Estimates in the Operator Norm

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Abstract: In this paper, we will obtain estimates of the distance between the q-k-eigenvalues of two q-k-normal matrices A and B in terms of $\|A - B\|$. Apart from the optimal matching distances, $s(L, M)$ and $h(L, M)$, Note that $s(L, M)$ is the smallest number $\delta$ such that every element of $L$ is within a distance $\delta$ of some element of $M$; and $h(L, M)$ is the smallest number $\delta$ for which this as well as the symmetric assertion with $L$ and $M$ interchanged, is true.

We will use the notation $\sigma(A)$ for both the subset of the quaternion plane that consists of all the q-k-eigenvalues on $n \times n$ matrix $A$, and for the unordered $n$-tuple whose entries are the q-k-eigenvalues of $A$ counted with multiplicity. Since we will be taking of the distances $s(\sigma(A), \sigma(B)), h(\sigma(A), \sigma(B))$, and $d(\sigma(A), \sigma(B))$, it will be clear which of the two objects is being represented by $\sigma(A)$.

I. Introduction

In this paper, we will obtain estimates of the distance between the q-k-eigenvalues of two q-k-normal matrices A and B in terms of $\|A - B\|$. Apart from the optimal matching distances $s(L, M)$ and $h(L, M)$. Note that $s(L, M)$ is the smallest number $\delta$ such that every element of $L$ is within a distance $\delta$ of some element of $M$; and $h(L, M)$ is the smallest number $\delta$ for which this as well as the symmetric assertion with $L$ and $M$ interchanged, is true.

We will use the notation $\sigma(A)$ for both the subset of the quaternion plane that consists of all the q-k-eigenvalues on $n \times n$ matrix $A$, and for the unordered $n$-tuple whose entries are the q-k-eigenvalues of $A$ counted with multiplicity. Since we will be taking of the distances $s(\sigma(A), \sigma(B)), h(\sigma(A), \sigma(B))$, and $d(\sigma(A), \sigma(B))$, it will be clear which of the two objects is being represented by $\sigma(A)$.

II. Definitions And Some Theorems

Definition 2.1:
If $L, M$ are closed subsets of a quaternion space $H_n$, let
$s(L, M) = \sup_{\lambda \in L} \text{dist}(\lambda, M) = \sup_{\lambda \in L} \inf_{\mu \in M} |\lambda - \mu|$

Definition 2.2:
The Hausdorff distance between $L$ and $M$ is defined as
$h(L, M) = \max(s(L, M), s(M, L))$

Definition 2.3:
The $d(\sigma(A), \sigma(B))$ is defined as $d(\sigma(A), \sigma(B)) \leq \|A - B\|$ if either $A$ and $B$ are both q-k-Hermitian or one is q-k-Hermitian and other q-k-Skew-Hermitian.

Theorem 2.4:
Let $A$ be q-k-normal and $B$ an arbitrary matrix of same order of $A$. Then
$s(\sigma(B), \sigma(A)) \leq \|A - B\|$
Proof:
Let $\delta = \|A - B\|$. For proving the theorem, we have to show that if $\beta$ is any q-k-eigenvalues of $B$, then $\beta$ is within a distance $\delta$ of some q-k-eigenvalue $\alpha_j$ of $A$.

By applying a translation, we assume that $\beta = 0$. If none of the $\alpha_j$ is within a distance $\delta$ of this, then $A^{-1}$ exists.

Since $A$ is q-k-normal.
Therefore, \( \|A^{-1}\| = \max_{1 \leq j \leq n} \left| \alpha_j \right| < \frac{1}{\delta} \).

Hence, \( \|A^{-1}(B - A)\| \leq \|A^{-1}\| \|B - A\| < \frac{1}{\delta} \delta = 1 \).

Since \( B = A(I + A^{-1}(B - A)) \), this shows that \( B \) is invertible. Then but \( B \) could not have a zero q-k-eigenvalue. Hence proved.

**Corollary 2.5:**

If \( A \) and \( B \) are \( n \times n \) q-k-normal matrices then \( h(\sigma(A), \sigma(B)) \leq \|A - B\| \).

**Proof:**

Since \( A \) and \( B \) are q-k-normal matrices of order \( n \times n \).

Let \( \sigma(A) \) and \( \sigma(B) \) be set of all q-k-eigenvalues of \( A \) and \( B \) respectively.

\[
s(\sigma(A), \sigma(B)) \leq \|A - B\| \tag{1}
\]

and \( h(\sigma(A), \sigma(B)) = \max(s(\sigma(A), \sigma(B)), s(\sigma(B), \sigma(A))) \).

From these two, one can conclude that \( h(\sigma(A), \sigma(B)) \leq \|A - B\| \).

**Remark 2.6:**

For \( n = 2 \), the corollary 2.5 will lead to \( d(\sigma(A), \sigma(B)) \leq \|A - B\| \).

**Theorem 2.7:**

For any two q-k-unitary matrices \( d(\sigma(A), \sigma(B)) \leq \|A - B\| \).

**Proof:**

The proof will use the marriage theorem and above. Let \( \{\lambda_1, \lambda_2, ..., \lambda_n\} \) and \( \{\mu_1, \mu_2, ..., \mu_n\} \) be the q-k-eigenvalues of \( A \) and \( B \) respectively.

Let \( \Lambda \) be any subset of \( \{\lambda_1, \lambda_2, ..., \lambda_n\} \).

Let \( \mu(\Lambda) = \left\{ \mu_j : \left| \mu_j - \lambda_j \right| \leq \delta \text{ and } \lambda_j \in \Lambda \right\} \).

By the marriage theorem, the assertion would be proved if we show that \( |\mu(\Lambda)| \geq |\Lambda| \).

Let \( I(\Lambda) \) be the set at all points on the unit ball \( T \) that are within distance of some point of \( \Lambda \). Then \( \mu(\Lambda) \) contains exactly those \( \mu_j \) that lie in \( I(\Lambda) \).

Let \( I(\Lambda) \) be written as a disjoint union of arcs \( I_1, ..., I_r \). For each \( k; k < r \), let \( J_k \) be the arc contained in \( I_k \) all whose points at least distance from the boundary of \( I_k \) then \( I_k = (J_k)_k \).

We have \( \sum_{k=1}^r m_A(J_k) \leq \sum_{k=1}^r m_B(I_k) = m_B(I(\Lambda)) \)

But all the elements of \( \Lambda \) are in some \( J_k \).

\( \Rightarrow |\Lambda| \leq |\mu(\Lambda)| \)

Similarly for, \( \mu \) is a subset of \( \{\mu_1, \mu_2, ..., \mu_n\} \).

\[
|\mu| \leq |\Lambda(\mu)| \leq |\Lambda| - |\mu| \leq |\Lambda - \mu| \leq |\Lambda(\mu) - \mu(\Lambda)|
\]
That is, \[ d(A, B) \leq \max_{1 \leq j \leq n} |\lambda_j - \mu_{\sigma(j)}| \] (2)

Hence proved.

**Remark 2.8:**

There is one difference between theorem 2.7 and most of our earlier results of this type. Now nothing is said about the order in which the q-k-eigenvalues of \( A \) and \( B \) are arranged for the optimal matching. No canonical order can be prescribed in general.

**Theorem 2.9:**

Let \( A \) and \( B \) be q-k-normal matrices with q-k-eigenvalues \( \{\lambda_1, \lambda_2, ..., \lambda_n\} \) and \( \{\mu_1, \mu_2, ..., \mu_n\} \) respectively. Then there exists a permutation \( \sigma \) such that

\[ \|A - B\| \leq \sqrt{2} \max_{1 \leq j \leq n} |\lambda_j - \mu_{\sigma(j)}| \] (2)

**Proof:**

Since \( A \) and \( B \) are q-k-normal matrices. So \( A \otimes I \) and \( I \otimes B \) are both q-k-normal and commute with each other. Hence \( A \otimes I - I \otimes B \) is q-k-normal. The q-k-eigenvalues of this matrix are all the differences \( \lambda_j - \mu_j; 1 \leq i, j \leq n \)

Hence \( \|A \otimes I - I \otimes B\| = \max_{i, j} |\lambda_i - \mu_j| \)

Since q-k-eigenvalues of \( B \) are q-k-eigenvalues of \( B^T \).

So \( \|A \otimes I - I \otimes B^T\| = \max_{i, j} |\lambda_i - \mu_j| \)

\[ \Rightarrow \|A - B\| = \|A \otimes I - I \otimes B^T\| \leq \sqrt{2} \|A \otimes I - I \otimes B\| \]

This is equivalent to (2)

Therefore, \( \|A - B\| \leq \sqrt{2} \max_{1 \leq j \leq n} |\lambda_j - \mu_{\sigma(j)}| \)

Hence proved.

**Remark 2.10:**

This is, in fact, true for all \( A, B \) and is proved below.

**Theorem 2.11:**

For all quaternion matrices \( A, B \) \( \|A - B\| \leq 2 \|A \otimes I - I \otimes B^T\| \) (3)

**Proof:**

We have to prove that for all \( x, y \) in \( H_n \)

\[ \|x, (A - B)y\| \leq \sqrt{2} \|A \otimes I - I \otimes B^T\| \|x\| \|y\| \]

Now, \[ \|x, (A - B)y\| = \|x^*y - x^*By\| \]

\[ = \|x^*Ay - x^*By\| \]

\[ = |tr(Ayx^* - yx^*B)| \]

\[ \leq \|Ayx^* - yx^*B\| \]

This matrix \( Ayx^* - yx^*B \) has rank atmost 2. So, \( \|Ayx^* - yx^*B\| \leq \sqrt{2} \|Ayx^* - yx^*B\|_2 \).
Let $\overline{x}$ be the vector whose components are the conjugates of the components of $x$. Then with respect to the standard basis $e_i \otimes e_j$ of $H_n \otimes H_n$, $(i, j)$-coordinate of the vector $(A \otimes I) (y \otimes \overline{x})$ is $\sum_k a_{ik}y_k \overline{x}_j$.

This is also $(i, j)$-entry of the matrix $Ay\overline{x}^*$. In the same way, the $(i, j)$-entry of $yx^*B$ is the $(i, j)$-coordinate of the vector $(I \otimes B^T) (y \otimes \overline{x})$.

Thus we have,

$$\|Ay\overline{x}^* - yx^*B\|_2 \leq \left\| (A \otimes I - I \otimes B^T) (y \otimes \overline{x}) \right\|_2$$

$$\leq \left\| A \otimes I - I \otimes B^T \right\|_2 \left\| y \otimes \overline{x} \right\|_2$$

$$= \left\| A \otimes I - I \otimes B^T \right\|_2 \|x\| \|\overline{x}\|$$

Hence proved.

References