Study of Limit Cycle for Fitzhugh-Nagumo System

Ali E. M. Saeed
Department of Mathematics, Alzaem Alazhari University, Sudan 1

Corresponding Author: Ali E. M. Saeed

Abstract: In the present study we have investigated the complete Fitzhugh-Nagumo system with \( I \neq 0 \): we have shown that one or two limit cycles may bifurcate at the origin. Bendixon's theorem has been used in our study to prove non-existence of limit cycles. We have also proved that the system has unique limit cycle through change of the parameters.

Keywords: Limit cycle, Liénard equation, Hopf-bifurcation, Fitzhugh-Nagumo system.

I. Introduction

In the present paper, we revisit the problem of bifurcation of limit cycles. We give criterion for the study model (Fitzhugh-Nagumo system) to have or not to have limit cycles with \( I \neq 0 \): We also demonstrate that the model exhibits a Hopf-bifurcation. Now we consider the following Liénard equation

\[
\dot{x} + f(x)\dot{x} + g(x) = 0.
\]

The above equation may be written into two dimensional autonomous dynamical system

\[
\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y.
\]

In Liénard plane above equations becomes

\[
\dot{x} = y - F(x), \quad \dot{y} = -g(x) \tag{1.1}
\]

where \( F(x) = \int_0^x f(t)dt \).

The main part of this paper is devoted to explain the existence and uniqueness of limit cycles of Fitzhugh-Nagumo system which is expressed through following differential system

\[
\dot{x} = y - A(x - B)(x - \lambda) + I; \quad \dot{y} = -\epsilon(x - \delta y); \tag{1.2}
\]

This system has been extensively studied with particular emphasis on bifurcation of limit cycles as well as in model been of certain phenomenon. Literature review indicates that, most of the articles studies the system taking some parameters as zeros, for instance see (Mattias 2006, Nikola & Dragana 2003, Enno 2006),(Arnaud 2002, Rabinovitich & Friedman 2009, Romel et al 2001 and Baili 2004)(LuoDingjun, et al 1997) investigated the particular case of taking \((1 + \lambda) = 0\); and proved the uniqueness of limit cycle. In (Ringkvist & Zhou 2009) there is a general analysis of the system for bifurcation of limit cycles from Hopf bifurcation. In this paper, we study the system (1.2) with all parameters not zeros and prove the uniqueness of limit cycle. The paper is organized as follows.

In section 2, we prosed the main system equations when all parameters are not zero. The sufficient conditions that the system has at least two limit cycles are shown by using Hopf-bifurcation methods.

The case of saddle point with limit cycle is presented, theorems and lemmas in section 4 along with the concluding remarks.

II. Main system equation

In this section, we investigate the Fitzhugh-Nagumo system with the parameters \( A, B, \delta, \epsilon, \lambda \) and \( I \) being not zeros. In particular, we study the system under the case \( B = 1 \) and \( \lambda \neq 1; \) where \( \delta \in (-1; 0); \) if \( \delta \neq 0 \).

In order to study the existence and non-existence of limit cycles we make change of variables to get Liénard type (1.1). Let \( x \rightarrow \infty \rightarrow x \) and \( y + \delta \epsilon x + \frac{n}{\delta} \rightarrow y \)

where \( x \) is the root of equilibrium equation \( ax^3 - \delta(1 + \lambda)x^2 - (\delta \lambda - 1)x - I\delta = 0 \).

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Then the system becomes,
\[ \begin{align*}
\dot{x} &= y - [x^3 + (3\alpha - (1 + \lambda))x^2 + (3\alpha^2 - 2(1 + \lambda)\alpha + \lambda + \delta \epsilon)x] \\
y &= -\delta \epsilon \left[ x^3 + (3\alpha - (1 + \lambda))x^2 + \left(3\alpha^2 - 2(1 + \lambda)\alpha + \lambda + \frac{1}{\delta}\right)x \right]
\end{align*} \quad (2.1) \]

We note that \( F(0) = 0; \ g(0) = 0 \): The other two roots of \( F(x) = 0 \) and \( g(x) = 0 \) respectively are
\[ \begin{align*}
x &= \frac{1}{2} \left[ -(3\alpha - (1 + \lambda)) \pm \sqrt{A^2 - 4\left(3\alpha^2 - 2(1 + \lambda)\alpha + \lambda + \frac{1}{\delta}\right)} \right] \\
y &= \frac{1}{2} \left[ -(3\alpha - (1 + \lambda)) \pm \sqrt{A^2 - 4\left(3\alpha^2 - 2(1 + \lambda)\alpha + \lambda - \frac{1}{\delta}\right)} \right]
\end{align*} \quad (2.2) \]
\[ \begin{align*}
w &= \frac{\delta \epsilon (3\alpha^2 - 2(1 + \lambda)\alpha + \lambda + \frac{1}{\delta})}{\sqrt{\bar{\mu}}} \\
w_2 &= \frac{\delta \epsilon \left[ 2A^2 - 3\left(3\alpha^2 - 2(1 + \lambda)\alpha + \lambda - \frac{1}{\delta}\right) \right]}{8\mu^2 \sqrt{\bar{\mu}}} \\
w_3 &= -\frac{15c_\delta \epsilon}{\mu^3 \sqrt{\bar{\mu}}}
\end{align*} \quad (2.5) \]

The system has unique singular point for \( (3\alpha - (1 + \lambda))^2 - 4(3\alpha^2 - 2(1 + \lambda)\alpha + \lambda - \frac{1}{\delta}) < 0 \), and for \( (3\alpha - (1 + \lambda))^2 - 4(3\alpha^2 - 2(1 + \lambda)\alpha + \lambda - \frac{1}{\delta}) > 0 \), we have three singular points.

### 2.1 Brief Note On Anti-Saddle Bifurcation Case

Let us consider the system (2.1) in the case \( O \) as anti-saddle i.e
\[ \delta \epsilon \left(3\alpha^2 - 2(1 + \lambda)\alpha + \lambda - \frac{1}{\delta}\right) > 0, \quad (2.4) \]

Since \( \delta \epsilon > 0 \) then \( 3\alpha^2 - 2(1 + \lambda)\alpha + \lambda - \frac{1}{\delta} \) must be greater than zero. Then we have \( \lambda - \frac{1}{\delta} > 0 \) and 
\[ (1 + \lambda)^2 - 3\left(\lambda - \frac{1}{\delta}\right) < 0. \]

The first three focal values are [8]:
\[ W_1 = \frac{3\alpha^2 - 2(1 + \lambda)\alpha + \lambda + \delta \epsilon}{\sqrt{\bar{\mu}}} \]
\[ W_2 = \frac{\delta \epsilon \left[ 2A^2 - 3\left(3\alpha^2 - 2(1 + \lambda)\alpha + \lambda - \frac{1}{\delta}\right) \right]}{8\mu^2 \sqrt{\bar{\mu}}} \]
\[ W_3 = -\frac{15c_\delta \epsilon}{\mu^3 \sqrt{\bar{\mu}}} \]

From Lemma 3.3.2 in [8] \( O \) is unstable (stable) strong focus when \( W_1 > 0 \) \( (W_1 < 0) \), unstable (stable) weak focus of order one when \( W_2 > 0 \) \( (W_2 < 0) \) and unstable (stable) weak focus of order two when \( W_3 > 0 \) \( (W_3 < 0) \). Thus, from Hopf bifurcation one stable limit cycle appears in the case \( W_1 > 0 \) \( \text{and} \) \( W_2 < 0 \). Therefore, \( F(x) \) has three critical points and the system has only one singular point \( g(0) = 0 \). To prove the uniqueness of limit cycle we can apply the following lemma:

**Lemma 2.1** [8]

Suppose that system (1.1) satisfies the following conditions:

1. There exist \( c_1 < p_1 < c_2 < q_1 < 0 < q_2 < p_2 < c_2 \) such that \( p_1, 0, p_2 \) are zero points of \( F(x), c_1, 0, c_2 \) are zero points of \( g(x), xg(x) > 0 \) \( \forall x \in (c_1, 0), (0, c_2) \) and \( q_1, q_2 \) are zero points of \( f(x), f(x) < 0 \) \( \forall x \in (q_1, q_2) \).

2. If the simultaneous equations

\[ F(u) = F(v); \ G(u) = G(v); \ c_1 < u < v < c_2 \]

have no solution \( (u, v) \), then system (1.1) has no closed orbit in the strip \( \{c_1 < x < c_2; -\infty < y < +\infty\} \) or if it has at most one solution and the function \( f(x)g(x) \) is monotonically decreasing (increasing) in \( x \in (c_1, p_2) \) or \( x \in (p_2, c_2) \); then (1.1) has at most one limit cycle in the strip \( \{c_1 < x < c_2; y \in (-\infty, \infty)\} \) and it is stable (unstable) if it exists.

where \( G(x) = \int_0^x g(x)dx. \)

**Lemma 2.2**

For \( (\lambda - \frac{1}{\delta})^2 + \frac{3}{4} - 3\delta \epsilon < 0 \) the system (2.1) has no limit cycles.

Proof

Considering the equation
\[ \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 3x^2 + 2(3\alpha - (1 + \lambda))x + (3\alpha^2 - 2(1 + \lambda)\alpha + \lambda + \delta \epsilon), \quad (2.6) \]
we define
\[ N(x) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 3x^2 + 2(3\alpha - (1 + \lambda))x \\
+ (3\alpha^2 - 2(1 + \lambda)\alpha + \lambda + \delta\varepsilon), \]
(2.7)

The discriminant of above polynomial root is
\[ \Delta = (1 + \lambda)^2 - 3(\lambda + \delta\varepsilon) = \left(\lambda + \frac{1}{2}\right)^2 - \frac{3}{4} - 3\delta\varepsilon. \]

In the lemma we have \(\Delta < 0\) and since \((1 + \lambda) \neq 0\) implies that \(N(x)\) isdefinite in sign and non-zero, then by Bendeixsons theorem [4], we concludefthat there are no limit cycles.

**Lemma 2.3**

If \(W_1 = 0\); then \(W_2\) not identical zero.

**Proof** Let \(W_1 = 0\); then we have \(3\alpha^2 - 2(1 + \lambda)\alpha + \lambda + \delta\varepsilon = 0\) substitute thisequation in the thevalue of \(W_2\); then we get
\[ W_2 = 2(1 + \lambda)^2 - 3(\lambda + \delta\varepsilon) - 3\left(\lambda - \frac{1}{\delta}\right) = 2\lambda^2 - 2\lambda + 2 - 3\left(\delta\varepsilon - \frac{1}{\delta}\right). \]

we define
\[ H(\lambda) = 2\lambda^2 - 2\lambda + 2 - 3\left(\delta\varepsilon - \frac{1}{\delta}\right). \]

Since \(\delta \in (-1; 0), \) so \(2 - 3\left(\delta\varepsilon - \frac{1}{\delta}\right) < 0\). All limit cycles would lie in one of thecases \(H(\lambda) < 0\) or \(H(\lambda) > 0\). For \(H(\lambda) < 0\) we have \(W_2 < 0\) and \(W_1 > 0\), and for \(H(\lambda) > 0\) we have \(W_2 > 0\) and \(W_1 < 0\). Thus, \(H(\lambda)\) cannot be identicalet zero and we havethe following lemma.

**Lemma 2.4**

For \(W_1 > 0\) and \(W_2 < 0\) the system (3) has at least one limit cycle surrounding \(O\).

**Proof** First let \(W_1 = 0\) when \(W_2 < 0\) then \(O\) is stable focus of order one, and when \(W_1\) increasing from zero one stable limit cycle appear surrounding \(O\), by Hopf-bifurcation.

From above lemma we have
\[ \alpha < \frac{1}{3}\sqrt{(1 + \lambda)^2 - 3(\lambda - \delta\varepsilon)} + 1 + \lambda = \rho_1 \]
and
\[ \alpha < \frac{1}{3}\sqrt{3(\lambda - \delta\varepsilon) - (1 + \lambda)^2} + 1 + \lambda = \rho_2 \]

Let \(\rho = \max(\rho_1, \rho_2)\), then we have the following theorem.

**Theorem 2.5** For \(\alpha < \frac{1}{3}\sqrt{\rho} + 1 + \lambda\), the system (2.1) has unique limitcycle.

**Proof** Now we apply lemma 2.1. From the condition of \(W_2 > 0\) we see that \(g(x)\) has only one zero point \(g(0) = 0\), then we can easily find that \(xg(x) > 0\). From the condition \(W_1\) we find that \(F(x)\) has three zeros, therefor \(f(x)\) has twozeros. And also from the case \(3\alpha^2 - 2(1 + \lambda)\alpha + \lambda + \delta\varepsilon < 0\) we deduce that \(f(x) < 0 \)for \(x \in (q_1, q_2)\) and \(f(x) > 0\) otherwise. So condition (1) satisfied. From the local position of \(f(x)\) and the case that \(xg(x) > 0\) we get \(\frac{f(x)}{g(x)}\) increasing for all \(x > p_2\). So we just need to prove the condition of the simultaneous equations \(F(u) = F(v); G(u) = G(v)\). After simplify and by putting \(s = u + v\) and \(r = uv\) we get,
\[ h(s) = \frac{1}{4}s^3 - \frac{1}{2}(3\alpha\alpha - 2(1 + \lambda))s^2 + \left(\frac{1}{3}(3\alpha - (1 + \lambda)^2 + \frac{1}{2}\delta\varepsilon - \frac{11}{2}\delta)\right)s \]
\[ + \frac{1}{3}(3\alpha - (1 + \lambda))(3\alpha^2 - 2(1 + \lambda)\alpha + \lambda + \delta\varepsilon). \]

Since \(h(s)\) with odd degree, then \(h(s)\) has at most one solution, so for prove the only solution let consider discriminant of \(h(s)\),
\[ h'(s) = \frac{3}{4}s^2 - (3\alpha\alpha - 2(1 + \lambda))s + \left(\frac{1}{3}(3\alpha - (1 + \lambda)^2 + \frac{1}{2}\delta\varepsilon - \frac{11}{2}\delta)\right), \]

it is easy to find \(\Delta = -\frac{3}{2}\left(\delta\varepsilon - \frac{1}{\delta}\right) < 0\).

And hence the theorem has been proved.
Remark 6 An inequality $\alpha < \frac{1}{3} \sqrt{p} + 1 + \lambda$ equivalent to $H(\lambda) < 0$.

2.2 A saddle bifurcation case

In this case we have

$$\delta \varepsilon \left(3\alpha^2 - 2(1 + \lambda)\alpha + \lambda + \frac{1}{\delta}\right) < 0,$$

we discuss saddle bifurcation in the case

$$\delta \varepsilon < 0 \text{ and } (3\alpha^2 - 2(1 + \lambda)\alpha + \lambda + \frac{1}{\delta}) > 0,$$

From these situations, we can find that the discriminant of the roots of $g(x)$

$$\Delta = \left(3\alpha - (1 + \lambda)\right)^2 - 4 \left(3\alpha^2 - 2(1 + \lambda)\alpha + \lambda + \frac{1}{\delta}\right) < 0.$$

Thus, the system (2.1) has unique singular point which is hyperbolic saddle at the origin, and therefore no limit cycle is possible. Thus we get the following result.

Theorem 2.7 For $\delta \varepsilon < 0$ and $(3\alpha^2 - 2(1 + \lambda)\alpha + \lambda - 1/\delta) > 0$ the system (2.3) has no limit cycles.

2.3 The existence of two limit cycles

Theorem 2.8 For $\delta \varepsilon > 0$ and $H(\lambda) > 0$ the system (2.1) has at least two limit cycles.

Proof Since from the system as $W_1 = W_3 = 0$, $O$ is stable weak focus of order two. Initially keep $W_1 = 0$ and let $W_2$ increases from zero, then one stable limit cycle $L_1$ bifurcates. Then, change $W_1$ to the negative such that $L_1$ does not disappear but $O$ change its stability again and unstable limit cycle $L_2$ bifurcates in the interior of $L_1$: Hence, the conclusion is obtained.

III. A special case of $\delta = 0$

In this case the system after $B = 1$ becomes

$$\begin{align*}
\dot{x} &= y - Ax(x - 1)(x - \lambda) + 1 \\
\dot{y} &= \varepsilon x.
\end{align*}$$

(3.1)

This system has unique singular point $(0; 1)$, so by putting $y = y + 1$; then we have the following Lifenard type

$$\begin{align*}
\dot{x} &= y - \left[Ax^3 - (1 + \lambda)x^2 + \lambda x\right] \\
\dot{y} &= \varepsilon x
\end{align*}$$

(3.2)

this system has the origin as unique singular point. The Jacobian is given by

$$f(0, 0) = \begin{bmatrix} \lambda & 1 \\ \varepsilon & 0 \end{bmatrix}$$

thendet($f$) = $-\varepsilon$; for $\varepsilon > 0$ $O$ is saddle and no limit cycle possible. Thus, for existence of limit cycles we must consider $\varepsilon < 0$:

Lemma 3.1

$A$ and $\lambda$ are both rotated parameters of system (3.1)

Proof Denote the right hand sides by $P$ and $Q$ respectively. Then we have

$$\begin{align*}
\frac{\partial Q}{\partial A} - \frac{\partial P}{\partial \lambda} &= -\varepsilon x^4 \geq 0 \text{ and } \frac{\partial Q}{\partial \lambda} - \frac{\partial P}{\partial A} = -\varepsilon x^2 \geq 0.
\end{align*}$$

The first three focal values (3.10) are see [8]

$$W_1 = \frac{-\lambda}{\sqrt{-\varepsilon}}, W_2 = \frac{3\varepsilon A}{8\varepsilon \sqrt{-\varepsilon}}, W_3 = 0.$$

For $\lambda \neq 0$ $O$ is strong focus stable(unstable) if $\lambda > 0(\lambda < 0)$ for $\lambda = 0$ then $O$ weak focus of order one stable(unstable) if $A > 0(A < 0)$ thus, we get the following results.

Lemma 3.2

To create limit cycles of system (3.2) we have $\lambda A \neq 0$.

Proof Consider the case $\lambda = 0$; then for $A = 0$ system has $O$ as center, and for $A \neq 0$ no limit cycle from rotated vector field. Similarly the case $A = 0$:

Lemma 3.3

If $\lambda < 0$, $A > 0(\lambda > 0, A < 0)$ stable (unstable) limit cycle for system (3.2) surrounding $O$ appears via a Hopf-bifurcation.
Remark 3.4  
From above lemma we deduced that a limit cycle can appear just in the two cases \( \lambda A \neq 0 \) and \( A \lambda < 0 \).

Lemma 3.5 [1,5]  
Let \( f(x) \) and \( g(x) \) be continuously differentiable functions for \( k_1 < x < k_2 \) where \( k_1 < 0 < k_2 \) such that for \( k_1 < x < k_2 \) the following conditions are satisfied:

1. \( g(x) > 0(< 0) \) for \( x > 0(< 0) \);
2. there exist \( x_0 \) such that \( f(x_0) = 0 \) and \( f(x) > 0(< 0) \) for \( x > 0(< 0) \);
3. \( f(x) \) is an increasing function both for \( x < 0 \) for \( x > x_0 \).

Then the Liénard system has at most one periodic orbit, and if exist it must be a limit cycle with negative characteristic exponent.

Theorem 3.6  
For \( \lambda < 0 \) and \( A > 0 \) system (3.2) has unique stable limit cycle.

Proof  
Now we apply lemma 3.5, its easily to see that \( f(x) \) and \( g(x) \) are continuously differentiable functions. Since \( \epsilon < 0 \) then condition (1) holds.

For second condition consider \( f(x) = 3A \lambda x - 2(1 + \lambda)x + A \Delta = 4(1 + \lambda)^2 - 12A \lambda > 0 \) so \( f(x) \) has two singular points \( x_1 < 0 < x_2 \); and since \( \lambda < 0 \) then we deduce that condition (2) satisfied. For third one let

\[
\frac{f(x)}{g(x)} = \frac{3A \lambda x - 2(1 + \lambda)x + A \Delta}{\Delta} = \frac{A \lambda x^2 + x \Delta}{\epsilon x^2} > 0 \quad \text{for all } x.
\]

Thus condition (3) holds and the theorem is proved.

IV. A saddle case with limit cycle

In this section we study the saddle case with the following quadratic system

\[
\begin{align*}
\dot{x} &= y + A(\lambda + 1)x^2 - \lambda Ax + I
\end{align*}
\]

\[
\begin{align*}
\dot{y} &= \epsilon(x - \delta y).
\end{align*}
\]

where \( \epsilon \in (0, 1), I \in R, (\lambda + 1) < 0 \).

For studying limit cycles, we may transform the system to the following Lienard system

\[
\begin{align*}
\dot{x} &= y - [-(\lambda + 1)x^2 + (-(\lambda + 1)\alpha + \lambda + \delta \epsilon)x]
\end{align*}
\]

\[
\begin{align*}
\dot{y} &= -\delta \epsilon \left[-(\lambda + 1)x^2 + (-(\lambda + 1)\alpha + \lambda - \frac{1}{\delta})x\right].
\end{align*}
\]

The system has two critical points with \( O \) as saddle, and \( C(c; 0) \) is an antisaddle such that

\[
c = \frac{-2(\lambda + 1)\alpha + \lambda - \frac{1}{\delta}}{(\lambda + 1)}
\]

Since \( O(0; 0) \) as saddle, and \( \delta \epsilon > 0 \) then we have \(-2(\lambda + 1)\alpha + \lambda - \frac{1}{\delta} < 0\), to study the existence of limit cycles we translate \( C \) to the origin, then we can find

\[
\begin{align*}
\dot{x} &= y - \left[-(\lambda + 1)x^2 + (2(\lambda + 1)\alpha - \lambda + \delta \epsilon + \frac{2}{\delta})x\right]
\end{align*}
\]

\[
\begin{align*}
\dot{y} &= -\delta \epsilon \left((\lambda + 1)x^2 - 2(\lambda + 1)\alpha + \lambda - \frac{1}{\delta})x\right).
\end{align*}
\]

Change \( (x, t) \) to \((-x, -t)\) then we get

\[
\begin{align*}
\dot{x} &= y - \left[-(\lambda + 1)x^2 + (2(\lambda + 1)\alpha - \lambda + \delta \epsilon + \frac{2}{\delta})x\right]
\end{align*}
\]

\[
\begin{align*}
\dot{y} &= -\delta \epsilon \left((\lambda + 1)x^2 - 2(\lambda + 1)\alpha + \lambda - \frac{1}{\delta})x\right).
\end{align*}
\]

The critical points of \( F(x) \) and \( g(x) \) respectively are

\[
O(0, 0); x_1 = \frac{-2(\lambda + 1)\alpha - \lambda + \delta \epsilon + \frac{2}{\delta}}{(\lambda + 1)}
\]

\[
O(0, 0); x_2 = \frac{2(\lambda + 1)\alpha + \lambda - \frac{1}{\delta}}{(\lambda + 1)}
\]
The first three focal values are [8]:

\[
W_1 = \frac{2(\lambda + 1)\alpha - \lambda + \delta \epsilon + \frac{\delta}{\delta}}{\lambda + 1}, \\
W_2 = -\frac{(\lambda + 1)^2 \alpha + \lambda - \frac{1}{\delta}}{8\mu^2 \sqrt{\mu}}, \\
W_3 = -0.
\]

Since \(W_2 < 0\) so for creating limit cycles \(W_1\) must be positive, for instance see[6]. Thus, for \(W > 0\) and from Hopf-bifurcation the system has stable limit cycle. Thus, for this situation we deduce that \(0 < x_1 < x_2\) and \(f(x)\) has minimum value and \(g(x)\) has maximum value. For uniqueness of limit cycle we got the following theorem;

**Theorem 4.1**  The system (4.12) has unique limit cycle.

**Proof**  Now we apply Lemma 3.5 for system (4.15), since \(f(x)\) and \(g(x)\) are polynomials function, then these functions are continuously differentiable, and easily to see that conditions 1 and 2 holds. For the third one and after simplifying the following theorem:

\[
\frac{f(x)}{g(x)} = \frac{-2(\lambda + 1)\alpha}{\delta \epsilon (\lambda + 1)x^2 + bx} - \frac{1}{(\lambda + 1)x^2 + bx}
\]

It's enough to prove that \(N(x) > 0, N(0) = -ab > 0\) and \(\Delta = 4\alpha(\lambda + 1)^2(a + 2b) = (--) = -\). Thus \(x < 0\) and \(N(x) > 0 \forall x\). Hence the theorem is proved.

**Concluding Remarks**

A complete FitzHugh-Nagumo system with \(I \neq 0\) is studied and analyzed in detail by adapting Hopf-bifurcation theory. It was shown that one or twolimit cycles bifurcates from the origin. Bendixon's theorem is used to prove nonexistence of limit cycles. Also we proved that the system has unique limit cycle under some change of parameters.

**References**