On Scalar Quasi - Weak Commutative Algebras

G.Gopalakrishnamoorthy¹, S.Anitha² and M.Kamaraj³

¹Principal, Sri krishnasamy Arts and Science College, Sattur – 626203, Tamilnadu.
²Lecturer, Raja Doraisingam Government Arts College, Sivagangai – 630 561, Tamil Nadu.
³Dept. of Mathematics, Government Arts and Science College,Sivakasi - 626124.

Abstract: The concepts of scalar commutativity defined in an Algebra A over a commutative ring R and Quasi - weak commutativity defined in a near- ring are mixed together to coin the concept of scalar quasi weak commutativity in an algebra A over a commutative ring R and its various properties are studied.

I. Introduction:

Koh,Luh,Putcha [5] called an algebra A over a commutative ring R to be scalar commutative if for every x,y ∈ A there exists an α ∈ R depending on x and y such that xy = αyx.Coughlin and Rich[1] and Coughlin,Kleinfield [2] have studied scalar commutativity in algebras over a field F.G.Gopalakrishnamoorthy,S.Geetha and M.kamaraj[3] have defined a near - ring N to be Quasi – weak commutative if xyz = yxz for all x,y,z ∈ N.They have obtained many interesting results of Quasi – weak commutativity in Near – rings. In this paper we call an algebra A over a commutative ring R to be scalar Quasi – weak Commutative, if for every x,y,z ∈ A ,there exists a scalar α ∈ R depending on x,y,z such that xyz = αyxz. We prove many interesting results.

II. Preliminaries:

In this section we give the basic definitions and various well known results which we use in the sequel.

2.1 Definition[5]:
Let A be an algebra over a commutative ring R.If for every x,y ∈ A,there exists an element α ∈ R depending on x,y such that xy = αyx, then A is said to be scalar commutative.If for every x,y ∈ A, there exists an element α ∈ R depending on x,y such that xy = -αyx, then A is said to anti-scalar commutative.

2.2 Definition[3]:
Let N be a near – ring inwhich xyz = yxz for all x,y,z ∈ N.Then N is called Quasi - weak commutative near-ring.If xyz = - yxz for all x,y,z ∈ N,then N is said to be Quasi - weak anti-commutative.

2.3 Lemma 3.5[4]:
Let N be a distributive near-ring.If xyz = ± yxz for all x,y,z ∈ N,then N is either quasi weak commutative or quasi weak anti-commutative.

III. Main Results:

3.1 Definition:
Let A be an algebra over a commutative ring R.If for every x,y,z ∈ A,there exists α ∈ R depending on x,y,z such that xyz = αyxz,then A is said to be scalar quasi weak commutative.
If xyz = -αyxz,then A is said to be scalar quasi weak anti-commutative.

3.2 Theorem:
Let A be an algebra (not necessarily associative) over a field F.If A is scalar quasi weak commutative,then A is either quasi weak commutative or quasi weak anti commutative.

Proof:
Suppose xyz = yxz for all x,y,z ∈ A,there is nothing to prove.Suppose not,we shall prove that xyz = - yxz for all x,y,z ∈ A. We shall first prove that if x,y,z ∈ A such that xyz ≠ yxz,then x²z = y²z = 0.
Let \( x,y,z \in A \) such that \( xyz \neq yzx \). Since \( A \) is scalar quasi weak commutative, there exists \( \alpha = \alpha(x,y,z) \in F \) such that
\[
xyz = \alpha yzx \quad \rightarrow (1)
\]
Also there exists \( \gamma = \gamma(x,y,z) \in F \) such that
\[
x(x+y)z = \gamma(x+y)zx \quad \rightarrow (2)
\]

(1) – (2) gives
\[
xyz - x^2z - yzx = \alpha yzx - yz^2 - \gamma yzx.
\]
i.e., \( yzx - x^2z = (\alpha - \gamma) yzx \).

\[
(\gamma - 1) - x^2z = (\alpha - \gamma) yzx
\]
i.e., \( (1 - \gamma) x^2z = \gamma - \alpha yzx \quad \rightarrow (3) \)

Now \( yzx \neq 0 \) for if \( yzx = 0 \), then from (1) we get \( xyz = 0 \) and so \( xyz = yzx \), contradicting our assumption that \( xyz \neq yzx \).

Also \( \gamma \neq 1 \). For \( \gamma = 1 \), then from (3) we get \( \alpha = \gamma = 1 \).

Then from (1) we get \( xyz = yzx \), again contradicting our assumption that \( xyz \neq yzx \).

Now from (3) we get
\[
x^2z = \frac{\gamma - \alpha}{1 - \gamma} yzx
\]
Similarly \( y^2z = \delta yzx \) for some \( \delta \in F \quad \rightarrow (4) \)

Now corresponding to each choice of \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in F \), there is an \( \eta \in F \) such that
\[
(\alpha_1 x + \alpha_2 y) (\alpha_3 x + \alpha_4 y) z = \eta (\alpha_1 x + \alpha_3 y) (\alpha_2 x + \alpha_4 y) z
\]
\[
(\alpha_1 x^2 + \alpha_3 x y + \alpha_2 x y + \alpha_4 y^2) z = \eta (\alpha_1 x^2 + \alpha_1 x y + \alpha_4 y x + \alpha_4 y^2) z.
\]
\[
\alpha_1 x^2 + \alpha_1 x y + \alpha_4 y x + \alpha_4 x y + \alpha_4 y^2 = \eta (\alpha_1 x^2 + \alpha_1 x y + \alpha_4 y x + \alpha_4 y^2) \quad \rightarrow (5)
\]
\[
\alpha_1 x^2 + \alpha_1 x y + \alpha_4 y x + \alpha_4 x y + \alpha_4 y^2 = \eta (\alpha_1 x^2 + \alpha_1 x y + \alpha_4 y x + \alpha_4 y^2) \quad \rightarrow (6)
\]
\[
(\alpha_1 \alpha_2 x^2 + \alpha_1 \alpha_2 x y + \alpha_4 \alpha_1 y x + \alpha_4 \alpha_1 x y + \alpha_4 \alpha_1 y^2) = \eta (\alpha_1 \alpha_2 x^2 + \alpha_1 \alpha_2 x y + \alpha_4 \alpha_1 y x + \alpha_4 \alpha_1 x y + \alpha_4 \alpha_1 y^2) \quad \rightarrow (7)
\]
If in (7) we choose \( \alpha_2 = 0, \alpha_1 = 1, \alpha_4 = -\beta \), the right hand side of (7) is zero where as the left hand side of (7) is
\[
(\beta \alpha_{-1} - \beta) xyz = 0
\]
i.e., \( \beta (\alpha^{-1} - 1) xyz = 0 \).

Since \( xyz \neq 0 \) and \( \alpha \neq 1 \), we get \( \beta = 0 \).

Hence from (4) we get \( x^2z = 0 \).

Also if in (7) we choose \( \alpha_1 = 0, \alpha_4 = \alpha_2 = 1, \alpha_1 = -\delta \), the right hand side of (7) is zero where as the left hand side of (7) is
\[
(-\delta + \delta \alpha^{-1}) xyz = 0
\]
i.e., \( \delta (\alpha^{-1} - 1) xyz = 0 \).

Since \( xyz \neq 0 \) and \( \alpha \neq 1 \), we get \( \delta = 0 \).

Hence from (5) we get \( y^2z = 0 \).

Then (6) becomes
\[
\alpha_1 \alpha_4 x y z + \alpha_2 \alpha_3 x y z = \eta (\alpha_1 \alpha_2 x y z + \alpha_4 \alpha_1 y x z)
\]
\[
\alpha_1 \alpha_4 x y z + \alpha_2 \alpha_3 x y z = \eta (\alpha_1 \alpha_2 x y z + \alpha_4 \alpha_1 x y z)
\]
i.e., \( (\alpha_1 \alpha_4 + \alpha_2 \alpha_3 \alpha^{-1}) x y z = \eta (\alpha_1 \alpha_2 + \alpha_4 \alpha_1 \alpha^{-1}) x y z \).

This is true for any choice of \( \alpha, \alpha_2, \alpha_3, \alpha_4 \in F \). Choose \( \alpha_1 = \alpha_3 = 1, \alpha_2 = \alpha_4 = 1 \) we get
\[
(1 - (\alpha^{-1})^2) x y z = 0.
\]

Since \( x y z \neq 0 \) and \( x - (\alpha^{-1})^2 = 0 \), Hence \( (\alpha^{-1})^2 = 1 \).

i.e., \( \alpha = \pm 1 \) since \( \alpha \neq 1 \), we get \( \alpha = -1 \).

i.e., \( x y z = -x y z \) for \( x, y, z \in A \).

Thus \( A \) is either quasi weak commutative or quasi weak anti commutative.

3.3 Lemma:
Let \( A \) be an algebra (not necessarily associative) over a commutative ring \( R \).

Suppose \( A \) is scalar quasi weak commutative. Then for all \( x, y, z \in A, \alpha \in R \), \( axyz = 0 \) iff \( \alpha yzx = 0 \). Also \( axyz = 0 \) if \( yzx = 0 \).

Proof:
Let \( x, y, z \in A, \alpha \in R \) such that \( \alpha yzx = 0 \).

DOI: 10.9790/5728-1403013037 www.iosrjournals.org
Since $A$ is scalar quasi weak commutative, there exists $\beta = \beta (\alpha y, x, z) \in R$ such that
\[(\alpha y) x z = \beta x (\alpha y) z = \beta x y z = 0\]
(ie) $\alpha y x z = 0$.

Similarly
If $\alpha y x z = 0$, then there exists $\gamma \in R$ such that
\[(\alpha y) x z = \gamma (\alpha y) x z = \gamma x y z = 0\]
(ie) $\alpha y x z = 0$.
Thus $\alpha y x z = 0$ iff $\alpha y x z = 0$.
Assume $\alpha y x z = 0$.

Since $A$ is scalar quasi weak commutative, there exists $\delta = \delta (y, x, z) \in R$ such that
\[x (\alpha y) x z = \delta x y x z = 0\]
(ie) $\alpha y x z = 0$.

3.4 Lemma:
Let $A$ be an algebra over a commutative ring $R$. Suppose $A$ is scalar quasi weak commutative. Let $x, y, z, u \in A$, $\alpha, \beta \in R$ such that
\[x u = u x, \quad y x z = \alpha y x z \quad \text{and} \quad (y + u) x z = \beta x (y + u) x z = 0.\]

Proof:
Given
\[(y + u) x z = \beta x (y + u) x z \quad \rightarrow (1)\]
\[y x z = \alpha x y z \quad \rightarrow (2)\]
and $x u = u x \quad \rightarrow (3)$
From (1) we get
\[y x z + u x z = \beta x y z + \beta x u z\]
\[\alpha x y z + u x z = \beta x y z + \beta x u z \quad \text{(using (2))}\]
i.e., $\alpha x y z + u x z = \beta x y z + \beta x u z \quad$ (using (3))
i.e., $x (\alpha y + u - \beta y - \beta u) x z = 0$.

By Lemma 3.3 we get
\[\alpha x y z + u x z - \beta x y z - \beta u x z = 0.\]

From (1) we get
\[y x z + u x z = \beta x y z + \beta x u z\]
\[y x z + u x z = \beta x y z + \beta x u z \quad \text{(using (3))}\]
i.e., $y x z - \beta x y z = \beta u x z - u x z$.

Multiply by $\alpha$,
\[\alpha x y z - \alpha \beta x y z = \alpha x u z - \alpha u x z\]
From (4) and (5) we get
\[\alpha x u z - \alpha u x z + u x z - \beta u x z = 0\]
i.e., $(\alpha x u z - \alpha u x z + u x z - \beta u x z) = 0$
i.e., $(\alpha x u z - \alpha u x z + u x z - \beta u x z) = 0$
i.e., $(\alpha x u z = \alpha u x z + u x z - \beta u x z) = 0$
Hence the Lemma.

3.5 Corollary:
Taking $u = x$, we get
\[(x^2 - \alpha x^2 - \beta x^2 + \alpha \beta x^2) z = 0.\]
\[(x - \alpha x)(x - \beta x) z = 0.\]

3.6 Theorem:
Let $A$ be an algebra over a commutative ring $R$. Suppose $A$ has no zero divisors. If $A$ is scalar quasi weak commutative, then $A$ is quasi weak commutative.

Proof:
Let $x, y, z \in A$.
Since $A$ is scalar quasi weak commutative, there exists scalars $\alpha = \alpha (y, x, z) \in R$ and $\beta = \beta (y + x, x, z) \in R$ such that
\[\alpha y x z = 0 \quad \text{iff} \quad \alpha y x z = 0.\]
(y+x) xz = \beta x (y+x) \quad \rightarrow (1)

and

yxz = \alpha x y z \quad \rightarrow

(2)

From (1) we get

yxz + x^2 z = \beta xyz + \beta x^2 z.

\alpha x y z + x^2 z - \beta y x z - \beta x^2 z = 0. \quad \text{(using (2))}

By Lemma 3.3,

(\alpha y + x - \beta y - \beta x) x z = 0.

\alpha y x z + x^2 z - \alpha \beta y x z - \beta x^2 z = 0.

From (1) we get

yxz + x^2 z = \beta xyz + \beta x^2 z.

yxz - \beta y x z = \beta x^2 z - x^2 z.

\alpha y x z - \alpha \beta y x z = \beta x^2 z - \alpha x^2 z \quad \rightarrow (4)

From (3) and (4) we get

\alpha x^2 z - \alpha x^2 z + x^2 z - \beta x^2 z = 0.

( x^2 - \alpha x^2 - \beta x^2 + \alpha \beta x^2 ) z = 0.

( x - \alpha x ) ( x - \beta x ) z = 0.

Since A has no zero divisors.

x = \alpha x \quad \text{or} \quad x = \beta x.

If \ x = \alpha x, then from (2) we get

yxz = xyz.

If \ x = \beta x, then from (1) we get

yxz + x^2 z = \beta xy z + \beta x^2 z.

yxz = \beta y x z.

Thus A is quasi weak commutative.

3.7 Definition:

Let R be any ring and x,y,z \in R. We define xyz – yxz as the quasi weak commutator of x,y,z.

(i.e) \quad xyz – yxz = (xy – yx)z

x y z = [x, y]z \quad \text{is called the quasi weak commutator of x, y, z.}

3.8 Theorem:

Let A be an algebra over a commutative ring R. Let A be a scalar quasi-weak commutative. If A has an identity, then the square of every quasi-weak commutator is zero.

(i.e.), (xyz – yxz)^2 = 0 \quad \text{for all } x,y,z \in A.

Proof:

Let x,y,z \in A. Since A is scalar quasi-weak commutative, there exists scalars

\alpha = \alpha(y,x,z) \in R \quad \text{and} \quad \beta = \beta(x,y+1,z) \in R \quad \text{such that}

yxz = \alpha x y z \quad \rightarrow (1)

and

x(y+1) = \beta (y+1)x z \quad \rightarrow (2)

From (2) we get

yxz + xz - \beta y x z - \beta x z = 0

yxz + xz - \alpha \beta y x z - \beta x z = 0

x(y+1 - \alpha \beta y - \beta ) z = 0

By Lemma 3.3,

(y+1 - \alpha \beta y - \beta ) x z = 0

yxz + xz - \alpha \beta y x z - \beta x z = 0

\alpha x y z + x z - \alpha \beta y x z - \beta x z = 0 \quad \text{(using (1))} \quad \rightarrow (3)

Also from (2) we get

yxz + x z = \beta y x z + \beta x z.
Multiplying by $\alpha$

\[
\alpha xyz + \alpha xz = \alpha \beta yxz + \alpha \beta xz
\]

\[
\alpha xyz - \alpha \beta yxz = \alpha \beta xz - \alpha xz \quad \rightarrow (4)
\]

From (3) and (4) we get

\[
xz - \beta xz + \alpha \beta xz - \alpha xz = 0
\]

(ie)

\[
x (z - \alpha z) = x(\beta z - \alpha \beta z)
\]

Multiplying by $(y+1)$ on the left we get

\[
(y+1) x (z - \alpha z) = (y+1) x (\beta z - \alpha \beta z)
\]

\[
(y+1) x (z - \alpha z) = \alpha \beta xz - \alpha xz
\]

(ie)

\[
x (y - 2z + \alpha^2 y)z = 0
\]

(ie)  \[
(yz - xz)^2 = (xyz - xz)^2 = (xyz - xz)(xyz - xz) = xyzxyz - \alpha xyzxyz - \alpha xyz + \alpha^2 xyzxyz
\]

\[
(yz - xz)^2 = x (y - 2z + \alpha^2 y)zxyz
\]

\[
(yz - xz)^2 = 0 \quad (using (5))
\]

Thus \((xz - yxz)^2 = 0\).

(ie) Square of every quasi-weak commutator is zero.

3.9 Definition:

Let $R$ be a P.I.D (Principal ideal domain) and $A$ be an algebra over $R$. Let $a \in R$.

The order of $a$, denoted as $O(a)$ is defined to be the generator of the ideal $I = \{ \alpha \in R / \alpha a = 0 \}$.

$O(a)$ is unique upto associates and $O(a)=1$ if and only if $a=0$.

3.10 Lemma:

Let $A$ be an algebra with identity over P.I.D. If $A$ is scalar quasi-weak commutative, $y \in R$ with $O(y) = 0$, then $y$ is in the center of $A$.

Proof:

Let $y \in R$ with $O(y) = 0$.

For every $x \in A$, there exists scalars $\alpha = \alpha(x,y,1) \in R$ and $\beta = \beta(y,x,1,1) \in R$ such that

\[
x y.1 = \alpha xy.1
\]

\[
xy = \alpha xy \quad \rightarrow (1)
\]

and

\[
y(x+1) = \beta(x+1)y.1
\]

(ie)

\[
y(x+1) = \beta(x+1)y \quad \rightarrow (2)
\]

From (2) we get

\[
yx + x = \beta xy + \beta y \quad (using(1))
\]

\[
yx + x - \beta xy - \beta y = 0
\]

Using Lemma 3.3 we get

\[
(x + 1 - \alpha \beta x - \beta y)x.1 = 0
\]

\[
xy + y - \alpha \beta xy - \beta y = 0 \quad \rightarrow (3)
\]

Also from (2) we get

\[
xy + y - \beta xy - \beta y = 0
\]

Multiply by $\alpha$

\[
\alpha xy + \alpha y - \alpha \beta xy - \alpha \beta y = 0
\]

\[
xy + \alpha y - \alpha \beta xy - \alpha \beta y = 0 \quad (using(1)) \quad \rightarrow (4)
\]

From (3) and (4) we get

\[
y - \beta y = -\alpha y + \alpha \beta y = 0
\]

\[
(y - \beta y) - (y - \ beta) = 0
\]

\[
(1 - \alpha) (1 - \beta) y = 0
\]

Since $O(y) = 0$ we get $\alpha = 1$ or $\beta = 1$

If $\alpha = 1$, from (1) we get $xy = yx$

If $\beta = 1$ from (2) we get $y(x+1) = x+1y$

\[
yx + y = xy + y
\]

\[
y = xy
\]
3.11 Lemma:

Let $A$ be an algebra with unity over a principal ideal domain $R$. If $A$ is scalar quasi-weak commutative, $x \in A$ such that $O(xz) = 0$, then $xyz = yxz$ for all $y, z \in A$.

Proof:

Let $x \in A$ with $O(xz) = 0$

For $y, z \in A$, there exist scalars $\alpha = \alpha(y, x, z) \in R$ and $\beta = \beta(x, y+1, z) \in R$ such that

\[
yxz = \alpha yxz
\]

(1)

From (2) we get

\[
yxz + xz = \beta xz + \beta xz
\]

(3)

By Lemma 3.3, we get

\[
(xy + x - \beta y - \beta z)xz = 0
\]

(4)

Multiplying (3) by $\alpha$, we get

\[
\alpha yxz + \alpha xz - \alpha \beta yz - \alpha \beta xz = 0
\]

(5)

(4) – (5) gives

\[
xz + \beta xz = \alpha xz + \alpha \beta xz = 0
\]

(6)

Since $O(xz) = 0$, we get $1 - \alpha = 0$ or $1 - \beta = 0$.

That is $\alpha = 1$ or $\beta = 1$.

If $\alpha = 1$, then from (1) we get $yxz = yxz$.

If $\beta = 1$, then from (3) we get

\[
yxz = yxz
\]

(ie) $yxz = yxz$.

3.12 Lemma:

Let $A$ be an algebra with identity over a P.I.D $R$. Suppose that $A$ is scalar quasi-weak commutative. Assume further that there exists a prime $p \in R$ and a positive integer $m \in Z^+$ such that $p^m A = 0$. Then $A$ is quasi-weak commutative.

Proof:

Let $x, y \in A$ such that $O(xy) = p^k$ for some $k \in Z^+$. We prove by induction on $k$ that $xy = yx$ for all $x, y \in A$.

If $k = 0$, then $O(xy) = p^0 = 1$ and so $xy = 0$.

So $xy = 0$. By Lemma 3.3, $yx = 0$.

Hence $xy = yx$ for all $x, y \in A$.

We first prove that for any $u, v \in A$,

\[
xyu - yxu = 0
\]

implies

\[
y(u)w - (u)yw = 0
\]

for all $w \in A$.

So, let $xy - yx \neq 0$. Since $A$ is scalar quasi-weak commutative, there exists scalars

\[
\alpha = \alpha(x, y, u) \quad \text{and} \quad \beta = \beta(x, y+1, u)
\]

such that

\[
xyu = \alpha yxu
\]

(1)

and

\[
x(y+1)u = \beta (y+1) xu
\]

(2)

From (2) we get

\[
xyu + xu = \beta yxu + \beta xu
\]

(3)

If $\alpha - \beta \neq 0$, then $(\beta - 1) xu = 0$ and so $\beta xu = xu$

(4)

So from (2) we get

\[
x(y+1)u = (y+1) \beta xu = (y+1) xu
\]

\[
xyu + xu = yx + xu
\]

i.e., $xy - yx = 0$, contradicting our assumption that $xyu \neq yxu$. 


Let \( A \) be an algebra with identity over a principal ideal domain \( R \). If \( A \) is scalar quasi-weak commutative, then \( A \) is quasi-weak commutative.

**Proof:**

Suppose \( A \) is not quasi-weak commutative, there exists \( x \in A \) such that \( xyz \neq yxz \) for all \( y, z \in A \).

Also \((x+1)yz \neq y(x+1)z\).

By Lemma 3.10 \( O(x) \neq 0 \) and \( O(x+1) \neq 0 \).

Hence \( O(1) \neq 0 \). Let \( O(1) = d \neq 0 \). Then \( d \) is not a unit and hence
\[
d = p_1^{t_1} p_2^{t_2} \ldots \ldots p_k^{t_k}
\]
for some primes \( p_1, p_2, \ldots, p_k \in A \) and some positive integers \( t_1, t_2, \ldots, t_k \).

Let \( A_i = \{ a \in A \mid p_i^{j}a = 0 \} \). Then each \( A_i \) is a non-zero sub-algebra of \( A \) and \( A = A_1 \oplus A_2 \oplus \ldots \oplus A_k \). Being sub-algebras of \( A \), each \( A_i \) is scalar quasi-weak commutative.

Being homomorphic image of \( A \), all the \( A_i \)'s have identity element 1.

By Lemma 3.12 each \( A_i \) is quasi-weak commutative and hence \( A \) is quasi-weak commutative, a contradiction. This contradiction proves that \( A \) is quasi-weak commutative.

**3.13 Theorem:**

Let \( A \) be an algebra with identity over a principal ideal domain \( R \). If \( A \) is scalar quasi-weak commutative, then \( A \) is quasi-weak commutative.
References

[3]. G.Gopalakrishnamoorthy,M.Kamaraj and S.Geetha,On Quasi weak commutative
[7]. Pliz Ginter, Near-Rings, North Holland, Ameteramc(1983)