m xz = β x^m yz + β x^{m+1} z\]
Let A be an algebra with identity over a P.I.D. R. Let \( \text{mez}^s \). Suppose that \( (x + y)^m = x^m + y^m \) for all \( x, y \in A \) and that every element of R is m-potent. Suppose that A is scalar quasi weak m-power commutative. Assume further that there exists a prime \( \text{p} \in \text{R} \) such that \( \text{p}^mA = 0 \). Then A is quasi weak m-power commutative.

**Proof:**

Let \( x, y \in A \) such that \( O(x^n) = \text{p}^k \) for some \( k \in \text{z}^s \).

We prove by induction on \( k \) that \( x^nyu = y^nxu \) for all \( u \in A \).

If \( k = 0 \), then \( O(x^n) = \text{p}^1 \) and so \( y^nx = 0 \).

So \( y^nxu = 0 \) for all \( u \in A \).

By Lemma 3.3 \( x^nyu = 0 \) for all \( u \in A \).

So assume that \( k > 0 \) and that the statements true for all \( 1 < k \).

If \( y^nxu = 0 \) then there is nothing to prove.

So, let \( y^nxu \neq 0 \). Since A is scalar quasi weak m-power commutative, there exists scalars \( \alpha = \alpha(x, y, u) \in R \) and \( \beta = \beta(x, y, x, u) \in R \).

Such that

\[
x^myu = \alpha y^nxu \quad \rightarrow (1)
\]

and

\[
x^m(y + x)u = \beta (y + x)^nu \quad \rightarrow (2)
\]

From (2) we get

\[
x^m(y + x)^nu = \beta (y + x)^nu \quad \rightarrow (3)
\]

\[
x^m u + x^{m+1} u = \beta (y^m + x^m) xu.
\]

i.e.,

\[
x^m u + x^{m+1} u - \beta y^mu + \beta x^{m+1} u = 0
\]

\[
(\alpha - \beta) y^nu = (\beta - 1) x^{m+1} u \quad \rightarrow (4)
\]

If \( (\alpha - \beta) y^nu = 0 \), we get \( (\beta - 1) x^{m+1} u = 0 \).
Since \( x^{m+1} u \neq 0, \beta = 1 \). Hence from (3) we get
\[
x^m y u = y^m x u ,\text{contradicting our assumption that } x^m y u \neq y^m x u .
\]
So \((\alpha - \beta)y^m x u \neq 0\). In particular \(\alpha - \beta \neq 0\).

Let \(\alpha - \beta = p^t \delta\).

For some \(t \in \mathbb{Z}^+\) and \(\delta \in R\) with \((\delta, p) = 1\). If \(t \geq k\), then since \(O(y^m x) = p^k\) we would get \((\alpha - \beta) y^m x u = 0\), again a contradiction.

Hence \(t < k\).

Since \(p^k y^m x u = 0\), by Lemma 3.5 \(p^k x^m y u = 0\).

From (4) we get
\[
p^{k t} (\beta - 1) x^{m+1} u = p^{k t} (\alpha - \beta) y^m x u
\]
\[
= p^{k t} p^t \delta y^m x u
\]
\[
= 0
\]

Let \(O(x^{m+1} u) = p^{t}A\). If \(i < k\), then by induction hypothesis, \(x^m y u = y^m x u\), a contradiction.

So \(i \geq k\).

Now \(p^i | p^i | p^{k t} (\beta - 1)\) and \(p^i | (\beta - 1)\).

Let \(\beta - 1 = p^t \gamma\) for some \(\gamma \in R\).

Then from (4) we get
\[
(\alpha - \beta) y^m x u = (\beta - 1) x^{m+1} u
\]
\[
p^t \delta y^m x u = p^t \gamma x^{m+1} u
\]
\[
p^t (\delta y^m - \gamma x^m) x u = 0.
\]
i.e., \(p^t (\delta y - \gamma x) x^m (x u) = 0\).

Hence by induction hypothesis
\[
(\delta y - \gamma x)^m (x u) w = (x u)^m (\delta y - \gamma x) w\text{ for all } w \in A.
\]

Taking \(u = 1\), we get
\[
(\delta y - \gamma x)^m x w = x^m (\delta y - \gamma x) w
\]
\[
(\delta y^m - \gamma x^m) x w = x^m (\delta y - \gamma x) w
\]
\[
\delta y^m x w - \gamma x^{m+1} w = \delta x^m y w - \gamma x^{m+1} w
\]
\[
\delta (y^m x w - x^m y w) = 0 \quad (6)
\]

Since \((\delta, p) = 1\), there exists \(\mu, \delta \in R\) such that \(\mu p^m + \gamma \delta = 1\).

\[
\therefore \mu p^m (y^m x w - x^m y w) + \gamma \delta (y^m x w - x^m y w) = y^m x w - x^m y w
\]
\[
0 + 0 = y^m x w - x^m y w \quad (\because p^m A = 0 \text{ and } (6))
\]
\[
\therefore y^m x w - x^m y w = 0 \quad \forall \text{ } w \in A.
\]

Hence the Lemma.

\textbf{References:}


\[
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