**g* γ-open functions and g* γ-closed functions in topology**

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**Abstract:** In this paper, we define and study g* γ-open functions and g* γ-closed functions and their various allied forms via g* γ-open sets due to Navalagi et al. (2018). Also, we define and study the concepts of g* γ-normal spaces.

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**Key words:** γ-closed sets, g* γ-closed sets, g* γ-open sets, g* γ-closed functions, g* γ-open functions, almost g* γ-irresolute functions, g* γ-normal space.

I. Introduction

In 1996, D. Andrijevic[2] defined and studied the concepts of b-open sets in topological spaces. b-open sets are also called as sp-open sets. Later in 1997, A. A. El-Atik[6], has introduced and studied the concept of γ-open sets in topology. It is known that b-open sets or sp-open sets are same as γ-open sets. In 2007, E. Ekici[8] has defined and studied the concept of γ-normal spaces in topology and concepts of g γ-closed sets and γ g-closed sets. In [15], author have defined and studied the concepts of g* γ-closed sets, g* γ-continuous functions, g* γ-irresolute functions and g* γ-R₀ spaces, g* γ-R₁ spaces in topology. The purpose of this paper is to define and study the concepts of g* γ-open functions, g* γ-closed functions, almost g* γ-irresolute functions, (g* γp) – openfunctions, (g* γp) – closed functions, (g* γs) – open functions and g* γ-normal spaces.

II. Preliminaries

In this paper (X, τ) and (Y, σ) (or X and Y) we always mean topological spaces on which no separation axioms are assumed. Unless otherwise mentioned. For a subset of X, Cl(A) and Int(A) represent the closure of A and the interior of A respectively.

The following definitions and results are useful in the sequel:

**Definition 2.1:** Let X be a topological space. A subset A is called :
(i)semiopen[10] if A ⊂ Cl(Int(A)),
(ii)preopen[12] if A ⊂ Int(Cl(A)),
(iii)b-open[2] or sp-open[1] or γ-open[6] if A ⊂ Cl(Int(A)) ∪ Int(Cl(A)).

The complement of semiopen (resp. peropen, b-open or sp-open or γ-open) set is called semiclosed[5] (resp. preclosed[12], b-closed[2] or sp-closed[1] or γ-closed[6]).

The family of all semiopen (resp. preopen, b-open or sp-open or γ-open) sets of a space X is denoted by SO(X)(resp. PO(X), BO(X), SPO(X) or γO(X)). And the family of all semiclosed(resp. preclosed, b-closed sp-closed or γ-closed) sets of a space X is denoted by SF(X)(resp. PF(X), BF(X) or SPF(X) or γF(X)).

**Definition 2.2:** Let A be a subset of a space X, then semi-interior[5](resp. pre-interior[13], semipre-interior[1], γ-interior[6]) of A is the union of all semiopen(resp. preopen, semipreopen, γ-open) sets contained in A and is denoted by sInt(A) (resp. pInt(A), spInt(A), γInt(A)).

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Definition 2.3: Let A be a subset of a space X, then the intersection of all semi-closed (resp. preclosed, semipre-closed, γ-closed) sets containing A is called semiclosure[5] (resp. preclosure[7], semipreclosure[1], γ-closure[9]) of A and is denoted by sCl(A) (resp. pCl(A), spCl(A), γCl(A)).

Definition 2.4: A subset A of a space X is said to be γ-open[8] if γCl(A) ⊂ U whenever A ⊂ U and U ∈ τ. The complement of γ-open set is said to be γ-open.

Definition 2.5: A subset A of a space X is said to be γ-g-closed[11] if γCl(A) ⊂ U whenever A ⊂ U and U ∈ γO(X).

The complement of γ-open set is said to be γ-open.

The definitions of be γ-open set and γ-g-closed set respectively, defined by E. Ekici[8] and El-Maghrabi[11] are the same.

Definition 2.6: A space X is said to be γ-normal[8], if for any pair of disjoint closed sets A and B, there exist disjoint γ-open sets U and V such that A ⊂ U and B ⊂ V.

Definition 2.7: A subset A of a space X is called γ-open[15] set if Cl(A) ⊂ U whenever A ⊂ U and U is γ-open set in X.

Definition 2.8: A subset A of a space X is called γ-open[15] set if F ⊂ Int(A) whenever F ⊂ A and F is γ-closed set in X.

The family of all γ-open sets in topological space X is denoted by γO(X) and that of, the family of all γ-open sets in topological space X is denoted by γF(X). And the family of all γ-open sets containing a point x of X will be denoted by γO(X,x).

Definition 2.9: Let A be a subset of a space X, then the intersection of all γ-open sets containing A is called the γ-open closure[15] of A and is denoted by γCl(A).

Definition 2.10: Let A be a subset of a space X, then the union of all γ-open sets contained in A is called the γ-interior[15] of A and is denoted by γInt(A).

Definition 2.11: A set U ⊂ X is said to be γ-neighbourhood[16](in brief, γ-nbd) of a point x ∈ X if and only if there exists A ∈ γO(x) such that A ⊂ U.

Definition 2.12: A function f: X → Y is called presemiopen[4](resp. presemiclosed[9]), if the image of each semiopen(resp. semiclosed) set of X is semiopen(resp. semiclosed) set in Y.

Definition 2.13: A function f: X → Y is called presemipreopen[14] (resp. presemipreclosed[14]), if the image of each semipreopen(resp. semipreclosed) set of X is semipreopen(resp. semipreclosed) set in Y.

Definition 2.14: A function f: X → Y is called M-preopen[13](resp. M-preclosed[13]), if the image of each preopen(resp. preclosed) set of X is preopen(resp. preclosed) set in Y.

Definition 2.15: A function f: X → Y is called semiopen[3](resp. preopen[13], semipreopen[14]), if the image of each open set of X is semiopen(resp. preopen, semipreopen) set in Y.

Definition 2.16: A function f: X → Y is called semiclosed[17](resp. preclosed[7]).
semiopen([13,14]), if the image of each open set of X is semiclosed(resp. preclosed, semipreclosed) set in Y.

**Definition 2.17:** A function $f: X \to Y$ is said to be strongly $g^* \gamma$-closed[15] if the image of each $g^* \gamma$-closed set of $X$ is closed set in $Y$.

**Definition 2.18:** A function $f: X \to Y$ is said to be always $g^* \gamma$-open[15] (resp. always $g^* \gamma$-closed[15]), if the image of each $g^* \gamma$-open(resp. $g^* \gamma$-closed) set of $X$ is $g^* \gamma$-open(resp. $g^* \gamma$-closed) set in $Y$.

**III. $g^* \gamma$-open functions and $g^* \gamma$-closed functions**

We recall the following:

**Definition 3.1:** A function $f: X \to Y$ is said to be $g^* \gamma$-open[1] if the image of open set of $X$ is $g^* \gamma$-open in $Y$.

We define the following:

**Definition 3.2:** A function $f: X \to Y$ is said to be $g^* \gamma$-closed if the image of closed set of $X$ is $g^* \gamma$-closed set in $Y$.

**Definition 3.3:** A function $f: X \to Y$ is said to be almost $g^* \gamma$-irresolute if for each $x$ in $X$ and each $g^* \gamma$-neighbourhood $V$ of $f(x)$, $g^* \gamma Cl(f^{-1}(V))$ is a $g^* \gamma$-neighbourhood of $x$.

We have the following characterizations:

**Lemma 3.4:** For a function $f: X \to Y$, the following are equivalent:

(i) $f$ is almost $g^* \gamma$-irresolute

(ii) $f^{-1}(V) \subseteq g^* \gamma Int(g \gamma Cl(f^{-1}(V)))$ for every $V \in g^* \gamma O(Y)$

**Proof:** Obvious.

**Theorem 3.5:** A function $f: X \to Y$ is strongly $g^* \gamma$-closed if and only if for each subset $A$ of $Y$ and for each $g^* \gamma$-open set $U$ in $X$ containing $f^{-1}(A)$, there exists a $g^* \gamma$-open set $V$ containing $A$ such that $f^{-1}(V) \subseteq U$.

**Proof:** Suppose that $f$ is strongly $g^* \gamma$-closed. Let $A$ be a subset of $Y$ and $U \in g^* \gamma O(X)$ containing $f^{-1}(A)$. Put $V = Y \setminus f(X \setminus U)$, then $V$ is a $g^* \gamma$-open set of $Y$ such that $A \subseteq V$ and $f^{-1}(V) \subseteq U$.

Conversely, let $K$ be any $g^* \gamma$-closed set of $X$. Then $f^{-1}(Y \setminus f(K)) \subseteq X \setminus K$ and $X \setminus K \in g^* \gamma O(X)$. There exists a $g^* \gamma$-open set $V$ of $Y$ such that $Y \setminus f(K) \subseteq V$ and $f^{-1}(V) \subseteq X \setminus K$. Therefore, we have $Y \setminus V \subseteq f(K)$ and $K \subseteq f^{-1}(Y \setminus V)$. Hence, we obtain $f(K) = Y \setminus V$ and $f(K)$ is $g^* \gamma$-closed set in $Y$. This shows that $f$ is strongly $g^* \gamma$-closed function.

**Theorem 3.6:** A function $f: X \to Y$ is almost $g^* \gamma$-irresolute if and only if $f(g^* \gamma Cl(U)) \subseteq g^* \gamma Cl(f(U))$ for every $U \in g^* \gamma O(U)$.

**Proof:** Let $U \in g^* \gamma O(X)$. Suppose $y \notin g^* \gamma Cl(f(U))$. Then there exists $V \in g^* \gamma O(Y, y)$ such that $V \cap f(U) = \emptyset$. Hence, $f^{-1}(V) \cap U = \emptyset$. Since $U \in g^* \gamma O(X)$, we have $g^* \gamma Cl(f^{-1}(V)) \subseteq g^* \gamma Cl(U) = \emptyset$. Then by Lemma 3.4, $f^{-1}(V) \cap g^* \gamma Cl(U) = \emptyset$ and hence $V \cap f(g^* \gamma Cl(U)) = \emptyset$. This implies that $y \notin f(g^* \gamma Cl(U))$. Hence $f(g^* \gamma Cl(U)) \subseteq g^* \gamma Cl(f(U))$.

Conversely, if $V \in g^* \gamma O(Y)$, then $M = X \setminus g^* \gamma Cl(f^{-1}(V)) \in g^* \gamma O(X)$. By hypothesis,
Some decompositions on $g^\gamma$-open functions and $g^\gamma$-closed functions:

We define the following:

**Definition 3.7:** A function $f : X \to Y$ is said to be $g^\gamma$-pre-open (in brief, $(g^\gamma)p$-open) if the image of each $g^\gamma$-open set of $X$ is preopen in $Y$.

**Definition 3.8:** A function $f : X \to Y$ is said to be $g^\gamma$-pre-closed (in brief, $(g^\gamma)p$-closed) if the image of each $g^\gamma$-closed set of $X$ is preclosed in $Y$.

**Definition 3.9:** A function $f : X \to Y$ is said to be $g^\gamma$-semi-open (in brief, $(g^\gamma)s$-open) if the image of each $g^\gamma$-open set of $X$ is semipreopen in $Y$.

**Definition 3.10:** A function $f : X \to Y$ is said to be $g^\gamma$-semi-closed (in brief, $(g^\gamma)s$-closed) if the image of each $g^\gamma$-closed set of $X$ is semipreclosed in $Y$.

**Definition 3.11:** A function $f : X \to Y$ is said to be $g^\gamma$-semipre-open (in brief, $(g^\gamma)sp$-open) if the image of each $g^\gamma$-open set of $X$ is semipreopen in $Y$.

**Definition 3.12:** A function $f : X \to Y$ is said to be $g^\gamma$-semipre-closed (in brief, $(g^\gamma)sp$-closed) if the image of each $g^\gamma$-closed set of $X$ is semipreclosed in $Y$.

Now we have the following decompositions

**Theorem 3.13:** Let $f : X \to Y$ and $g : Y \to Z$ be two functions. The following statements are valid:

(i) If $f$ is $(g^\gamma)p$-open and $g$ is M-preopen then $g \circ f$ is $(g^\gamma)p$-open function.
(ii) If $f$ is $(g^\gamma)s$-open and $g$ is presemiopen then $g \circ f$ is $(g^\gamma)s$-open function.
(iii) If $f$ is $(g^\gamma)sp$-open and $g$ is presemiopen then $g \circ f$ is $(g^\gamma)sp$-open function.

**Proof:** (i) Let $V$ be any $g^\gamma$-open set in $X$. Since $f$ is $(g^\gamma)p$-open function, $g(V)$ is preopen in $Y$. Again, $g$ is M-preopen function and $g(V)$ is preopen set in $Y$, then $g(f(V)) = (g \circ f)(V)$ is preopen in $Z$. This shows that $g \circ f$ is $(g^\gamma)p$-open function.

(ii) Obvious.

(iii) Obvious.

**Theorem 3.14:** Let $f : X \to Y$ and $g : Y \to Z$ be two functions. The following statements are valid:

(i) If $f$ is $g^\gamma$-open and $g$ is $(g^\gamma)s$-open then $g \circ f$ is semipreopen function.
(ii) If $f$ is $g^\gamma$-open and $g$ is $(g^\gamma)p$-open then $g \circ f$ is preopen function.
(iii) If $f$ is $g^\gamma$-open and $g$ is $(g^\gamma)sp$-open then $g \circ f$ is semipreclosed function.
Proof: (i) Let $V$ be any open set in $X$. Since $f$ is $g^*\gamma$-open function, $g(V)$ is $g^*\gamma$-open set in $Y$. Again, $g$ is $(g^*\gamma s)$–open function and $g(V)$ is $g^*\gamma$-open set in $Y$, then $g(f(V))=(g\circ f)(V)$ is semiopen set in $Z$. Thus $g\circ f$ is semiopen function.

(ii) Obvious.

(iii) Obvious.

Theorem 3.15: Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two functions. The following statements are valid:

(i) If $f$ is $(g^*\gamma s)$–closed and $g$ is presemiclosed then $g\circ f$ is $(g^*\gamma s)$–closed function.
(ii) If $f$ is $(g^*\gamma p)$–closed and $g$ is $M$-preclosed then $g\circ f$ is $(g^*\gamma p)$–closed function.
(iii) If $f$ is $(g^*\gamma sp)$–closed and $g$ is presemipreclosed then $g\circ f$ is $(g^*\gamma sp)$–closed function.

Proof: (i) Let $V$ be any $g^*\gamma$-closed set in $X$. Since $f$ is $(g^*\gamma s)$–closed function, $g(V)$ is $g^*\gamma$-closed set in $Y$. Again, $g$ is presemiclosed function and $g(V)$ is semiclosed set in $Y$, then $g(f(V))=(g\circ f)(V)$ is semiclosed in $Z$. This shows that $g\circ f$ is $(g^*\gamma s)$–closed function.

(ii) Obvious.

(iii) Obvious.

Theorem 3.16: Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two functions. The following statements are valid:

(i) If $f$ is $g^*\gamma$-closed and $g$ is $(g^*\gamma sp)$–closed then $g\circ f$ is semipreclosed function.
(ii) If $f$ is $g^*\gamma$-closed and $g$ is $(g^*\gamma p)$–closed then $g\circ f$ is preclosed function.
(iii) If $f$ is $g^*\gamma$-closed and $g$ is $(g^*\gamma s)$–closed then $g\circ f$ is semiclosed function.

Proof: (i) Let $V$ be any closed set in $X$. Since $f$ is $g^*\gamma$-closed function, $g(V)$ is $g^*\gamma$-closed set in $Y$. Again, $g$ is $(g^*\gamma sp)$–closed function and $g(V)$ is $g^*\gamma$-closed set in $Y$, then $g(f(V))=(g\circ f)(V)$ is semiclosed in $Z$. This shows that $g\circ f$ is $(g^*\gamma s)$–closed function.

(ii) Obvious.

(iii) Obvious.

Now, we define the following:

Definition 3.17: A function $f : X \rightarrow Y$ is said to be pre-$g^*\gamma$-open (in brief, $(p, g^*\gamma)$–open) if the image of each preopen set of $X$ is $g^*\gamma$-open set in $Y$.

Definition 3.18: A function $f : X \rightarrow Y$ is said to be pre-$g^*\gamma$-closed (in brief, $(p, g^*\gamma)$–closed) if the image of each preclosed set of $X$ is $g^*\gamma$-closed set in $Y$.

Definition 3.19: A function $f : X \rightarrow Y$ is said to be semi-$g^*\gamma$-open (in brief, $(s, g^*\gamma)$–open) if the image of each semipopen set of $X$ is $g^*\gamma$-open set in $Y$.

Definition 3.20: A function $f : X \rightarrow Y$ is said to be semi-$g^*\gamma$-closed (in brief, $(s, g^*\gamma)$–closed) if the image of each semiclosed set of $X$ is $g^*\gamma$-closed set in $Y$.
Definition 3.21: A function \( f : X \rightarrow Y \) is said to be semipre-\( g^* \gamma \)-open (in brief, (sp, \( g^* \gamma \))-open) if the image of each semipreopen set of \( X \) is \( g^* \gamma \)-open set in \( Y \).

Definition 3.22: A function \( f : X \rightarrow Y \) is said to be semipre-\( g^* \gamma \)-closed (in brief, (sp, \( g^* \gamma \))-closed) if the image of each semipreclosed set of \( X \) is \( g^* \gamma \)-closed set in \( Y \).

We have the following characterizations:

Lemma 3.23: A function \( f : X \rightarrow Y \) is (sp, \( g^* \gamma \))-closed if and only if \( \text{spCl}(f(A)) \subseteq f(g^* \gamma \text{Cl}(A)) \) for every subset \( A \) of \( X \).

Proof: Assume \( f \) is (sp, \( g^* \gamma \))-closed and \( A \) be any arbitrary subset of \( X \). Then \( \text{spCl}(A) \) is an semipreclosed set and hence \( f(\text{spCl}(A)) \) is \( g^* \gamma \)-closed set in \( Y \) and so \( \text{spCl}(f(A)) \subseteq f(g^* \gamma \text{Cl}(A)) \).

Conversely, if \( A \) is semipreclosed in \( X \) and by hypothesis, \( \text{spCl}(f(A)) \subseteq f(g^* \gamma \text{Cl}(A)) = f(A) \). \( f(A) = g^* \gamma \text{Cl}(f(A)) \) which implies that \( f \) is (sp, \( g^* \gamma \))-closed function.

Theorem 3.24: If a function \( f : X \rightarrow Y \) is (sp, \( g^* \gamma \))-closed then for each subset \( B \) of \( Y \) and semipreopen set \( V \) of \( X \) containing \( f^{-1}(B) \), there exists an \( g^* \gamma \)-open set \( U \) in \( Y \) containing \( B \), such that \( f(U) \subseteq V \).

The routine proof of this theorem is omitted.

Lemma 3.25: A function \( f : X \rightarrow Y \) is (s, \( g^* \gamma \))-closed if and only if \( \text{sCl}(f(A)) \subseteq f(g^* \gamma \text{Cl}(A)) \) for every subset \( A \) of \( X \).

Proof is similar to the proof of lemma 3.23.

Theorem 3.26: If a function \( f : X \rightarrow Y \) is (s, \( g^* \gamma \))-closed then for each subset \( B \) of \( Y \) and semipopen set \( V \) of \( X \) containing \( f^{-1}(B) \), there exists an \( g^* \gamma \)-open set \( U \) in \( Y \) containing \( B \), such that \( f(U) \subseteq V \).

Proof of this theorem is easy and hence omitted.

Lemma 3.27: A function \( f : X \rightarrow Y \) is (p, \( g^* \gamma \))-closed if and only if \( \text{pCl}(f(A)) \subseteq f(g^* \gamma \text{Cl}(A)) \) for every subset \( A \) of \( X \).

Proof is similar to the proof of lemma 3.23.

Theorem 3.28: If a function \( f : X \rightarrow Y \) is (p, \( g^* \gamma \))-closed then for each subset \( B \) of \( Y \) and preopen set \( V \) of \( X \) containing \( f^{-1}(B) \), there exists an \( g^* \gamma \)-open set \( U \) in \( Y \) containing \( B \), such that \( f(U) \subseteq V \).

Proof of this theorem is omitted.

Now we have the following decompositions:

Theorem 3.29: Let \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) be two functions. The following statements are valid:

DOI: 10.9790/5728-1404010108 www.iosrjournals.org 6 | Page
(i) If \( f \) is presemiopen and \( g \) is \((sp, g^* \gamma)\)-open then \( g \circ f \) is \((sp, g^* \gamma)\)-open function.

(ii) If \( f \) is presemiopen and \( g \) is \((s, g^* \gamma)\)-open then \( g \circ f \) is \((s, g^* \gamma)\)-open function.

(iii) If \( f \) is \( M \)-preopen and \( g \) is \((p, g^* \gamma)\)-open then \( g \circ f \) is \((p, g^* \gamma)\)-open function.

**Proof:** (i) Let \( V \) be any semipreopen set in \( X \). Since \( f \) is presemiopen function, \( g(V) \) is semipreopen set in \( Y \). Again, \( g \) is \((sp, g^* \gamma)\)-open function and \( g(V) \) is semipreopen set in \( Y \), then \( g(f(V))=(g \circ f)(V) \) is \( g^* \gamma \)-open set in \( Z \). Thus \( g \circ f \) is \((sp, g^* \gamma)\)-open function.

(ii) Obvious.

(iii) Obvious.

**Theorem 3.30:** Let \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) be two functions. The following statements are valid:

(i) If \( f \) is presemiclosed and \( g \) is \((s, g^* \gamma)\)-closed then \( g \circ f \) is \((s, g^* \gamma)\)-closed function.

(ii) If \( f \) is \( M \)-preclosed and \( g \) is \((p, g^* \gamma)\)-closed then \( g \circ f \) is \((p, g^* \gamma)\)-closed function.

(iii) If \( f \) is presemipreclosed and \( g \) is \((sp, g^* \gamma)\)-closed then \( g \circ f \) is \((sp, g^* \gamma)\)-closed function.

**Proof:** (i) Let \( V \) be any semiclosed set in \( X \). Since \( f \) is presemiclosed function, \( g(V) \) is semiclosed set in \( Y \). Again, \( g \) is \((sp, g^* \gamma)\)-closed function and \( g(V) \) is semiclosed set in \( Y \), then \( g(f(V))=(g \circ f)(V) \) is \( g^* \gamma \)-closed set in \( Z \). Thus \( g \circ f \) is \((sp, g^* \gamma)\)-closed function.

(ii) Obvious.

(iii) Obvious.

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**Definition 3.31:** A space \( X \) is said to be \( g^* \gamma \)-normal, if for any pair of disjoint closed sets \( A \) and \( B \) of \( X \), there exist disjoint \( g^* \gamma \)-open sets \( U \) and \( V \) such that \( A \subseteq U \) and \( B \subseteq V \).

**Remark 3.32:** Every \( \gamma \)-normal space is \( g^* \gamma \)-normal space.

**Characterizations of \( g^* \gamma \)-normal space:**

**Theorem 3.33:** For a space \( X \), the following are equivalent:

(i) \( X \) is \( g^* \gamma \)-normal.

(ii) For every pair of open sets \( U \) and \( V \) whose union is \( X \), there exists a \( g^* \gamma \)-closed sets \( A \) and \( B \) such that \( A \subseteq U \) and \( B \subseteq V \) and \( A \cup B = X \).

(iii) For every closed set \( H \) and every open set \( K \) containing \( H \), there exists a \( g^* \gamma \)-closed set \( U \) such that \( H \subseteq U \subseteq g^* \gamma \text{Cl}(U) \subseteq K \).

**Proof:** (i) \( \Rightarrow \) (ii) Let \( U \) and \( V \) be a pair of open sets in a \( g^* \gamma \)-normal space \( X \) such that \( X = U \cup V \). Then \( X \setminus U \) and \( X \setminus V \) are disjoint closed sets. Since \( X \) is \( g^* \gamma \)-normal, there exists disjoint \( g^* \gamma \)-open sets \( U_1 \) and \( V_1 \) such that \( X \setminus U \subseteq U_1 \) and \( X \setminus V \subseteq V_1 \). Let \( A = X \setminus U_1 \), \( B = X \setminus V_1 \). Then \( A \) and \( B \) are \( g^* \gamma \)-closed sets such that \( A \subseteq U \), \( B \subseteq V \) and \( A \cup B = X \).

(ii) \( \Rightarrow \) (iii) Let \( H \) be a closed set and \( K \) be an open set containing \( H \). Then \( X \setminus H \) and \( K \) are open sets whose union is \( X \). Then by (ii), there exists \( g^* \gamma \)-closed sets \( M_1 \) and \( M_2 \) such that \( M_1 \subseteq X \setminus H \) and \( M_2 \subseteq K \) and \( M_1 \cup M_2 = X \). Then \( H \subseteq X \setminus M_1 \), \( X \setminus K \subseteq X \setminus M_2 \) and...
Let \( U = X \setminus M_1 \) and \( V = X \setminus M_2 \). Then \( U \) and \( V \) are disjoint \( \gamma \)-open sets such that \( H \subseteq U \cap V \subseteq K \). As \( X \setminus V \) is \( \gamma \)-closed set, we have \( \gamma \text{ Cl}(U) \subseteq X \setminus V \) and \( H \subseteq U \cap \gamma \text{ Cl}(U) \subseteq K \).

(iii) \( \Rightarrow \) (i) Let \( H_1 \) and \( H_2 \) be any two disjoint closed sets of \( X \). Put \( K = X \setminus H_2 \) then \( H_2 \cap K = \emptyset, H_1 \subseteq K \) where \( K \) is an open set. Then by (iii), there exists a \( \gamma \)-open set \( U \) of \( X \) such that \( H_1 \subseteq U \cap \gamma \text{ Cl}(U) \subseteq K \). It follows that \( H_2 \subseteq X \setminus \gamma \text{ Cl}(U) = V \) (say), then \( V \) is \( \gamma \)-open and \( U \cap V = \emptyset \). Hence \( H_1 \) and \( H_2 \) are separated by \( \gamma \)-open sets \( U \) and \( V \). Therefore \( X \) is \( \gamma \)-normal space.

**Theorem 3.34:** If \( f : X \rightarrow Y \) is a always \( \gamma \)-open continuous almost \( \gamma \)-irresolute function from a \( \gamma \)-normal space \( X \) into a space \( Y \), then \( Y \) is \( \gamma \)-normal.

**Proof:** Let \( A \) be a closed subset of \( Y \) and \( B \) be an open set containing \( A \). Then by continuity of \( f \), \( f^{-1}(A) \) is closed and \( f^{-1}(B) \) is an open set of \( X \) such that \( f^{-1}(A) \subset f^{-1}(B) \).

As \( X \) is \( \gamma \)-normal, there exists a \( \gamma \)-open set \( U \) in \( X \) such that \( f^{-1}(A) \subset U \subset \gamma \text{ Cl}(U) \subset f^{-1}(B) \) by theorem 3.33. Then, \( f(f^{-1}(A)) \subset f(U) \subset f(\gamma \text{ Cl}(U)) \subset f(f^{-1}(B)) \).

Since \( f \) is always \( \gamma \)-open almost \( \gamma \)-irresolute surjection, we obtain \( A \subset f(U) \subset \gamma \text{ Cl}(f(U)) \subset B \). Then again by theorem 3.33, the space \( Y \) is \( \gamma \)-normal.

**Theorem 3.35:** If \( f : X \rightarrow Y \) is an always \( \gamma \)-closed continuous function from a \( \gamma \)-normal space \( X \) onto a space \( Y \), then \( Y \) is \( \gamma \)-normal.

**Proof:** Let \( F_1 \) and \( F_2 \) be disjoint closed sets. Then \( f^{-1}(F_1) \) and \( f^{-1}(F_2) \) are closed sets. Since \( X \) is \( \gamma \)-normal, then there exist disjoint \( \gamma \)-open sets \( U \) and \( V \) such that \( f^{-1}(F_1) \subset U \) and \( f^{-1}(F_2) \subset V \). By theorem 3.5, there exist \( \gamma \)-open sets \( A \) and \( B \) such that \( F_1 \subseteq A, F_2 \subseteq B, f^{-1}(A) \subset U \) and \( f^{-1}(B) \subset V \). Also, \( A \) and \( B \) are disjoint. Hence, \( Y \) is \( \gamma \)-normal space.

**References**


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