Nonlinear Difference Equations and Simulation for Zooplankton-Fish Model with Noise

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Abstract: In this paper, nonlinear difference equations for zooplankton-fish population model with noise is considered. The model is on predation of phytoplanktivore fishes on zooplankton, this is to understand the individual behaviour of the organisms as well as interaction with the environment. The model is a nonlinear logistic type of model incorporating nonlinear feeding functions. The conditions for the existence of the equilibrium points are obtained through some nonlinear equations and Diophantine equations. The conditions for local stability for the dual population investigated and results obtained. Simulation made for the dual populations when the ocean is polluted with chemical substances and oil spillage using Gaussian noise. The noise accounts for pollution of the ocean that may lead to species migration from the pollutants source. It is observed that the risk factor increases with time and makes the species to be endangered and some kind of chemo taxis effect is experienced whereby the survived species tend to migrate to region with lower concentrations of pollutants.

Keywords: Zooplankton-fish, model, stability, Simulation, pollution, environmental risk.

I. Introduction

Modelling integrates understanding of field observation, laboratory experiments’ theory and computation together within wide range of physical and biological processes. All activities in the marine ecosystem have some mathematical connections hence modelling the economic resources, chemical and biological processes of the marine ecosystem is of great concern to researchers in the recent times [1,4,6,8]. There is a growing need among researchers, environmental conservatives and government agencies to understand the pollution of the ocean and attendant environmental changes that accompany it and the biodiversity of the marine ecosystem[3,6,11].

The fish constitute great percentage of human sources of protein and the population of fishes (school) can be affected by global change in temperature, salinity, turbulence and mixing in the ocean ([8,12]). The ocean is constantly polluted by chemical substances that are found to affect the biodiversity and the food chain system of the marine biomes. Consequently, the global economic resources from the ocean are depleting with time ([5,10,11,12, and 13]).

The motivation for this paper is to study the predation of phytoplankton fishes on zooplankton and to simulate the model when the ocean is polluted by spillages from crude oil and other chemical substances. The model we will consider is a kind prey-predator model. There several prey-predator models for studying predation of phytoplankton fishes on zooplankton. There are also individual based models (IBMs) used for investigating the dynamic of different spatial and temporal pattern and transportation of fish eggs and larvae ([4]). There are models of trophodynamics of species, surrogate of biomass and multispecies models and circulation of nutrients in aquatic systems ([2]).

The main food item of pelagic fishes is zooplankton which has also been influenced by the change of salinity that is thought to have caused a decrease of large zooplankton in the Gulf of Finland. It is found that the distribution pattern of phytoplanktivore fish changes according to current [8]. Horst et. al [4] considered plankton interaction-diffusion model to study the influence of fish on the spatial-temporal pattern for the zooplankton population. The local properties such as the excitation and stability of spatially uniform stability in the absence of diffusion were analysed.

Arnfinn [1] studied the effect of fish predation on zooplankton community in an oligotrophic lake with addition of artificial fertilizer. It was found that there was increase in the biomass of zooplankton and fish predation and the distribution pattern of planktivore fish change according to current direction. Nutrient–phytoplankton–zooplankton–detritus (NPZD) models have been developed for more than 40 years (e.g. Andrew [2]; Katja et.al, [6]; Ute et al. [12]).
The lack of general models for zooplankton population has become a major setback to carbon flux models for simulating the effects of zooplankton grazing and faecal production on carbon storage and export flux, and for food-web models to simulate biomass transfer from lower to higher tropic level organisms including fish populations ([2,7,9]).

The modelling of marine zooplankton has made great progress over the two last decades covering a large range of representations from detailed individual processes to functional groups. A new challenge is to dynamically represent zooplankton within marine food webs coupling lower tropic levels to fish and to thereby further our understanding of the role of zooplankton in global change ([1,8,11,12]).

The weight and composition of zooplankton varies significantly, and it can be attributed to age, seasonal difference, geographical distribution, or environmental conditions (Makoso [9]). Eddies and fronts are generally thought to have phytoplankton abundance than adjacent waters through an increase in phytoplankton stock (Thibault et al. [11]).

The eggs and dry weight of zooplankton are found to be nonlinear in the Bariri reservoir in the Sao Paulo state in Brazil. From field experiment (see Gozalez [3]) in the middle of Tieli river in Sao Paulo in Brazil, the diameter of cladoceran and copepod species sizes are found to be 108.6 micron (Keratella tropical) to 2488.6 micron (females of Argyrodiaptomus azevedol) while dry weight varies between 0.025 micron(K.tropical) and 51.250 micron (female azevedol) egg diameter.

In this paper, nonlinear difference equations for zooplankton – fish population model with noise is proposed. The model is on predation of phytoplanktivore fishes on zooplankton; our aim is to understand the individual organism, behaviour as well as interaction with the environment. The conditions for the existence of the equilibrium points obtained through some nonlinear equations and Diophantine equations and conditions for local stability for the dual population are obtained. The model considered is a type nonlinear stochastic equation treated as a type of Prey-predator model with pursuit and search parameters as against the continuous and impulsive version found in the literature (see for example [13]).

The use of nonlinear difference equations with nonlinear feeding functions used in this paper is expected to offer interesting platform for research on prey-predator models for species and should open-up the window of future research on species in polluted environments.

II. Preliminary Definitions and Notations

2.1 Notations
We will make use of the following notations

- $u_1$: Velocity of the fish
- $u_2$: Velocity of the zooplankton
- $d_1$: Search distance at the period $t$ by the fish
- $d_2$: Escape distance at the period $t$ of the zooplankton
- $a_1 = m_r r^2 u_2 x_1$, $r$ is the search radius and $m_r$ is mass of zooplankton
- $b_1 = m_r r^2 u_1$, $m_y$ is mass of the fish
- $a_y$: Volumetric search velocity for fish
- $a_{2y}$: Volumetric escape velocity for the zooplankton
- $x_1$: Population of zooplankton at time $t$ measured in months
- $y_1$: Population of the fish at time $t$ measured in months.

\[ \mu \text{ is the expected value for } u_i \]

\[ E((y_t - \mu)(y_{t-\tau} - \mu)) = Cov(y_t, y_{t-\tau}) = \gamma \tau \text{ Auto covariance with lag } \tau \text{ while } d = (\ldots) \text{ is the common divisor of (\ldots).} \]

2.2 Nonlinear Stochastic Equations
Consider the stochastic equation form

\[ Z_t = f(Z_{t-1}) + \varepsilon_t, -\infty < t < \infty \]  

(1)
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\( f \) is an arbitrary function such that \( f : \mathbb{R} = (-\infty, \infty) \rightarrow \mathbb{R}^n \). It is a special case of linear model in which \( f(y) = 0 \). Analyses of the properties of eq. (1) have been subject of many researches in the recent times. Removing the noise from the model, we obtain the following deterministic model

\[
z_t = f(z_{t-1}) (t \geq 1)
\]  

(2)

where \( z_t \) in the lower case is for the deterministic form as against the stochastic form in the eq. (1) where \( z_t \) is used. We are not going to make discussion on this but will focus on some features of the deterministic model but later on we will consider the simulation for the fish-zooplankton model which special form of the eq. (2).

**Definition 1**

A solution \( z^* \) of the equation \( z = f(z) \) is an equilibrium point of the difference equation in the eq. (2), if \( f(z^*) = 0 \). In many cases \( z^* = 0 \) is an equilibrium point this is a trivial case, but for non-trivial case the equilibrium point \( z^* \neq 0 \) are often be of interest. An equilibrium point \( z^* \) is said to be asymptotically stable or an asymptotical stable limit point for (almost) all initial value, \( z_0 \) such that \( z_t \rightarrow z^* \) as \( t \rightarrow \infty \).

**Definition 2**

An equilibrium point \( z^* \) is said to be weakly stable if given an infinitesimal perturbation from the equilibrium point \( z \) the process returns to the equilibrium point i.e. given that \( z_0 = z^* + h \Rightarrow z_t \rightarrow z^* \) as \( t \rightarrow \infty \), \( h \rightarrow 0 \) \((t \geq 0)\)

**Definition 3**

A process is said to be weakly stationary (or covariance) stationary or in wide sense stationary if \( \{u_t\} \) is constant over time and \( \text{cov}(u_t, u_{t-\tau}) \) depends only on the lag \( \tau \) and not \( t \). If \( \{u_t\} < \infty \), \( \text{cov}(u_t, u_{t-\tau}) < \infty \) then strictly stationary process implies weakly stationary process but weakly stationary process does not imply strictly stationary process.

**Definition 4**

Let \( u_t \) be independently and identically distributed (i.i.d), with common variance \( \sigma^2 \) such that \( \gamma_0 = \sigma^2 \), \( \gamma_\tau = 0 \) \((\gamma \neq 0)\), \( \phi_0 = 1 \) and \( \phi_\tau = 0 \) \((\tau \neq 0)\).

\( u_t \) is said to be white noise process if it’s stationary and the \( u_t \) is pairwise uncorrelated.

A sequence \( (\xi_n)_{n=0}^{\infty} \) of radium variables is said to be convergent in mean to random variable \( \xi \) if

\[
E((\xi_n - \xi)) \xrightarrow{n \rightarrow \infty} 0
\]

We will simply write \( \xi_n \xrightarrow{m} \xi \).

**III. Statement of Problem**

The model we will consider is a nonlinear logistic type of model for the zooplankton-fish population which incorporated nonlinear feeding functions \( F_i, i = 1, 2 \). The feeding function \( F_i \) takes into consideration the search rate and search velocity of the fish on the zooplankton and the escape rate of the zooplankton from the fish. The model also includes escape velocity of the zooplankton from predation of fish and feeding rate function \( F_j \) for the zooplankton.

Now consider the nonlinear zooplankton-fish population.
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\[ x_t = a_1 \left( \frac{k_1 - x_{t-1}}{k_1} \right) F_1 + b_1 y_{t-1} + \varepsilon_{1t} \]
\[ y_t = a_2 \left( \frac{k_2 - y_{t-1}}{k_2} \right) F_2 + b_2 y_{t-1} + \varepsilon_{2t} \]

(3)

Where the feeding functions \( F_i, i = 1, 2 \) are given as

\[ F_1 = \left( \frac{1}{1 + \sum_t a_{it} x_{t-1}} \right)^{-1} \left[ \frac{1}{1 + \sum_t a_{it} x_{t-1}} + t \right] \]

and

\[ F_2 = \left( \frac{1}{1 + \sum_t b_{it} y_{t-1}} \right)^{-1} \left[ \frac{1}{1 + \sum_t b_{it} y_{t-1}} + t \right] \]

To ensure capture of zooplankton by the fish, we must have \( \sum_t a_{it} x_{t-1} = \sum_t b_{it} y_{t-1} \) for every \( t \). The noise accounts for pollution of the ocean by chemical substances or other forms of pollutants that may lead to chemotaxis effect of the dual population, that is, forcing the species to migrate toward the region of low chemical concentration or away from the region of high chemical concentration.

Now consider the model in the eq. (3) and let \( z_t = (x_t, y_t)^T \) then the model can be represented in deterministic form for \( z = (x, y), \varepsilon_{it} = 0, i = 1, 2 \). Let \( f(x, y) = \begin{pmatrix} a_1 \left( \frac{k_1 - x}{k_1} \right) F_1 + b_1 xy \\ a_2 \left( \frac{k_2 - y}{k_2} \right) F_2 + b_2 xy \end{pmatrix} \). If \( x = 0, y = 0 \) then

\[ x_t = a_1 F_1 = a_1 t, \quad y_t = a_2 F_2 = a_2 t. \]

Therefore the per capita growth rate of the zooplankton is \( \frac{a_1}{t} - 1 \), and the per capita growth rate of the fish is \( \frac{a_2}{t} - 1 \).

It is worthy of note that the presence of fishes is causing the zooplankton population to decrease.

IV. Methods

Assuming that \( f \) is continuously differentiable and \( h_{t-1} \) is infinitesimally small and from the eq. (2)

\[ z_t = f(z_{t-1}) = f(z^*) + \frac{\partial f}{\partial z} f_{t-1} = z^* + \frac{\partial f}{\partial z} h_{t-1} \]

where \( \frac{\partial f}{\partial z} \) is the matrix of partial derivative with \( (i, j) \) element \( \frac{\partial f_i}{\partial z_j} \) are evaluated at the equilibrium points; if \( \lambda_i \) are the eigenvalues to \( \frac{\partial f}{\partial z} \) and \( \left| \lambda_i \left( \frac{\partial f}{\partial z} \right) \right| < 1 \), then \( h_t \rightarrow 0 \) as \( t \rightarrow 0^+ \)

In order to obtain the equilibrium points for the model in the eq. (3), we set the right hand side of the equation to zero and we have the following equations:
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\[
a_1 \left( \frac{k_1 - x^*}{k_1} \right) F_1 + b_1 x^* y^* = 0
\]
\[
a_2 \left( \frac{k_2 - x^*}{k_2} \right) F_2 + b_2 x^* y^* = 0
\]

We found that the equilibrium points are

\( E_1 = (0,0), E_2 = (0, y^*), E_3 = (x^*, 0) = (k_1, 0) \) and \( E_4 = (x^*, y^*) \).

To obtain \( E_4 \) involves solving the following nonlinear equation

\[
b_1 a_1 \left( \frac{k_1 - x^*}{k_1} \right) F_1 - b_2 a_2 \left( \frac{k_2 - y^*}{k_2} \right) F_2 = 0.
\]

Solutions to the eq. (5) is intractable because of the nonlinear nature of the feeding functions \( F_i, i = 1, 2 \).

4.2 Risk Assessment as result of chemical pollutions

The estimation of the risk for various environmental hazard factors in the pollution of the ocean and the equivalent environmental cost can be computed from the following formulae

\[
\text{Risk} = u_i = \left( \sum_{t} f_{it} \right) \left( \sum_{t} w_{it} \right), \ i = 0, 1, 2, \ldots, n
\]

And the equivalent environmental cost

\[
u^e_i = \left( \sum_{t} f_{it} \right) \left( \sum_{t} w_{it} \right)^e, \ i = 0, 1, 2, \ldots, n, e \geq 1
\]

Where \( \sum_{t} f_{it} \) are the cumulative weight frequencies of the types of hazard present at the period \( t \) and \( n \) is the number of hazard factors.

\( \sum_{t} w_{it} \) are the cumulative weights of the types of hazard at present at the period \( t \).

For example, weight \( W_1 \) is for chemical hazard and \( W_2 \) is for biological hazard, etc.

Each of the weights can be estimated as number of death of the specie at the given period when the hazard occurs divided by total death recorded during estimation period. This is expressible in the time scale of month or year. For example, if there are cases of hazard that occurred in given month as a result chemical pollution and a total 10 hazards occurred during experiment period, then \( W_1 = \frac{2}{10} \).

V. Results

Let the Jacobian matrix, \( J \left( x^*, y^* \right) = A \) and \( A \) is \( n \times n \) matrix and \( E_t = (e_{it}, e_{nt}) \) is the noise variable. Then we can find the approximate solution to the eq. (1) using the following iterative equations with noise.

\[
Z_t = AZ_{t-1} + u_t E_{t-1}
\]
For \( t = 0, 1, 2, \ldots, k \), we have

\[
\begin{align*}
z_1 &= A z_1 + u_0 E_0, \\
z_2 &= A z_1 + u_1 E_1 = A (A z_0 + u_0 E_0), \\
z_3 &= A^2 z_0 + A u_0 E_0 + u_1 E_1, \\
z_4 &= A z_2 + u_2 E_2 = A^3 z_0 + A^2 u_0 E_0 + A u_1 E_1, \\
\vdots \\
z_{t+1} &= A^t z_0 + A^{t-1} u_0 E_0 + A u_1 E_1 + \ldots + u_{t-1} E_{t-1}.
\end{align*}
\]

In the Lemma 1, we intend to estimate the bound for the ratio \( \frac{F_1}{F_2} \) of the feeding functions for given two equilibrium points \( x^* \) and \( y^* \) and the upper bound for \( |k_2 - y^*| \).

**Lemma 1**

Let \( x^* \) and \( y^* \) be equilibrium points for the eq. (5) such that

\[
\begin{align*}
b_1 a_1 \left( \frac{k_1 - x^*}{k_1} \right) F_1 - b_2 a_2 \left( \frac{k_1 - y^*}{k_2} \right) F_2 &= 0.
\end{align*}
\]

Assume that \( F_1 \) and \( F_2 \) are such that

\[
\left| \frac{F_1}{F_2} \right| \leq \frac{e^y + \frac{d_2}{u_{2i}} + t_r}{e^{-x} \left( \sum a_i x_i \right)^{-1}}.
\]

Then

\[
|k_2 - y^*| \leq \frac{k_2 a_2 b_2}{k_1 b a_2} \left| \frac{F_1}{F_2} \right| \left| k_1 - x^* \right| \leq \frac{k_2 a_2 b_2}{k_1 b a_2} \left[ e^x + \frac{d_2}{u_{2i}} + t_r \right] \left| k_1 - x^* \right|.
\]

**Proof.**

For the proof, we process as follows:

\[
\frac{F_1}{F_2} = \frac{e^{-y} \left( \sum b_i y_i \right)^{-1} + \frac{d_2 b_2 y_i}{u_{2i}} + t_r}{e^{-x} \left( \sum a_i x_i \right)^{-1} + \frac{a_2 y_i d_i}{u_{2i}} + t_r}.
\]

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From eq. (5), we have

\[
\begin{aligned}
\frac{|F_1|}{|F_2|} &\leq e^x \left( \sum \alpha_i x_i e^{-x} \left( \sum \alpha_i x_i \right)^{-1} \right) \\
&\leq e^x \left( e^x + \frac{d_2}{u_{2t}} + t_r \right) \sum \alpha_i x_i
\end{aligned}
\]

Then if \(|k_1 - x^*| \leq r_x\), then it implies that

\[
\begin{aligned}
|k_2 - y^*| &\leq \left(1 + x^*\right) \left[ \left(1 + y^*\right) + \frac{d_2}{u_{2t}} + t_r \right] \\
&\leq e^x \left( e^x + \frac{d_2}{u_{2t}} + t_r \right) \sum \alpha_i x_i
\end{aligned}
\]

Using Taylor series, we have

\[
e^x \approx 1 + x^*, e^x \approx 1 + x^* \text{ and } \tau = \sum \alpha_i x^*
\]

Therefore,

\[
|k_2 - y^*| \leq \left(1 + x^*\right) \left[ \left(1 + y^*\right) + \frac{d_2}{u_{2t}} + t_r \right] \tau.
\]

This ends the proof of the Lemma.

In the next Lemma, we will determine the conditions for solvability of the Diophantine equations characterising the equilibrium equations in the eq. (4) when \(a_i, b_i, F_i, i = 1, 2\) are integers.

**Lemma 2**

Let \(\alpha_1, a_1 b_2, \alpha_2, a_2 b_2\) such that \(n, \alpha_1, \alpha_2, F_1, F_2, x^*\) and \(y^*\) are integers such that

\[
\frac{\alpha_1}{k_1} F_1 x^* + \frac{\alpha_2}{k_2} F_2 x^* = \alpha_1 - \alpha_2 \pmod{n}.
\]

Then the above Diophantine equation is solvable if and only if \(d \left( \frac{\alpha_1}{k_1} F_1, \frac{\alpha_2}{k_2} F_2 \right)\) is such that \(d \left( \alpha_2 - \alpha_2 \right)\).

However, if \(d \left( \frac{\alpha_1}{k_1} F_1, \frac{\alpha_2}{k_2} F_2 \right) = 1\) then there exist \(x^*, y^* \in \mathbb{Z}\) such that \(\frac{\alpha_1}{k_1} F_1 x^* + \frac{\alpha_2}{k_2} F_2 x^* = 1\).

**Proof**

If \(x^*, y^* \in \mathbb{Z}\) are equilibrium point too.

Therefore, \((\alpha_1 - \alpha_2) x^*, (\alpha_1 - \alpha_2) y^*\) are equilibrium points.

Therefore,

\[
\frac{\alpha_1 F_1}{k_1} (\alpha_2 - \alpha_1) x^* + \frac{\alpha_2 F_2}{k_2} (\alpha_2 - \alpha_1) y^* = \alpha_2 - \alpha_1.
\]

If \(x_0^*\) and \(y_0^*\) are another solution to the eq. (5), then
\[
\left(\frac{\alpha_1 F_1}{k_1} x^* + \frac{\alpha_2 F_2}{k_2} y^* \right) - \left(\frac{\alpha_1 F_1}{k_1} x_0^* + \frac{\alpha_2 F_2}{k_2} y_0^* \right) = \alpha_2 - \alpha_1 - \alpha_3 - \alpha_4 = 0
\]

Therefore,
\[
\left(\frac{\alpha_1 F_1}{k_1} (x^* - x_0^*) + \frac{\alpha_2 F_2}{k_2} (y^* - y_0^*) \right) = 0.
\]

This is equivalent to
\[
-\frac{\alpha_1 F_1}{k_1} \frac{\alpha_2 F_2}{k_2} = \frac{y^* - y_0^*}{x^* - x_0^*}
\]
or
\[
-\frac{\alpha_1}{k_1} F_1 t = x^* - x_0^*
\]
\[
-\frac{\alpha_2}{k_2} F_2 t = y^* - y_0^*.
\]

Therefore,
\[
y^* = y_0^* + \frac{\alpha_2}{k_2} F_2 t, x^* = x_0^* + \frac{\alpha_1}{k_1} F_1 t, t \in \mathbb{R}.
\]

For every \(d = \left(\frac{\alpha_1}{k_1} F_1, \frac{\alpha_2}{k_2} F_2\right)\) since \(t\) is arbitrary, hence, the Diophantine equation has infinitely many solutions. However, if \(d = \left(\frac{\alpha_1}{k_1} F_1, \frac{\alpha_2}{k_2} F_2\right) \neq 1\) then eq. (1) transforms to
\[
\frac{\alpha_1 F_1}{k_1} \frac{\alpha_2 F_2}{k_2} \frac{x^*}{d} + \frac{\alpha_2 F_2}{k_2} \frac{y^*}{d} = \alpha_2 - \alpha_1
\]

And solution is \(x^* = x_0^* - \frac{k_1}{d} t\) and \(y^* = y_0^* - \frac{k_2}{d} t\).

This ends the proof.

In the 4.2 section we will consider linearization method for the nonlinear zooplankton-fish model when the underlying variables and the feeding functions are continuous.

**4.2 Linearization Method of the Nonlinear Equations**

Let \(F_i, i = 1, 2\) be continuous functions, therefore, we can linearize the nonlinear equation in the eq. (1). Let the Jacobean of \(f\) with respect \((x, y)\) be defined by

\[J(x, y) = \frac{\partial f}{\partial z} = \frac{\partial (f_1, f_2)}{\partial (x, y)} = \begin{pmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} \\
\frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial y}
\end{pmatrix}.
\]

\[J(x, y) = \begin{pmatrix}
a_1 \frac{F_1}{k_1} + a_1 \left(\frac{k_1 - x}{k_1} \right) \frac{\partial F_1}{\partial x} + b_1 y & \left(\frac{k_1 - x}{k_1} \right) \frac{\partial F_1}{\partial y} + b_2 x \\
a_1 \left(\frac{k_1 - x}{k_1} \right) \frac{\partial F_1}{\partial x} + b_1 y & -a_1 \frac{F_2}{k_1} + a_2 \left(\frac{k_1 - x}{k_1} \right) \frac{\partial F_2}{\partial y} + b_2 y
\end{pmatrix}.
\]

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Without ambiguity we will drop $t$ in the indices.

Therefore, for the equilibrium $E = (k_1, 0), E(0, k_2)$ and $E(0, 0)$ and we obtain the corresponding Jacobian for $J(0, 0)$ and $J(k_1, 0)$ as follows:

$$J(0, 0) = \begin{pmatrix}
a_1 - \frac{a_1 \sum a}{k_1 (1 + t_1)(1 + \sum b)^2} & 0 \\
0 & -\frac{a_1 \sum b}{k_2 (1 + t_1)(1 + \sum b)^2}
\end{pmatrix}$$

$$E(k_1, 0)$$ and $$E(0, k_2)$$

$$J(k_1, 0) = \begin{pmatrix}
a_1 F_1(k_1, 0) & b_1 k_1 \\
b k_1 & -\frac{a_1}{k_1} F_2(k_1, 0) + a_1 \frac{\partial F_2(k_1, 0)}{\partial y} - b x
\end{pmatrix}$$

$$J(0, k_2) = \begin{pmatrix}
a_1 F_1(0, k_2) + a_1 \frac{(k_2 - 0)}{k_1} \frac{\partial F_1(0, k_2)}{\partial x} & \frac{a}{k_1} \frac{\partial F_2(0, k_2)}{\partial x} \\
\frac{a}{k_1} \frac{\partial F(0, k_2)}{\partial y} & -\frac{a}{k_2} \frac{\partial F_2(0, k_2)}{\partial y} + a_2 (k_2 - y)
\end{pmatrix}$$

where

$$F_1(k_1, 0) = (1 + t_1)^{-1}$$

$$F_1 = \left[ e^{-x} + \frac{adx}{1 + \sum \alpha x} + t_1 \right]^{-1}$$

Therefore,

$$\frac{\partial F_1}{\partial y} = 0$$

$$\frac{\partial F_2}{\partial x} = 0$$

$$\frac{\partial F_2}{\partial y} = \frac{-F_2^2}{(1 + \sum ub)^2} \left[ bsd_2 u \sum bx - bsd_2 \sum bu + \sum bye^{-y} + due^{-y} \right]$$

In our next Theorem, we determine the conditions for weakly stability of the equilibrium points for the model

**Theorem 1**

Let there exist constants $a, k_1$ and the resting time $t$, such that the following inequalities are satisfied:
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\[
a(1+t_r)(1+\sum a)^2 - ak_i(1+t_r)(1+\sum a) < k_1(1+t_r)(1+\sum a)^2 \\
-ak_i\sum a < k_1(1+t_r)(1+\sum a)
\]

\[
-ak_1(1+t_r)(1+\sum b)^2 + ak_2\sum b < (1+t_r)(1+\sum b)^2 \\
-ak_2\sum b < k_2(1+t_r)(1+\sum b).
\]

Then the prey-predator model is locally stable at the equilibrium points \( E = (0,0) \) and \( E = (0,k_2) \) and stable if \( \frac{a_1}{k_1}F_2(0,k_2) < 1 \).

**Proof**
The eigenvalues for eq. (5) are

\[
\lambda_1 = \frac{a}{k_1} - \frac{a\sum b}{(1+t_r)(1+\sum a)^2}
\]

and

\[
\lambda_2 = \frac{a}{k_1} + \frac{a\sum b}{(1+t_r)(1+\sum b)^2}.
\]

Therefore, for the equilibrium point \( E(0,k_2) \) the condition for local stability is to require that \( |\lambda_i| < 1, i=1,2 \) for the eigenvalues of \( J(0,k_2) \) which is obtained from

\[
\begin{vmatrix}
\frac{a}{k_1}(1+t_r)^{-1} - \lambda & b_1k_1 \\
-b_1k_1 & \frac{-a}{k_2}(1+t_r)^{-1} - b_2k_1 - \lambda
\end{vmatrix} = 0.
\]

Applying the condition that \( |\lambda_i| < 1, i=1,2 \) for the eigenvalues \( J(k_1,0) \) and \( J(0,k_2) \) leads to the inequalities given in the hypothesis and hence the stability of the equilibrium points are established.

**VI. Numerical Simulation of the Model**

We will simulate the model using Maple 18 software by making use of Maple Library on Linear Algebra and Statistics. We consider a hypothetical situation and let us choose the following parameters:

\[
k_1 = 2000, k_2 = 200, t_r = \frac{1}{50}, b_1 = 0.02, y_0 = 30.
\]

That is, we start the simulation with 30 fishes and the resting time to be \( t_r = 1/50 \) hours.

Then the matrix \[
\begin{bmatrix}
\frac{a_1}{k_1(1+t_r)} & b_1k_1 \\
-b_1k_1 & \frac{-a_2}{k_2(1+t_r)^{-1}} - b_2k_1
\end{bmatrix}
\]

after inputting the parameters into Matrix \( A \) (see the Appendix) we have
When we invoke the Maple Code \texttt{EigenPlot(A)}; we obtain the eigenvalues and eigenvectors to the matrix \( A \) as

- Eigenvalue: 0.5391e-1
  - Multiplicity: 1
  - Eigenvector: \(<0.9998, 0.2113e-1>\>

- Eigenvalue: -0.2727
  - Multiplicity: 1
  - Eigenvector: \(<-0.2113e-1, 0.9998>\>

We also make use of \texttt{Eigenvalues(A, output = 'list')}; And the set of eigenvalues were obtained to be

\[
\begin{bmatrix}
0.5390929749 \\
-0.2726530609
\end{bmatrix}
\]

We invoked the Maple Code \texttt{EigenPlot(A)}; we obtained the Figures in the Fig.1& Fig.2.

![Fig. 1: Eigenplot for matrix A](image1)

![Fig. 2: Eigenplot2 for matrix A](image2)

We note that the equilibrium is locally stable since the eigenvalues 0.5391 and -0.27727. The field of eigenvalues are toward the inside of the circle with radius at the centre \((z=0)\)and modulus of the eigenvalues are less than one (unity) as long as the absolute value of \(|u_tE_t|, t = 0, 1, 2,...|\) are very small. We note that \(|A| \leq 1\) and \(z_t \to u_tE_t\) as \(t \to \infty\)

Suppose we consider the hypothetical situation where the environment is polluted with spillage from oil or chemical substance and the data from the field is obtain and recorded in the Table 1. The data are obtained as described in the section 3 and \(u_t\) and \(u'_t\) are obtain using the eq.(5) and the eq.(6). The data in Table 1 were plotted to obtain in the Fig.3.
Table 1: Environmental risks and frequencies of various hazard weights

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_t$</td>
<td>0.5000</td>
<td>0.3333</td>
<td>0.2500</td>
<td>0.8000</td>
<td>0.6000</td>
<td>0.2857</td>
<td>0.5455</td>
</tr>
<tr>
<td>$w_t$</td>
<td>0.07142</td>
<td>0.01645</td>
<td>0.4444</td>
<td>0.3750</td>
<td>0.4000</td>
<td>0.5000</td>
<td>0.4615</td>
</tr>
<tr>
<td>$u_t$</td>
<td>0.03541</td>
<td>0.1958</td>
<td>0.7343</td>
<td>2.9282</td>
<td>3.3034</td>
<td>5.002</td>
<td>7.4999</td>
</tr>
<tr>
<td>$u^s_t$</td>
<td>0.0035</td>
<td>0.9191</td>
<td>1.6021</td>
<td>1.4182</td>
<td>4.3142</td>
<td>8.0841</td>
<td>14.129</td>
</tr>
</tbody>
</table>

Figure 3: Bar Charts environmental risks and frequencies of various hazard weights for the fish

In the Figure 3 the blue bars are frequencies of the occurrence of hazards while the brown coloured bars are the weights values for the hazards. The green colour is the risk values and the purple is the colour of the environmental risk cost. We note as time increases the risk factor increases with time as the figure shows, this make the species to be endangered and some kind of chemotaxis effect is experiences whereby the survived species tend to migrate to region with lower concentrations of pollutants.

The matrix plots in the Fig.4 we plotted using the data in the Table1 for us to have 3-D view of the frequencies of the fish and the corresponding risk assessment in the eq.(5) and the eq.(6). In Matrix Plot 1 and Matrix plot 2 the gap between the histogram are of length 0.15 and 0.25 inches respectively while the Matrix Plot4 used the plot colour of map sin xy. The first set of histograms in the Matrix Plots are the values of $u_t$ in the Table 1 and next histograms are the values of $u_t$, $w_t$ and $f_t$ respectively.

Fig.3: Matrix plots environmental risks and frequencies of various hazard weights for the fish
Table 2: Environmental risks and frequencies of various hazard weights for the zooplankton

<table>
<thead>
<tr>
<th>t</th>
<th>f_t</th>
<th>W_t</th>
<th>u_t</th>
<th>u_t^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.400</td>
<td>0.0200</td>
<td>0.0080</td>
<td>0.0004</td>
</tr>
<tr>
<td>1</td>
<td>0.3750</td>
<td>0.0305</td>
<td>0.1783</td>
<td>0.0449</td>
</tr>
<tr>
<td>2</td>
<td>0.5556</td>
<td>0.3700</td>
<td>0.7067</td>
<td>0.5354</td>
</tr>
<tr>
<td>3</td>
<td>0.7500</td>
<td>0.3734</td>
<td>1.8820</td>
<td>1.7367</td>
</tr>
<tr>
<td>4</td>
<td>0.5330</td>
<td>0.4750</td>
<td>3.6057</td>
<td>4.6038</td>
</tr>
<tr>
<td>5</td>
<td>0.4615</td>
<td>0.5071</td>
<td>4.3236</td>
<td>5.6784</td>
</tr>
<tr>
<td>6</td>
<td>0.5334</td>
<td>0.4602</td>
<td>6.7342</td>
<td>11.0906</td>
</tr>
</tbody>
</table>

Fig. 4: Environmental risks and frequencies of various hazard weights for the zooplankton

The Fig 4 is Bar charts generated from Environmental risks and frequencies for various hazard weights and periods for the zooplankton. We generate the white noise from environmental risk factor and environmental risk cost factor

for the simulation using the vectors:
Nonlinear Difference Equations and Simulation for Zooplankton-Fish Model with Noise

\[ u_{t,E} = \begin{bmatrix} 0.03541 & 0.1958 & 0.7343 & 2.9282 & \ldots & 7.499 \end{bmatrix} \]

\[ u^i_{t,E} = \begin{bmatrix} 0.035 & 0.9191 & 1.6021 & 1.4182 & \ldots & 14.129 \end{bmatrix} \]

Where
\[ X_i, Y_i \in N(0,1), i = 1, 2, \ldots, 6 \]

We computed iterative solutions to the model using the environmental risk values in the Table 1 and the following iterative equation:

\[ z_{i+1} = A^t z_0 + A^{t-1} u_0 E_0 + \ldots + u_{t,E} E_{t-1}. \]

Using Maple 17, we obtained:

\[ z[1] := \text{MatrixMultiply}(p[1], z[0]) + X1; \]
\[ z_1 := \begin{bmatrix} 0.269162204350000 + 0.03541_R \\ 0.0208746397850000 + 0.03541_R0 \end{bmatrix} \]
\[ z[2] := \text{MatrixMultiply}(p[2], z[0]) + X2; \]
\[ z_2 := \begin{bmatrix} 0.0146151212725266 + 0.1958_R \\ -0.00383127051856490 + 0.1958_R0 \end{bmatrix} \]

We computed iterative solutions to the model using the environmental risk cost values in the Table 1 and the following iterative equation:

\[ z_{i+1} = A^t z_0 + A^{t-1} u^i_0 E_0 + \ldots + u^i_{t,E} E_{t-1}. \]

We obtained:

\[ z[1] := \text{MatrixMultiply}(p[1], z[0]) + Y1; \]
\[ z_1 := \begin{bmatrix} 0.269162204350000 + 0.9191_R \\ 0.0208746397850000 + 0.9191_R0 \end{bmatrix} \]
\[ z[2] := \text{MatrixMultiply}(p[2], z[0]) + Y2; \]
Nonlinear Difference Equations and Simulation for Zooplankton-Fish Model with Noise

\[ z_2 := \begin{bmatrix} 0.0146151212725266 + 1.6021 \cdot R \\ -0.00383127051856490 + 1.6021 \cdot R_0 \end{bmatrix} \]

\[ z[3] = MatrixMultiply(p[3], z[0]) + Y3 \]

\[ z_3 := \begin{bmatrix} 0.000759323441765264 + 1.4182 \cdot R \\ 0.00114489315471157 + 1.4182 \cdot R_0 \end{bmatrix} \]

\[ z[6] := MatrixMultiply(p[6], z[0]) + Y6 \]

\[ z_6 := \begin{bmatrix} 6.06146730548798 \cdot 10^7 + 14.129 \cdot R \\ -0.000228676245564868 + 14.129 \cdot R_0 \end{bmatrix} \]

Where \( R, R_0 \in N(0,1) \).

We note that the iteration solutions for the fish and zooplankton depend on the values of random numbers \( R, R_0 \in N(0,1) \). Therefore the model has infinitely many solutions. From the solution is obvious that population of the two species tends to be decreasing with time. For survival of the species there got to be migration from region of pollution to a region that would sustain the growth and flourishing of the species.

VII. Conclusion

Simulation is made for the zooplankton and fish populations when the ocean is polluted with chemical substances and oil spillage and the noise is Gaussian. We note that the iteration solutions for the fish and zooplankton depend on the values of random numbers \( R, R_0 \in N(0,1) \). Therefore the model has infinitely many solutions. From the solution is obvious that population of the two species tends to be decreasing with time. We emphasised that the noise accounts for pollution of the ocean by the chemical substances and oil spillage that may lead to species migration from the pollutants source. It was observed that the environmental risk factor used increases with time and this will make the species to be endangered and some kind of chemo taxis effect will be experienced whereby the survived species tend to migrate to region with lower concentrations of pollutants. The use of nonlinear difference equations with nonlinear feeding functions used in this paper will offer interesting platform for research on prey-predator models for species and should open the window of opportunities for future research on species in polluted environments.

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References


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Appendix

# involving the Linear Algebra and plots library
> with (LinearAlgebra) :
> with (DETools) :
> with (plots) :

# input the value of the matrix A

\[
A := \begin{bmatrix}
\frac{a_1}{k_1 (1 + r)} & b_1 k_1 \\
-\frac{a_2}{k_2 (1 + r)} - b_2 k_1 & b_1 k_1
\end{bmatrix}
\]

# input the value of the parameters

\[
\begin{align*}
& a[1] := 0.05; \\
& a_1 := 0.05; \\
& b[1] := 0.23; \\
& b_1 := 0.23; \\
& k[1] := 0.03; \\
& k_1 := 0.03; \\
& k[2] := 0.12; \\
& k_2 := 0.12; \\
& b[2] := 0.123; \\
& b_2 := 0.123; \\
& a[2] := 1; \\
& a_2 := 1
\end{align*}
\]

# computes the eigenvalues of matrix A

> Eigenvalues(A, output = 'list');

\[
\begin{bmatrix}
0.05390929749 \\
-0.2726530609
\end{bmatrix}
\]

> \( r(r) := 30; \)
> \( t_r := 30; \)

> \[
C := \begin{bmatrix}
0.05376344087 & 0.0069 \\
0.0069 & -0.2725072043
\end{bmatrix}
\]

> C := \[
0.05376344087 & 0.0069 \\
0.0069 & -0.2725072043
\]

> # invoking Student[LinearAlgebra] library facility to obtain eigenvalues and eigenvectors
\[ \text{with(Student[LinearAlgebra])} ; \]
\[ \text{infolevel[Student[LinearAlgebra]]} := 1 ; \]
\[ \text{CharacteristicPolynomial} \left( \begin{bmatrix} 0.05376344087 & 0.0069 \\ 0.0069 & -0.2725072043 \end{bmatrix}, x \right) \]
\[ \text{EigenPlot}(C) ; \]
\[ A := \begin{bmatrix} 0.05376344087 & 0.0069 \\ 0.0069 & -0.2725072043 \end{bmatrix} ; \]
\[ \text{EigenPlot}(A) ; \]
Eigenvalue: .5391e-1
Multiplicity: 1
Eigenvector: < .9998, .2113e-1 >
Eigenvalue: -.2727
Multiplicity: 1
Eigenvector: < -.2113e-1, .9998 >

# inputting the data from Table 1 to generate the matrix plots
\[ \text{with(plots)} ; \]
\[ \text{with(LinearAlgebra)} : \]
\[ A := \begin{bmatrix} 0.5000 & 0.3333 & 0.25 & 0.800 & 0.600 & 0.2857 & 0.5455 \\ 0.07142 & 0.01645 & 0.4444 & 0.3750 & 0.4000 & 0.5000 & 0.4615 \\ 0.03541 & 0.1958 & 0.7343 & 2.9282 & 3.3034 & 5.0002 & 7.4999 \\ 0.0035 & 0.9191 & 1.6021 & 1.4182 & 4.3142 & 8.0841 & 14.129 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} \]
\[ A := \begin{bmatrix} 0.5000 & 0.3333 & 0.25 & 0.800 & 0.600 & 0.2857 & 0.5455 \\ 0.07142 & 0.01645 & 0.4444 & 0.3750 & 0.4000 & 0.5000 & 0.4615 \\ 0.03541 & 0.1958 & 0.7343 & 2.9282 & 3.3034 & 5.0002 & 7.4999 \\ 0.0035 & 0.9191 & 1.6021 & 1.4182 & 4.3142 & 8.0841 & 14.129 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} \]

# input the Maple codes to generate matrix plots of various gap lengths
\[ \text{matrixplot}(A, \text{heights} = \text{histogram}, \text{axes} = \text{boxed}) ; \]
\[ \text{matrixplot}(A, \text{heights} = \text{histogram}, \text{axes} = \text{frame}, \text{gap} = 0.15) ; \]
\[ \text{matrixplot}(A) \]

\[ F := (x, y) \rightarrow \sin(xy) ; \]
\[ \text{matrixplot}(A, \text{heights} = \text{histogram}, \text{axes} = \text{frame}, \text{gap} = 0.25, \text{color} = F) ; \]

# generation random sample for Wight noise
\[ \text{with(LinearAlgebra)} ; \]
\[ \text{with(Statistics)} ; \]
\[ X[1] := \text{RandomVariable}(\text{Normal}(0, 1)) ; \]
\[ X[2] := \text{RandomVariable}(\text{Normal}(0, 1)) ; \]
\[ X[1] := 0.03541 \cdot X[1] ; \]
\[ X[2] := 0.03541 \cdot R \]
\[ X[1] := 0.03541 \cdot R \]
Nonlinear Difference Equations and Simulation for Zooplankton-Fish Model with Noise

\[
X_2 := \begin{bmatrix} 0.1958 \_R \\ 0.1958 \_R0 \end{bmatrix}
\]
\[
X_3 := 0.7343 \cdot \begin{bmatrix} X[1] \\ X[2] \end{bmatrix}
\]
\[
X_6 := 7.4999 \cdot \begin{bmatrix} X[1] \\ X[2] \end{bmatrix}
\]
\[
A := \begin{bmatrix} 0.05376344087 & 0.0069 \\ 0.0069 & -0.2725072043 \end{bmatrix}
\]
\[
A := \begin{bmatrix} 0.05376344087 & 0.0069 \\ 0.0069 & -0.2725072043 \end{bmatrix}
\]

# input the value of the initial data
\[
> z[0] := \begin{bmatrix} 5 \\ 0.05 \end{bmatrix}
\]
\[
z_0 := \begin{bmatrix} 5 \\ 0.05 \end{bmatrix}
\]

# generate the power matrix for matrix A of various power for n= 1,2,...4
\[
> p[1] := MatrixPower(A, 1);
\]
\[
p_1 := \begin{bmatrix} 0.05376344087 & 0.0069 \\ 0.0069 & -0.2725072043 \end{bmatrix}
\]
\[
> p[2] := MatrixPower(A, 2);
\]
\[
p_2 := \begin{bmatrix} 0.00293811757418199 & -0.0015093319676670 \\ -0.0015093319676670 & 0.0743077863954020 \end{bmatrix}
\]
\[
> p[3] := MatrixPower(A, 3);
\]
\[
p_3 := \begin{bmatrix} 0.000147548919891739 & 0.000431576846131408 \\ 0.000431576846131408 & -0.0202598215189095 \end{bmatrix}
\]
\[
> p[4] := MatrixPower(A, 4);
\]
\[
p_4 := \begin{bmatrix} 0.0000109106178683386 & -0.000116589712232628 \\ -0.000116589712232628 & 0.00552392520197331 \end{bmatrix}
\]
\[
> MatrixMatrixMultiply(p[1], z[0]);
\]
\[
\begin{bmatrix} 0.269162204350000 \\ 0.0208746397850000 \end{bmatrix}
\]

# computes iterative solutions to the model using the environmental risk values in the Table 1
\[
> z[1] := MatrixMatrixMultiply(p[1], z[0]) + X1;
\]
\[
z_1 := \begin{bmatrix} 0.269162204350000 + 0.03541 \_R \\ 0.0208746397850000 + 0.03541 \_R0 \end{bmatrix}
\]
\[
> z[2] := MatrixMatrixMultiply(p[2], z[0]) + X2;
\]
\[
z_2 := \begin{bmatrix} 0.146151212725266 + 0.1958 \_R \\ -0.0038317051856490 + 0.1958 \_R0 \end{bmatrix}
\]
\[
> X3 := \begin{bmatrix} 0.7343 \_R \\ 0.7343 \_R0 \end{bmatrix}
\]
\[
> z[3] := MatrixMatrixMultiply(p[3], z[0]) + X3;
\]
### Nonlinear Difference Equations and Simulation for Zooplankton-Fish Model with Noise

\[ z_3 := \begin{bmatrix} 0.000759323441765264 + 0.7343 \_ R \\ 0.00114489315471157 + 0.7343 \_ R0 \end{bmatrix} \]

\[ > p[6] := MatrixPower(A, 6); \]

\[ p_6 := \begin{bmatrix} 2.08029257877952 \times 10^{-7} & -0.0000867999117681921 \\ -0.0000867999117681921 & 0.000410646626552184 \end{bmatrix} \]

\[ > X6 := \begin{bmatrix} 7.4999 \_ R \\ 7.4999 \_ R0 \end{bmatrix} \]

\[ > z[6] := MatrixMatrixMultiply(p[6], z[0]) + X6; \]

\[ z_6 := \begin{bmatrix} 6.06146730548798 \times 10^{-7} + 7.4999 \_ R \\ -0.0000228676245564868 + 7.4999 \_ R0 \end{bmatrix} \]

```
restart

with(LinearAlgebra):
with(Statistics):

\[ Y[1] := RandomVariable(Normal(0, 1)); \]

\[ Y[2] := RandomVariable(Normal(0, 1)); \]

\[ # \text{generate the Wight noise using environmental risk cost formula} \]

\[ > Y1 := 0.9191 \cdot \begin{bmatrix} Y[1] \\ Y[2] \end{bmatrix}; \]

\[ Y1 := \begin{bmatrix} 0.9191 \_ R \\ 0.9191 \_ R0 \end{bmatrix} \]

\[ > Y2 := 1.6021 \cdot \begin{bmatrix} Y[1] \\ Y[2] \end{bmatrix}; \]

\[ Y2 := \begin{bmatrix} 1.6021 \_ R \\ 1.6021 \_ R0 \end{bmatrix} \]

\[ > Y3 := 1.4182 \cdot \begin{bmatrix} Y[1] \\ Y[2] \end{bmatrix}; \]

\[ Y3 := \begin{bmatrix} 1.4182 \_ R \\ 1.4182 \_ R0 \end{bmatrix} \]

\[ > Y6 := 14.129 \cdot \begin{bmatrix} Y[1] \\ Y[2] \end{bmatrix}; \]

\[ Y6 := \begin{bmatrix} 14.129 \_ R \\ 14.129 \_ R0 \end{bmatrix} \]

\[ > A := \begin{bmatrix} 0.05376344087 & 0.0069 \\ 0.0069 & -0.2725072043 \end{bmatrix}; \]

\[ A := \begin{bmatrix} 0.05376344087 & 0.0069 \\ 0.0069 & -0.2725072043 \end{bmatrix} \]

\[ > z[0] := \begin{bmatrix} 5 \\ 0.05 \end{bmatrix}; \]

\[ z_0 := \begin{bmatrix} 5 \\ 0.05 \end{bmatrix} \]

\[ > p[1] := MatrixPower(A, 1); \]
```
Nonlinear Difference Equations and Simulation for Zooplankton-Fish Model with Noise

\[ p_1 := \begin{bmatrix} 0.05376344087 & 0.0069 \\ 0.0069 & -0.2725072043 \end{bmatrix} \]

\[ p_2 := \text{MatrixPower}(A, 2); \]
\[ p_2 := \begin{bmatrix} 0.0029381175741899 & -0.00150933196766700 \\ -0.00150933196766700 & 0.0743077863954020 \end{bmatrix} \]

\[ p_3 := \text{MatrixPower}(A, 3); \]
\[ p_3 := \begin{bmatrix} 0.000147548919891739 & 0.0000431576846131408 \\ 0.000431576846131408 & -0.0202598215189095 \end{bmatrix} \]

\[ p_6 := \text{MatrixPower}(A, 6); \]
\[ p_6 := \begin{bmatrix} 2.08029257877952 \times 10^{-7} & -0.00000867999117681921 \\ -0.00000867999117681921 & 0.000410646626552184 \end{bmatrix} \]

# computes iterative solutions to the model using the environmental risk cost values in the Table 1

\[ \text{MatrixMatrixMultiply}(p[1], z[0]); \]
\[ 0.269162204350000 + 0.9191_1_R \]
\[ 0.0208746397850000 + 0.9191_0_R \]

\[ z[1] := \text{MatrixMatrixMultiply}(p[1], z[0]) + Y1; \]
\[ z_1 := \begin{bmatrix} 0.269162204350000 + 0.9191_1_R \\ 0.0208746397850000 + 0.9191_0_R \end{bmatrix} \]

\[ \text{MatrixMatrixMultiply}(p[2], z[0]); \]
\[ 0.0164151212725266 + 1.6021_1_R \]
\[ -0.00383127051856490 + 1.6021_0_R \]

\[ z[2] := \text{MatrixMatrixMultiply}(p[2], z[0]) + Y2; \]
\[ z_2 := \begin{bmatrix} 0.0164151212725266 + 1.6021_1_R \\ -0.00383127051856490 + 1.6021_0_R \end{bmatrix} \]

\[ \text{MatrixMatrixMultiply}(p[3], z[0]); \]
\[ 0.000759323441765264 + 1.4182_1_R \]
\[ 0.00114489315471157 + 1.4182_0_R \]

\[ z[3] := \text{MatrixMatrixMultiply}(p[3], z[0]) + Y3 \]
\[ z_3 := \begin{bmatrix} 0.000759323441765264 + 1.4182_1_R \\ 0.00114489315471157 + 1.4182_0_R \end{bmatrix} \]

\[ \text{MatrixMatrixMultiply}(p[6], z[0]) + Y6 \]
\[ z_6 := \begin{bmatrix} 6.06146730548798 \times 10^{-7} + 14.129_1_R \\ -0.0000228676245564868 + 14.129_0_R \end{bmatrix} \]

> 
> \text{N} := 4; 
> \text{N} := 4 

\[ (A, i) \rightarrow \text{MatrixPower}(A, i, z[0]); \]
\[ z_{i+1} := (A, i) \rightarrow \text{LinearAlgebra:-MatrixPower}(A, i, z_0) \]

\[ \text{for} i \text{ from 0 to 4 while } i < N - 1 \text{ do } H(A, i) := \text{unapply(} \text{MatrixMatrixMultiply}(z[i+1], \text{z}[0])); \text{end do; } \]

\[ H := (A, i) \rightarrow \text{unapply(} \text{LinearAlgebra:-MatrixMultiply}(z_{i+1}, z_0)) \]
\[ H := (A, i) \rightarrow \text{unapply(} \text{LinearAlgebra:-MatrixMultiply}(z_{i+1}, z_0)) \]
\[ H := (A, i) \rightarrow \text{unapply(} \text{LinearAlgebra:-MatrixMultiply}(z_{i+1}, z_0)) \]

> 

\[ H(A, 0) \]
\[ (\rightarrow \text{Vector}(2, [(1) = \text{`+`}(\text{`*`}(5, \text{`*`(z[1]))}), (2) = \text{`+`}(\text{`*`}(0.5\cdot1, \text{`*`(z[1]))}))]) \]

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\( H(z[1], z[0]) \);