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Nonlinear Difference Equations and Simulation for Zooplankton-Fish Model with Noise

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Abstract: *In this paper, nonlinear difference equations for zooplankton–fish population model with noise is considered. The model is on predation of phytoplanktivore fishes on zooplankton, this is to understand the individual behaviour of the organisms as well as interaction with the environment. The model is a nonlinear logistic type of model incorporating nonlinear feeding functions. The conditions for the existence of the equilibrium points are obtained through some nonlinear equations and Diophantine equations. The conditions for local stability for the dual population investigated and results obtained .Simulation made for the dual populations when the ocean is polluted with chemical substances and oil spillage using Gaussian noise. The noise accounts for pollution of the ocean that may lead to species migration from the pollutants source. It is observed that the risk factor increases with time and makes the species to be endangered and some kind of chemo taxis effect is experienced whereby the survived species tend to migrate to region with lower concentrations of pollutants.*

Keywords: *Zooplankton-fish, model, stability, Simulation, pollution, environmental risk.*

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I. Introduction

Modelling integrates understanding of field observation, laboratory experiments' theory and computation together within wide range of physical and biological processes. All activities in the marine ecosystem have some mathematical connections hence modelling the economic resources, chemical and biological processes of the marine ecosystem is of great concern to researchers in the recent times [1,4,6,8]. There is a growing need among researchers, environmental conservatives and government agencies to understand the pollution of the ocean and attendant environmental changes that accompany it and the biodiversity of the marine ecosystem([3,6,11]).

The fish constitute great percentage of human sources of protein and the population of fishes (school) can be affected by global change in temperature, salinity, turbulence and mixing in the ocean ([8,12]). The ocean is constantly polluted by chemical substances that are found to affect the biodiversity and the food chain system of the marine biomes. Consequently, the global economic resources from the ocean are depleting with time ([5, 10, 11, 12, and 13]).

The motivation for this paper is to study the predation of phytoplankton fishes on zooplankton and to simulate the model when the ocean is polluted by spillages from crude oil and other chemical substances. The model we will consider is a kind prey-predator model. There several prey-predator models for studying predation of phytoplankton fishes on zooplankton. There are also individual based models (IBMs) used for investigating the dynamic of different spatial and temporal pattern and transportation of fish eggs and larvae ([4]). There are models of trophodynamics of species, surrogate of biomass and multispecies models and circulation of nutrients in aquatic systems ([2]).

The main food item of pelagic fishes is zooplankton which has also been influenced by the change of salinity that is thought to have caused a decrease of large zooplankton in the Gulf of Finland. It is found that the distribution pattern of phytoplanktivore fish changes according to current [8]. Horst et. al. [4] considered plankton interaction-diffusion model to study the influence of fish on the spatial-temporal pattern for the zooplankton population. The local properties such as the excitation and stability of spatially uniform stability in the absence of diffusion were analysed.

Arnfinn [1] studied the effect of fish predation on zooplankton community in an oligotrophic lake with addition of artificial fertilizer .It was found that there was increase in the biomass of zooplankton and fish predation and the distribution pattern of planktivore fish change according to current direction. Nutrient–phytoplankton–zooplankton–detritus (NPZD) models have been developed for more than 40 years (e.g. Andrew [2]; Katja et.al, [6]; Ute et al. [12]).

The lack of general models for zooplankton population has become a major setback to carbon flux models for simulating the effects of zooplankton grazing and faecal production on carbon storage and export flux, and for food-web models to simulate biomass transfer from lower to higher tropic level organisms including fish populations([2,7,9]).

The modelling of marine zooplankton has made great progress over the two last decades covering a large range of representations from detailed individual processes to functional groups. A new challenge is to dynamically represent zooplankton within marine food webs coupling lower tropic levels to fish and to thereby further our understanding of the role of zooplankton in global change ([1,8,11 ,12]).

The weight and composition of zooplankton varies significantly, and it can be attributed to age, seasonal difference, geographical distribution, or environmental conditions (Makoso [9]).eddies and fronts are generally thought to have phytoplankton abundance than adjacent waters through an increase in phytoplankton stock (Thibault et al. [11])

The eggs and dry weight of zooplankton are found to be nonlinear in the Bariri reservoir in the Sao Paulo state in Brazil. From field experiment(see Gozalez [3]) in the middle of Tieli river in Sau Paulo in Brazil, the diameter of cladoceran and copepod species sizes are found to be 108.6 micron (Keratella tropical) to 2488.6 micron (females of Argyrodiaptomus azevedol) while dry weight varies between 0.025 micron(K.tropical) and 51.250 micron (female azevedol) egg diameter.

In this paper, nonlinear difference equations for zooplankton –fish population model with noise is proposed. The model is on predation of phytoplanktivore fishes on zooplankton; our aim is to understand the individual organism, behaviour as well as interaction with the environment. The conditions for the existence of the equilibrium points obtained through some nonlinear equations and Diophantine equations and conditions for local stability for the dual population are obtained. The model considered is a type nonlinear stochastic equation treated as a type of Prey-predator model with pursuit and search parameters as against the continuous and impulsive version found in the literature (see for example [13]).

The use of nonlinear difference equations with nonlinear feeding functions used in this paper is expected to offer interesting platform for research on prey-predator models for species and should open-up the window of future research on species in polluted environments.

II. Preliminary Definitions and Notations

2.1 **Notations**

We will make use of the following notations

u_{1t} Velocity of the fish

u_{2t} Velocity of the zooplankton

d_{1t} Search distance at the period *t* by the fish

d_{2t} Escape distance at the period *t* of the zooplankton

$a_1 = m_x r^2 u_{2t} x_t$, *r* is the search radius and m_x is mass of zooplankton

$b_1 = m_y r^2 u_{1t}$, m_y is mass of the fish

a_{1t} Volumetric search velocity for fish

a_{2t} Volumetric escape velocity for the zooplankton

x_t Population of zooplankton at time t measured in months

y_t Population of the fish at time t measured in months.

$E\{u_t\} = \mu$ The expected value for u_t

$E\left((y_t - \mu)(y_{t-\tau} - \mu\right) = Cov\left(y_t, y_{t-\tau}\right) = \gamma_\tau$ Auto covariance with lag τ while $d = (.,.)$ is the common divisor of $(.,.)$.

2.2 **NONLINEAR STOCHASTIC EQUATIONS**

Consider the stochastic equation form

$$Z_t = f\left(Z_{t-1}\right) + \in_t, -\infty < t < \infty \tag{1}$$

f is an arbitrary function such that $f : R = (-\infty, \infty) \rightarrow R^n$. It is a special case of linear model in which $f(y) = \Theta \text{y}$. Analyses of the properties of eq. (1) have been subject of many researches in the recent times. Removing the noise from the model, we obtain the following deterministic model

$$z_t = f\left(z_{t-1}\right)(t \geq 1) \tag{2}$$

where z_t in the lower case is for the deterministic form as against the stochastic form in the eq. (1) where Z_t is used .We are not going to make discussion on this but will focus on some features of the deterministic model but later on we will consider the simulation for the fish-zooplankton model which special form of the eq. (2).

Definition 1

A solution z^* of the equation $z = f(z)$ is an equilibrium point of the difference equation in the eq. (2), if $f(z^*) = 0$ .In many cases $z^* = 0$ is an equilibrium point this is a trivial case, but for non-trivial case the equilibrium point $z^* \neq 0$ are often be of interest. An equilibrium point z^* is said to be asymptotically stable or an asymptotical stable limit point for (almost) all initial value, z_0 such that $z_t \rightarrow z^*$ as $t \rightarrow \infty$.

Definition 2

An equilibrium point z* is said to be weakly stable if given an infinitesimal perturbation from the equilibrium point z the process returns to the equilibrium point i.e. given that $z_0 = z^* + h \Rightarrow z_t \rightarrow z^*$ as $t \rightarrow \infty$,h $\rightarrow 0$ $(t \geq 0)$

Definition 3

A process is said to be weakly stationary (or covariance) stationary or in wide sense stationary if $E\{u_t\}$ is constant over time and $\text{cov}(u_t, u_{t-\tau})$ depends only on the lag τ and not t. If $E\{u_t\} < \infty$, $\text{cov}(u_t, u_{t-\tau}) < \infty$ then strictly stationary process implies weakly stationary process but weakly stationary process does not imply strictly stationary process.

Definition 4

Let u_t be independently and identically distributed (i.i.d), with common variance σ^2 such that $\gamma_0 = \sigma^2$, $\gamma_\tau = 0$ $(\gamma \neq 0), \varphi_0 = 1$ and $\varphi_\text{r} = 0$ $(\tau \neq 0)$.

u_t is said to be white noise process if it's stationary and the u_t is pairwise uncorrelated.

A sequence $(\xi_n)_{n \in \aleph}$ of radium variables is said to be convergent in mean to random variable ξ if $E\{(\xi_n - \xi)\} \xrightarrow[n \to \infty]{} 0$. We will simply write $\xi_n \xrightarrow{m} \xi$.

III. Statement of Problem

The model we will consider is a nonlinear logistic type of model for the zooplankton-fish population which incorporated nonlinear feeding functions $F_i, i = 1, 2$ .The feeding function F_1 takes into consideration the search rate and search velocity of the fish on the zooplankton and the escape rate of the zooplankton from the fish. The model also includes escape velocity of the zooplankton from predation of fish and feeding rate function F_2 for the zooplankton.

Now consider the nonlinear zooplankton-fish population

$$\left.\begin{aligned} x_t &= a_1\left(\frac{k_1 - x_{t-1}}{k_1}\right)F_1 + b_1 y_{t-1} + \in_{1t} \\ y_t &= a_2\left(\frac{k_2 - y_{t-1}}{k_2}\right)F_2 + b_2 y_{t-1} + \in_{2t} \end{aligned}\right\} (3)$$

Where the feeding functions $F_i, i = 1, 2$ are given as

$$F_1 = \left\{\left(\frac{1}{1+\sum_t a_{1t}x_{t-1}}\right) + \frac{d_{1t}a_{1t}x_{t-1}}{\sum_t a_{1t}u_{1t}x_{t-1}} + t_r\right\}^{-1}$$

and

$$F_2 = \left\{\left(\frac{1}{1+\sum_t b_{1t}y_{t-1}}\right) + \frac{d_{2t}b_{1t}y_{t-1}}{\sum_t b_{1t}u_{2t}y_{t-1}} + t_r\right\}^{-1}$$

To ensure capture of zooplankton by the fish, we must have $\sum_t a_{1t}x_{t-1} = \sum_t b_{1t}y_{t-1}$ for every t. The noise accounts for pollution of the ocean by chemical substances or other forms of pollutants that may lead to chemo taxis effect of the dual population, that is, forcing the species to migrate toward the region of low chemical concentration or away from the region of high chemical concentration.

Now consider the model in the eq. (3) and let $z_t = (x_t, y_t)^T$ then the model can be represented in deterministic form for $z = (x, y), \in_{it} = 0, i = 1, 2.$ Let $f(x,y) = \begin{pmatrix} a_1\left(\frac{k_1 - x}{k_1}\right)F_1 + b_1 xy \\ a_2\left(\frac{k_2 - y}{k_2}\right)F_2 + b_2 xy \end{pmatrix}$. If $x = 0, y = 0$ then

$x_t = a_1F_1 = a_1t_r, y_t = a_2F_2 = a_2t_r$. Therefore the per capita growth rate of the zooplankton is $\frac{a_1}{t_r} - 1$, and the per capita growth rate of the fish is $\frac{a_2}{t_r} - 1.$

It is worthy of note that the presence of fishes is causing the zooplankton population to decrease.

IV. Methods

Assuming that f is continuously differentiable and h_{t-1} is infinitesimally small and from the eq. (2) $z_t = f(z_{t-1}) = f(z^*) + \frac{\partial f}{\partial z} f_{t-1} = z^* + \frac{\partial f}{\partial z} h_{t-1}$ where $\frac{\partial f}{\partial z}$ is the matrix of partial derivative with (i, j) element $\frac{\partial f_j}{\partial z_j}$ are evaluated at the equilibrium points; if λ_i are the eigenvalues to $\frac{\partial f}{\partial z}$ and $\left|\lambda_i\left(\frac{\partial f}{\partial z}\right)\right| < 1$, then $h_t \to 0$ as $t \to 0^+$

In order to obtain the equilibrium points for the model in the eq. (3), we set the right hand side of the equation to zero and we have the following equations:

$$\left.\begin{aligned} a_1\left(\frac{k_1 - x^*}{k_1}\right)F_1 + b_1 x^* y^* = 0 \\ a_2\left(\frac{k_2 - x^*}{k_2}\right)F_2 + b_2 x^* y^* = 0 \end{aligned}\right\}. \quad (4)$$

We found that the equilibrium points are

$E_1 = (0,0), E_2 = (0, y^*) = (0, k_2), E_3 = (x^*, 0) = (k_1, 0)$ and $E_4 = (x^*, y^*)$.

To obtain E_4 involves solving the following nonlinear equation

$$b_1 a_1\left(\frac{k_1 - x^*}{k_1}\right)F_1 - b_2 a_2\left(\frac{k_1 - y^*}{k_2}\right)F_2 = 0. \quad (5)$$

Solutions to the eq. (5) is intractable because of the nonlinear nature of the feeding functions $F_i, i = 1,2$ .Solving the eq. (5) for x^* and y^* can be done by the use of approximation method. If we have that $a_i, k_i,\ i = 1,2,\ b_2, x^*, F_1$ and F_2 are integers we can use number theory to solve it. However, let $z_t = (x_t, y_t)$, x_t and y_t are random variables and u_t t can be obtained through experimental fittings, time series or some kind of functions.

4.2 Risk Assessment as result of chemical pollutions

The estimation of the risk for various environmental hazard factors in the pollution of the ocean and the equivalent environmental cost can be computed from the following formulae

$$\text{Risk } = u_i = \left(\sum_t f_{it}\right)\left(\sum_t w_{it}\right),\ i = 0,1,2,\ldots,n \quad (6)$$

And the equivalent environmental cost

$$u_{it}^{\varepsilon} = \left(\sum_t f_{it}\right)\left(\sum_t w_{it}\right)^{\varepsilon},\ i = 0,1,2,\ldots,n, e \geq 1 \quad (7)$$

Where $\left(\sum_t f_t\right)$ are the cumulative weight frequencies of the types of hazard present at the period *t* and *n* is the number of hazard factors.

$\left(\sum_t w_t\right)$ are the cumulative weights of the types of hazard at present at the period *t*.

For example, weight w_1 is for chemical hazard and w_2 is for biological hazard, etc.

Each of the weights can be estimated as number of death of the specie at the given period when the hazard occurs divided by total death recorded during estimation period. This is expressible in the time scale of month or year. For example, if there are cases of hazard that occurred in given month as a result chemical pollution and a total 10 hazards occurred during experiment period, then $w_1 = \frac{2}{10}$.

V. Results

Let the Jacobean matrix, $J(x^*, y^*) = A$ and A is $n \times n$ matrix and $E_t = (\varepsilon_{1t}, \varepsilon_{1t})$ is the noise variable. Then we can find the approximate solution to the eq. (1) using the following iterative equations with noise.

$$Z_t = AZ_{t-1} + u_t E_{t-1}$$

For $t = 0,1,2,\ldots,k$,we have

$$
\begin{aligned}
z_1 &= Az_1 + u_0E_0 \\
z_2 &= Az_1 + u_1E_1 = A\left(Az_0 + u_0E_1\right) \\
&= A^2z_0 + Au_0E_0 + u_1E_1 \\
z_3 &= Az_2 + u_2E_2 = A^3z_0 + A^2u_0E_0 + Au_1E_1 \\
&\vdots \\
z_{t+1} &= A^tz_0 + A^{t-1}u_0E_0 + Au_1E_1 + \ldots + u_{t-1}E_{t-1}.
\end{aligned}
$$

In the Lemma1, we intend to estimate the bound for the ratio $\frac{F_1}{F_2}$ of the feeding functions for given two equilibrium points x^* and y^* and the upper bound for $\left|k_2 - y^*\right|$.

Lemma 1

Let x^* and y^* be equilibrium points for the eq. (5) such that

$$b_1a_1\left(\frac{k_1 - x^*}{k_1}\right)F_1 - b_2a_2\left(\frac{k_1 - y^*}{k_2}\right)F_2 = 0$$

Assume that F_1 and F_2 are such that

$$\left|\frac{F_1}{F_2}\right| \le \frac{e^y + \frac{d_2}{u_{2t}} + t_r}{e^{-x}\left(\sum_t a_t x_t\right)^{-1}}$$

Then

$$\left|k_2 - y^*\right| \le \frac{k_2a_1b_2}{k_1b_1a_2}\left|\frac{F_1}{F_2}\right|\left|k_1 - x^*\right|$$

$$\le \frac{k_2a_1b_2}{k_1b_1a_2}\left[\frac{e^y + \frac{d_2}{u_{2t}} + t_r}{e^{-x}\left(\sum_t \alpha_t x_t\right)^{-1}}\right]\left|k_1 - x^*\right|.$$

Proof.

For the proof, we process as follows:

$$\frac{F_1}{F_2} = \frac{e^{-y}\left(\sum_t b_t y_t\right)^{-1} + \frac{d_2b_{2t}y_t}{\sum_t b_{2t}u_{2t}y_t} + t_r}{e^{-x}\left(\sum_t \alpha_t x_t\right)^{-1} + \frac{a_{2t}x_td_t}{\sum_t a_{1t}x_tu_{2t}} + t_r}.$$

From eq. (5), we have

$$\left|\frac{F_1}{F_2}\right| \le \frac{e^{y} + \frac{\frac{d_2}{u_{2t}}}{\sum_t b_{2t} u_t y_t} \frac{d_2}{u_{2t}} + t_r}{e^{-x}\left(\sum_t \alpha_t x_t\right)^{-1}}$$

$$\le e^{x^*}\left(e^{y^*} + \frac{d_2}{u_{2t}} + t_r\right)\sum_t \alpha_t x_t$$

Then if $\left|k_1 - x^*\right| \le r_x$, then it implies that

$$\left|k_2 - y^*\right| \le \left(1 + x^*\right)\left[\left(1 + y^*\right) + \frac{d_2}{u_{2t}} + t_r\right]\tau$$

Using Taylor series, we have

$e^{y} \approx 1 + y^*, e^{x^*} \approx 1 + x^*$ and $\tau = \sum_t \alpha_t x^*$

Therefore,

$$\left|k_2 - y^*\right| \le \left(1 + x^*\right)\left[\left(1 + y^*\right) + \frac{d_2}{u_{2t}} + t_r\right]\tau. \qquad \square$$

This ends the proof of the Lemma.

In the next Lemma, we will determine the conditions for solvability of the Diophantine equations characterising the equilibrium equations in the eq. (4) when $a_i, b_i, F_i, i = 1,2$ are integers.

Lemma 2

Let $\alpha_1 = a_1 b_2,\ \alpha_2 = a_2 b_2$ such that $n,\ \alpha_1, \alpha_2, F_1, F_2, x^*$ and y^* are integers such that

$$\frac{\alpha_1}{k_1} F_1 x^* + \frac{\alpha_2}{k_2} F_2 x^* = \alpha_1 - \alpha_2 \ (\text{mod } n).$$

Then the above Diophantine equation is solvable if and only if $d\left(\frac{\alpha_1}{k_1} F_1,\ \frac{\alpha_2}{k_2} F_2\right)$ is such that $d\left(\alpha_2 - \alpha_2\right)$.

However, if $d\left(\frac{\alpha_1}{k_1} F_1,\ \frac{\alpha_2}{k_2} F_2\right) = 1$ then there exist $x^*, y^* \in \square$ such that $\frac{\alpha_1}{k_1} F_1 x^* + \frac{\alpha_2}{k_2} F_2 x^* = 1$.

Proof

If $x^*, y^* \in \square$ are equilibrium point too.

Therefore, $\left(\alpha_1 - \alpha_2\right) x^*, \left(\alpha_1 - \alpha_2\right) y^*$ are equilibrium points.

Therefore,

$$\frac{\alpha_1 F_1}{k_1}\left(\alpha_2 - \alpha_1\right) x^* + \frac{\alpha_2 F_2}{k_2}\left(\alpha_2 - \alpha_1\right) y^* = \alpha_2 - \alpha_1.$$

If x_0^* and y_0^* are another solution to the eq. (5), then

$$\left(\frac{\alpha_1 F_1}{k_1}x^* + \frac{\alpha_2 F_2}{k_2}y^*\right) - \left(\frac{\alpha_1 F_1}{k_1}x_0{}^* + \frac{\alpha_2 F_2}{k_2}y_0{}^*\right) = \alpha_2 - \alpha_1 - \alpha_2 - \alpha_1 = 0$$

Therefore,

$$\left(\frac{\alpha_1 F_1}{k_1}(x^* - x_0{}^*) + \frac{\alpha_2 F_2}{k_2}(y^* - y_0{}^*)\right) = 0.$$

This is equivalent to

$$-\frac{\alpha_1 F_1}{k_1} / \frac{\alpha_2 F_2}{k_2} = \frac{y^* - y_0^*}{x^* - x_0^*}$$

or $\frac{-\alpha_1}{k_1}F_1 t = x^* - x_0^*$

$$\frac{-\alpha_2}{k_2}F_2 t = y^* - y_0^*.$$

Therefore,

$$y^* = y_0^* + \frac{\alpha_2 F_2}{k_2}t,\ x^* = x_0^* + \frac{\alpha_1 F_1}{k_1}t,\ t \in \square.$$

For every $d = \left(\frac{\alpha_1}{k_1}F_1, \frac{\alpha_2}{k_2}F_2\right)$ since t is arbitrary, hence, the Diophantine equation has infinitely many solutions. However, if $d = \left(\frac{\alpha_1}{k_1}F_1, \frac{\alpha_2}{k_2}F_2\right) \neq 1$ then eq. (1) transforms to

$$\frac{\frac{\alpha_1 F_1}{k_1}}{d}x^* + \frac{\frac{\alpha_2 F_2}{k_2}}{d}y^* = \frac{\alpha_2 - \alpha_1}{d}$$

And solution is $x^* = x_0^* - \frac{k_1}{d} - t$ and $y^* = y_0^* - \frac{k_2}{d} - t$. □

This ends the proof.

In the 4.2 section we will consider linearization method for the nonlinear zooplankton-fish model when the underlying variables and the feeding functions are continuous.

4.2 Linearization Method of the Nonlinear Equations

Let $F_i,\ i = 1,2$ be continuous functions, therefore, we can linearize the nonlinear equation in the eq. (1). Let the Jacobean of *f* with respect (*x*,*y*) be defined by

$$J(x,y) = \frac{\partial f}{\partial z} = \frac{\partial(f_1, f_2)}{\partial(x,y)} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} \\ \frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial y} \end{pmatrix}.$$

$$J(x,y) = \begin{pmatrix} \frac{a_1}{k_1}F_1 + a_1\left(\frac{k_1 - x}{k_1}\right)\frac{\partial F_1}{\partial x} + b_1 y & \left(\frac{k_1 - x}{k_1}\right)\frac{\partial F_1}{\partial y} + b_2 x \\ a_1\left(\frac{k_1 - x}{k_1}\right)\frac{\partial F_1}{\partial x} + b_1 y & -\frac{a_1}{k_1}F_2 + a_2\left(\frac{k_1 - x}{k_1}\right)\frac{\partial F_2}{\partial y} + b_2 y \end{pmatrix}.$$

Without ambiguity we will drop t in the indices.
Therefore, for the equilibrium $E=(k_1,0), E(0,k_2)$ and $E(0,0)$ and we obtain the corresponding Jacobean for $J(0,0)$ and $J(k_1,0)$ as follows:

$$J(0,0)=\begin{pmatrix} \frac{a_1}{k_1}-\frac{a_1\sum a}{(1+t_r)(1+\sum b)^2} & 0 \\ 0 & -\frac{a_1}{k_2}+\frac{a_1\sum b}{(1+t_r)(1+\sum b)^2} \end{pmatrix}$$

$E(k_1,0)$ and $E(0,k_2)$

$$J(k_1,0)=\begin{pmatrix} \frac{a_1}{k_1}F_1(k_1,0) & b_1k_1 \\ b_1k_1 & \frac{-a_1}{k_1}F_2(k_1,0)+a_1\frac{\partial F_2(k_1,0)}{\partial y}-b_2x \end{pmatrix}$$

$$J(0,k_2)=\begin{pmatrix} \frac{a_1}{k_1}F_1(0,k_2)+a_1(\frac{k_1-0}{k_1})\frac{\partial F_1(0,k_2)}{\partial x} & a_1\frac{\partial F_1(0,k_2)}{\partial x} \\ a_2\frac{\partial F(0,k_2)}{\partial y} & -a_2\frac{\partial F_2(0,k_2)}{\partial y}+\frac{a_2}{k_2}(k_2-y) \end{pmatrix}$$

where

$$F_1(k_1,0)=(1+t_r)^{-1}$$

$$F_1=\left[\frac{e^{-x}}{1+\sum\alpha x}+\frac{adx}{u\sum\alpha x}+t_r\right]^{-1}$$

Therefore,

$$\frac{\partial F_1}{\partial y}=0$$

$$\frac{\partial F_2}{\partial x}=0$$

$$\frac{\partial F_2}{\partial y}=\frac{-F_2^2}{(1+\sum ub)^2}\left[bsd_2u\sum bx-bsd_2\sum bu+\sum bye^{-y}+due^{-y}\right]$$

In our next Theorem, we determine the conditions for weakly stability of the equilibrium points for the model

Theorem 1

Let there exist constants a, k_1 and the resting time t_r such that the following inequalities are satisfied:

$$a\left(1+t_r\right)\left(1+\sum a\right)^2 - ak_1\sum a < k_1\left(1+t_r\right)\left(1+\sum a\right)^2$$
$$-ak_1\sum a < k_1\left(1+t_r\right)\left(1+\sum a\right)$$
$$-a\left(1+t_r\right)\left(1+\sum b\right)^2 + ak_2\sum b < \left(1+t_r\right)\left(1+\sum b\right)^2$$
$$-ak_2\sum b < k_2\left(1+t_r\right)\left(1+\sum b\right).$$

Then the prey-predator model is locally stable at the equilibrium points

$E=(0,0)$ and $E=\left(0,k_2\right)$ and stable if $\left|\frac{a_1}{k_1}F_2\left(0,k_2\right)\right|<1.$

Proof
The eigenvalues for eq. (5) are

$$\lambda_1=\frac{a}{k_1}-\frac{a\sum b}{(1+t_r)(1+\sum a)^2}$$

and

$$\lambda_2=\frac{a_1}{k_1}+\frac{a\sum b}{(1+t_r)(1+\sum b)^2}.$$

Therefore, for the equilibrium point $E(0,k_2)$ the condition for local stability is to require that $\left|\lambda_i\right|<1, i=1,2$ for the eigenvalues of $J\left(0,k_2\right)$ which is obtained from

$$\left|\begin{pmatrix} \frac{a_1}{k_1}(1+t_r)^{-1}-\lambda & b_1k_1 \\ b_1k_1 & \frac{-a_2}{k_2}(1+t_r)^{-1}-b_2k_1-\lambda \end{pmatrix}\right|=0.$$

Applying the condition that $\left|\lambda_i\right|<1, i=1,2$ for the eigenvalues $J(k_1,0)$ and $J(0,k_2)$ leads to the inequalities given in the hypothesis and hence the stability of the equilibrium points are established.

VI. Numerical Simulation of the Model

We will simulate the model using Maple 18 software by making use of Maple Library on Linear Algebra and Statistics. We consider a hypothetical situation and let us choose the following parameters:

$$k_1=2000, k_2=200, t_r=\frac{1}{50}, b_1=0.02, y_0=30.$$

That is, we start the simulation with 30 fishes and the resting time to be $t_r=1/50$ hours.

Then the matrix $\begin{bmatrix} \frac{a_1}{k_1(1+t_r)} & b_1k_1 \\ b_1k_1 & \frac{-a_2}{k_2}(1+t_r)^{-1}-b_2k_1 \end{bmatrix}$ after inputting the parameters into Matrix *A* (see the Appendix) we have

$A := \begin{bmatrix} 0.05376344087 & 0.0069 \\ 0.0069 & -0.2725072043 \end{bmatrix}$: When we invoke the Maple Code $EigenPlot(\text{A});$ we obtain the eigenvalues and eigenvectors to the matrix A as
Eigenvalue: .5391e-1
Multiplicity: 1
Eigenvector: < .9998, .2113e-1 >

Eigenvalue: -.2727
Multiplicity: 1
Eigenvector: < -.2113e-1, .9998 >

We also make use $Eigenvalues(A, output='list');$ And the set of eigenvalues were obtain to be

$$\begin{bmatrix} 0.05390929749 \\ -0.2726530609 \end{bmatrix}$$

We invoked the Maple Code $EigenPlot(\text{A});$ we obtained the Figures in the Fig.1& Fig.2.

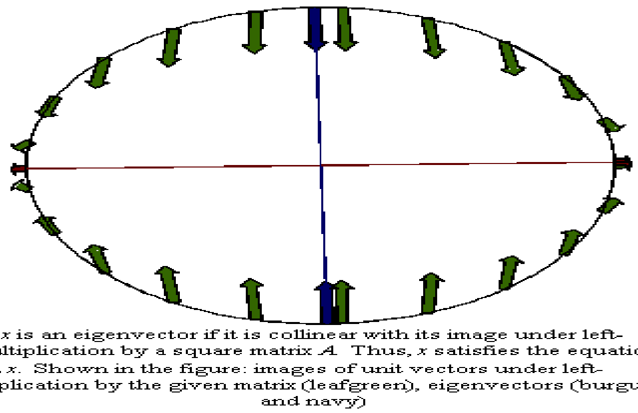

Fig. 1: Eigenplot for matrix A

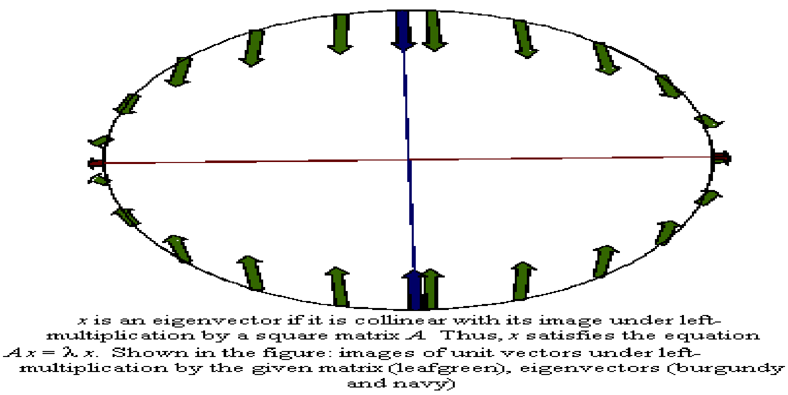

Fig. 2: Eigenplot2 for matrix A

We note that the equilibrium is locally stable since the eigenvales 0.5391 and -0.27727.The field of eigenvalues are toward the inside of the circle with radius at the centre (z=0)and modulus of the eigenvaluesare less than one(unity) as long as the absolute value of $|u_t E_t|, t = 0,1,2,...$ are very small.We note that $\|A\| \le 1$ and $z_t \to u^* E, \text{as } t \to \infty$.
Suppose we consider the hypothetical situation where the environment is polluted with spillage from oil or chemical substance and the data from the field is obtain and recorded in the Table 1.The data are obtained as described in the section 3 and u_t and u_t^s are obtain using the eq.(5) and the eq.(6). The data in Table 1 were plotted to obtain in the Fig.3.

Table 1: Environmental risks and frequencies of various hazard weights

t	0	1	2	3	4	5	6
f_t	0.5000	0.3333	0.2500	0.8000	0.6000	0.2857	0.5455
w_t	0.07142	0.01645	0.4444	0.3750	0.4000	0.5000	0.4615
u_t	0.03541	0.1958	0.7343	2.9282	3.3034	5.002	7.4999
u_t^s	0.0035	0.9191	1.6021	1.4182	4.3142	8.0841	14.129

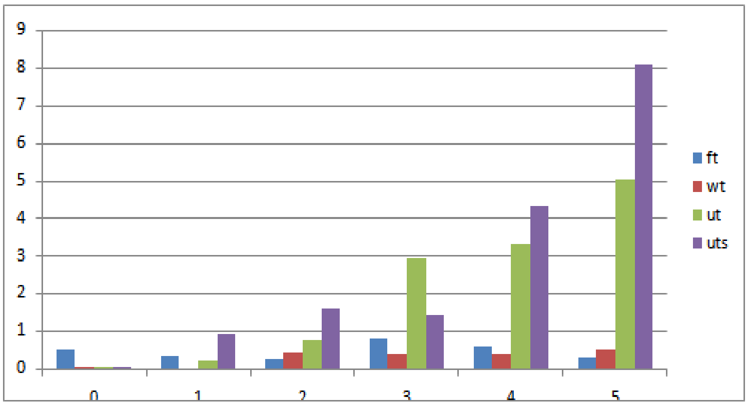

Figure 3: Bar Charts environmental risks and frequencies of various hazard weights for the fish

In the Figure 3 the blue bars are frequencies of the occurrence of hazards while the brown coloured bars are the weights values for the hazards. The green colour is the risk values and the purple is the colour of the environmental risk cost. We note as time increases the risk factor increases with time as the figure shows, this make the species to be endangered and some kind of chemotaxis effect is experiences whereby the survived species tend to migrate to region with lower concentrations of pollutants.

The matrix plots in the Fig.4 we plotted using the data in the Table1 for us to have 3-D view of the frequencies of the fish and the corresponding risk assessment in the eq.(5) and the eq.(6). In Matrix Plot 1 and Matrix plot 2 the gap between the histogram are of length 0.15 and 0.25 inches respectively while the Matrix Plot4 used the plot colour of map sin *xy*.The first set of histograms in the Matrix Plots are the values of u_t^s in the Table 1 and next histograms are the values of u_t , w_t and f_t respectively.

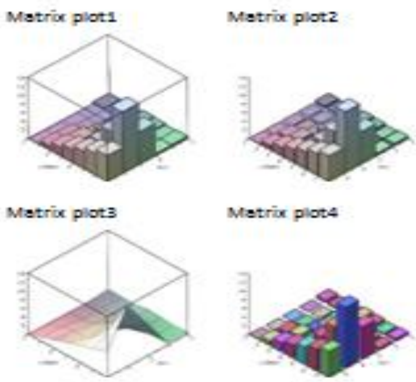

Fig.3: Matrix plots environmental risks and frequencies of various hazard weights for the fish

Table 2: Environmental risks and frequencies of various hazard weights for the zooplankton

T	0	1	2	3	4	5	6
f_t	0.400	0.3750	0.5556	0.7500	0.5330	0.4615	0.5334
w_t	0.0200	0.0305	0.3700	0.3734	0.4750	0.5071	0.4602
u_t	0.0080	0.1783	0.7067	1.8820	3.6057	4.3236	6.7342
u_t^s	0.0004	0.0449	0.5354	1,7367	4.6038	5.6784	11.0906

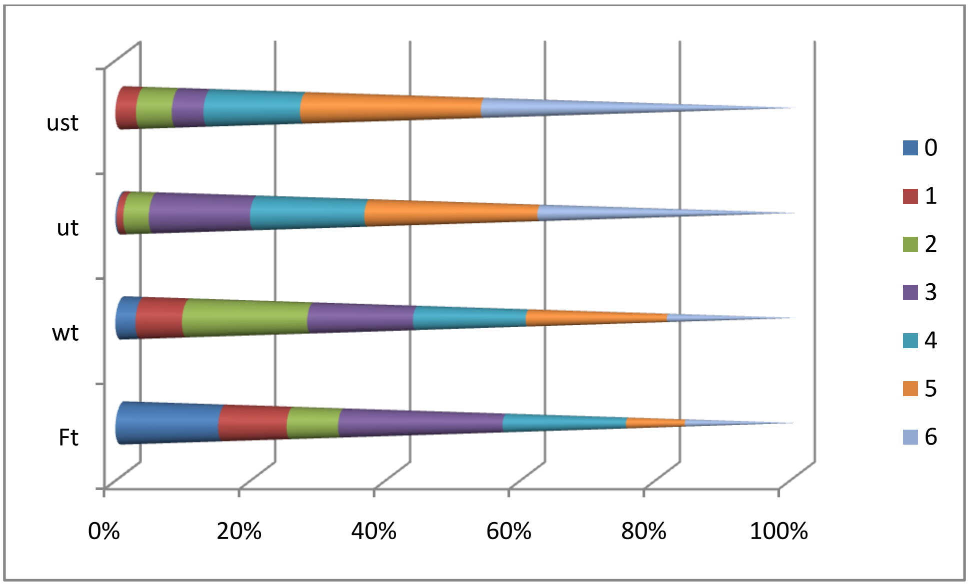

Fig. 4: Environmental risks and frequencies of various hazard weights for the zooplankton

The Fig 4 is Bar charts generated from Environmental risks and frequencies for various hazard weights and periods for the zooplankton. We generate the white noise from environmental risk factor and environmental risk cost factor

for the simulation using the vectors:

$$u_{tE_t=}\begin{bmatrix}0.03541 & 0.1958 & 0.7343 & 2.9282 & \ldots & 7.499\end{bmatrix}\begin{bmatrix}X_1\\X_2\\X_3\\X_4\\\vdots\\X_6\end{bmatrix}$$

$$u^s{}_{tE_t=}\begin{bmatrix}0.035 & 0.9191 & 1.6021 & 1.4182 & \ldots & 14.129\end{bmatrix}\begin{bmatrix}Y_1\\Y_2\\Y_3\\Y_4\\\vdots\\Y_6\end{bmatrix}$$

Where

$X_i, Y_i \in N(0,1), i = 1,2,\ldots,6$

We computed iterative solutions to the model using the environmental risk values in the Table 1 and the following iterative equation:

$z_{t+1} = A^t z_0 + A^{t-1}u_0E_0 + Au_1E_1 + \ldots + u_{t-1}E_{t-1}.$

Using Maple 17, see the Appendix, we obtained:

$>z[1] := MatrixMatrixMultiply(p[1], z[0]) + X1;$

$$z_1 := \begin{bmatrix}0.269162204350000 + 0.03541\ \_R\\0.0208746397850000 + 0.03541\ \_R0\end{bmatrix}$$

$>z[2] := MatrixMatrixMultiply(p[2], z[0]) + X2;$

$$z_2 := \begin{bmatrix}0.0146151212725266 + 0.1958\ \_R\\-0.00383127051856490 + 0.1958\ \_R0\end{bmatrix}$$

>

$$X3 := \begin{bmatrix}0.7343\ \_R\\0.7343\ \_R0\end{bmatrix}$$

$>z[3] := MatrixMatrixMultiply(p[3], z[0]) + X3;$

$$z_3 := \begin{bmatrix}0.000759323441765264 + 0.7343\ \_R\\0.00114489315471157 + 0.7343\ \_R0\end{bmatrix}$$

Where $R, R_0 \in N(0,1)$.

We computed iterative solutions to the model using the environmental risk cost values in the Table 1 and the following iterative equation:

$z_{t+1} = A^t z_0 + A^{t-1}u^s{}_0E_0 + Au^s{}_1E_1 + \ldots + u^s{}_{t-1}E_{t-1}.$

We obtained:

$z[1] := MatrixMatrixMultiply(p[1], z[0]) + Y1;$

$$z_1 := \begin{bmatrix}0.269162204350000 + 0.9191\ \_R\\0.0208746397850000 + 0.9191\ \_R0\end{bmatrix}$$

$>z[2] := MatrixMatrixMultiply(p[2], z[0]) + Y2;$

$$z_2 := \begin{bmatrix} 0.0146151212725266 + 1.6021\_R \\ -0.00383127051856490 + 1.6021\_R0 \end{bmatrix}$$

$>z[3] := MatrixMatrixMultiply(p[3], z[0]) + Y3$

$$z_3 := \begin{bmatrix} 0.000759323441765264 + 1.4182\_R \\ 0.00114489315471157 + 1.4182\_R0 \end{bmatrix}$$

$>z[6] := MatrixMatrixMultiply(p[6], z[0]) + Y6$

$$z_6 := \begin{bmatrix} 6.06146730548798\ 10^{-7} + 14.129\_R \\ -0.0000228676245564868 + 14.129\_R0 \end{bmatrix}$$

Where $R, R_0 \in N(0,1)$.

We note that the iteration solutions for the fish and zooplankton depend on the values of random numbers $R, R_0 \in N(0,1)$.Therefore the model has infinitely many solutions. From the solution is obvious that population of the two species tends to be decreasing with time. For survival of the species there got to be migration from region of pollution to a region that would sustain the growth and flourishing of the species.

VII. Conclusion

Simulation is made for the zooplankton and fish populations when the ocean is polluted with chemical substances and oil spillage and the noise is Gaussian. We note that the iteration solutions for the fish and zooplankton depend on the values of random numbers $R, R_0 \in N(0,1)$.Therefore the model has infinitely many solutions. From the solution is obvious that population of the two species tends to be decreasing with time. We emphasised that the noise accounts for pollution of the ocean by the chemical substances and oil spillage that may lead to species migration from the pollutants source. It was observed that the environmental risk factor used increases with time and this will make the species to be endangered and some kind of chemo taxis effect will be experienced whereby the survived species tend to migrate to region with lower concentrations of pollutants. The use of nonlinear difference equations with nonlinear feeding functions used in this paper will offer interesting platform for research on prey-predator models for species and should open the window of opportunities for future research on species in polluted environments.

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References

[1] Arnfinn Langeland Interactions between zooplankton fish in a fertilizer lake. Holarctic Ecology 5: 273-310, Copenhagen 1982.
[2] Andrew M. Edwards, ‘‘Adding detritus to a nutrient-phytoplankton-zooplankton model’’. A Dynamical systems approach. Journal of Plankton Research (2001) 23 (4): 389-413. DOI:10.1093/PLANKT/23.4.389.
[3] Gonzalez E.J., Matsumura Tundisi J.G.Size, and dry weight of main zooplankton species in Bariri reservoir (SP, Brazil).Braz J.Biol 2008.Feb 68(1), 69-75.
[4] Horst M, Birgit R., Malaak Alexander B.M., Dmitry A.T. Sergiel V. P. Spatio- temporal pattern formation in coupled models of plankton dynamics and fish school motion. Nonlinear Analysis Real World Application 1 (2000) 53-67.
[5] Katja Fennel, Martin Losch and Jens Schröter, Manfred Wenzel, Testing a marine ecosystem model: sensitivity analysis and parameter optimization. Systems, Volume, February 2001, Pages 45–63
[6] Kylie A. Pitt , Anne-Laure Clement , Rod M. Connolly , Delphine Thibault-Botha Predation by jellyfish on large and emergent zooplankton: implications for benthic-pelagic coupling. Estuarine, Coastal and Shelf Science 76 (2008) 827-833.
[7] Kormlovs G., Sidrevics L., and Dippner J.W. Fish and zooplankton interaction in the central Baltic Sea. ICES journal of marine science 58.579-588, 2001.
[8] Markin R.P., Adrian J.D.Jowitt, Julia H.S. & GeoH L.P. The practical importance of the interactions between fish, zooplankton and macrophytees in shallow lake restoration. Hydrobiology 395/396,199-210, 1999.
[9] Makoto Omori, ‘‘Weight and chemical composition of some important oceanic zooplankton in the North Pacific Ocean’’, Marine biology, May 1969, vol.3, issue 1, pp4-10.
[10] Oyelami B.O. and Asere A. A Mathematical modelling: An application to corrosion in a petroleum industry. Proceeding of National Mathematical Centre of the Workshop on Mathematical modelling of Environmental Problems.Vol.5, No.1. (2005), 48-67.www.nmcabuja.org/proceedings/workshop_Oyelami_Asere.pdf.

[11] Thibault D., Gaudy R.,J Le Fevre, Zooplankton biomass, feeding and metabolism in a geostrophic frontal area (Almeria-Oran front, Western Mediterranean) .Significance to pelagic food webs. Journal of Marine Systems, Vol.5, issue 3-5, August Pp. 297-311.http://dx.doi.org/10.1016/0924-7963(94)90052-3

[12] Ute Daewe, Myron A. Peck, Corinna Schrum and Michael A. St John .How best to include the effects of climate-driven forcing on prey fields in larval fish individual-based models. J. Plankton Res. (2008) 30 (1): 1-5.doi: 10.1093/plankt/fbm094

[13] Sam O. Ale and Benjamin O. Oyelami. Mathematical modelling of the Exploitations of biological Resources in Forestry and Fishery. Proceeding of National Mathematical Centre of the Workshop on Mathematical Modelling of Environmental Problems.Vol.5, No.1.2005, 6-34.www.nmcabuja.org/proceedings/workshop_Ale_Oyelami.pdf

Appendix

involving the Linear Algebra and plots library

>*with* (*LinearAlgebra*) :

>*with*(*DETools*) :

>*with* (*plots*) :

input the value of the matrix A

$$> A := Matrix\left(\left[\left[\frac{a[1]}{k[1]}\cdot(1+t[r])^{-1}, b[1]\cdot k[1]\right], \left[b[1]\cdot k[1], -\frac{a[2]}{k[2]}\cdot(1+t[r])^{-1} - b[2]\cdot k[1]\right]\right]\right);$$

$$A := \begin{bmatrix} \frac{a_1}{k_1(1+t_r)} & b_1 k_1 \\ b_1 k_1 & -\frac{a_2}{k_2(1+t_r)} - b_2 k_1 \end{bmatrix}$$

input the value of the parameters

$>a[1] := 0.05;$

$a_1 := 0.05$

$>b[1] := .23;$

$b_1 := 0.23$

$>k[1] := 0.03;$

$k_1 := 0.03$

$>k[2] := 0.12;$

$k_2 := 0.12$

$>b[2] := 0.123;$

$b_2 := 0.123$

$>a[2] := 1;$

$a_2 := 1$

computes the eigenvalues of matrix A

>*Eigenvalues*(*A*, *output* ='*list*');

$$\begin{bmatrix} 0.05390929749 \\ -0.2726530609 \end{bmatrix}$$

$>t[r] := 30;$

$t_r := 30$

$$>C := \begin{bmatrix} 0.05376344087 & 0.0069 \\ 0.0069 & -0.2725072043 \end{bmatrix};$$

$$C := \begin{bmatrix} 0.05376344087 & 0.0069 \\ 0.0069 & -0.2725072043 \end{bmatrix}$$

invoking Student[LinearAlgebra] library facility to obtain eigenvalues and eigenvectors

> with(Student[LinearAlgebra]):
> infolevel[Student[LinearAlgebra]] := 1:

$$CharateristicPolynomial\left(\begin{bmatrix} 0.05376344087 & 0.0069 \\ 0.0069 & -0.2725072043 \end{bmatrix}, x\right)$$

>$EigenPlot(C);$

$$>A := \begin{bmatrix} 0.05376344087 & 0.0069 \\ 0.0069 & -0.2725072043 \end{bmatrix}:$$

>$EigenPlot($ A);
Eigenvalue: .5391e-1
Multiplicity: 1
Eigenvector: < .9998, .2113e-1 >
Eigenvalue: -.2727
Multiplicity: 1
Eigenvector: < -.2113e-1, .9998 >

inputting the data from Table 1 to generate the matrix plots
>$with(plots):$
>$with(LinearAlgebra):$

$$>A := \begin{bmatrix} 0.5000 & 0.3333 & 0.25 & 0.800 & 0.600 & 0.2857 & 0.5455 \\ 0.07142 & 0.01645 & 0.4444 & 0.3750 & 0.4000 & 0.5000 & 0.4615 \\ 0.03541 & 0.1958 & 0.7343 & 2.9282 & 3.3034 & 5.002 & 7.4999 \\ 0.0035 & 0.9191 & 1.6021 & 1.4182 & 4.3142 & 8.0841 & 14.129 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

$$A := \begin{bmatrix} 0.5000 & 0.3333 & 0.25 & 0.800 & 0.600 & 0.2857 & 0.5455 \\ 0.07142 & 0.01645 & 0.4444 & 0.3750 & 0.4000 & 0.5000 & 0.4615 \\ 0.03541 & 0.1958 & 0.7343 & 2.9282 & 3.3034 & 5.002 & 7.4999 \\ 0.0035 & 0.9191 & 1.6021 & 1.4182 & 4.3142 & 8.0841 & 14.129 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

input the Maple codes to generate matrix plots of various gap lengths
>$matrixplot(A, heights = histogram, axes = boxed);$

>$matrixplot(A, heights = histogram, axes = frame, gap = 0.15)$

>$matrixplot(A)$

>$F := (x, y) \rightarrow \sin(x\,y):$
>$matrixplot(A, heights = histogram, axes = frame, gap = 0.25, color = F)$

generation random sample for Wight noise
>$with(LinearAlgebra):$
>$with(Statistics):$
>$X[1] := RandomVariable(Normal(0, 1)):$
>$X[2] := RandomVariable(Normal(0, 1)):$

$$>X1 := 0.03541 \cdot \begin{bmatrix} X[1] \\ X[2] \end{bmatrix};$$

$$X1 := \begin{bmatrix} 0.03541\_R \\ 0.03541\_R0 \end{bmatrix}$$

$$>X2 := 0.1958 \cdot \begin{bmatrix} X[1] \\ X[2] \end{bmatrix};$$

$$X2 := \begin{bmatrix} 0.1958\_R \\ 0.1958\_R0 \end{bmatrix}$$

$$X3 := 0.7343 \cdot \begin{bmatrix} X[1] \\ X[2] \end{bmatrix};$$

$$X6 := 7.4999 \cdot \begin{bmatrix} X[1] \\ X[2] \end{bmatrix};$$

$$>A := \begin{bmatrix} 0.05376344087 & 0.0069 \\ 0.0069 & -0.2725072043 \end{bmatrix};$$

$$A := \begin{bmatrix} 0.05376344087 & 0.0069 \\ 0.0069 & -0.2725072043 \end{bmatrix}$$

input the value of the initial data

$$>z[0] := \begin{bmatrix} 5 \\ 0.05 \end{bmatrix};$$

$$z_0 := \begin{bmatrix} 5 \\ 0.05 \end{bmatrix}$$

generate the power matrix for matrix A of various power for n= 1,2,…4

$>p[1] := MatrixPower(A, 1);$

$$p_1 := \begin{bmatrix} 0.05376344087 & 0.0069 \\ 0.0069 & -0.2725072043 \end{bmatrix}$$

$>p[2] := MatrixPower(A, 2);$

$$p_2 := \begin{bmatrix} 0.00293811757418199 & -0.00150933196766700 \\ -0.00150933196766700 & 0.0743077863954020 \end{bmatrix}$$

$>p[3] := MatrixPower(A, 3);$

$$p_3 := \begin{bmatrix} 0.000147548919891739 & 0.000431576846131408 \\ 0.000431576846131408 & -0.0202598215189095 \end{bmatrix}$$

$>p[4] := MatrixPower(A, 4);$

$$p_4 := \begin{bmatrix} 0.0000109106178683386 & -0.000116589712232628 \\ -0.000116589712232628 & 0.00552392520197331 \end{bmatrix}$$

$>MatrixMatrixMultiply(p[1], z[0]);$

$$\begin{bmatrix} 0.269162204350000 \\ 0.0208746397850000 \end{bmatrix}$$

computes iterative solutions to the model using the environmental risk values in the Table 1

$>z[1] := MatrixMatrixMultiply(p[1], z[0]) + X1;$

$$z_1 := \begin{bmatrix} 0.269162204350000 + 0.03541\_R \\ 0.0208746397850000 + 0.03541\_R0 \end{bmatrix}$$

$>z[2] := MatrixMatrixMultiply(p[2], z[0]) + X2;$

$$z_2 := \begin{bmatrix} 0.0146151212725266 + 0.1958\_R \\ -0.00383127051856490 + 0.1958\_R0 \end{bmatrix}$$

>

$$X3 := \begin{bmatrix} 0.7343\_R \\ 0.7343\_R0 \end{bmatrix}$$

$>z[3] := MatrixMatrixMultiply(p[3], z[0]) + X3;$

$$z_3 := \begin{bmatrix} 0.000759323441765264 + 0.7343\ \_R \\ 0.00114489315471157 + 0.7343\ \_R0 \end{bmatrix}$$

>p[6] := MatrixPower(A, 6);

$$p_6 := \begin{bmatrix} 2.08029257877952\ 10^{-7} & -0.00000867999117681921 \\ -0.00000867999117681921 & 0.000410646626552184 \end{bmatrix}$$

>

$$X6 := \begin{bmatrix} 7.4999\ \_R \\ 7.4999\ \_R0 \end{bmatrix}$$

>z[6] := MatrixMatrixMultiply(p[6], z[0]) + X6;

$$z_6 := \begin{bmatrix} 6.06146730548798\ 10^{-7} + 7.4999\ \_R \\ -0.0000228676245564868 + 7.4999\ \_R0 \end{bmatrix}$$

>restart

>with(LinearAlgebra) :

>with(Statistics) :

>Y[1] := RandomVariable(Normal(0, 1)) :

>Y[2] := RandomVariable(Normal(0, 1)) :

generate the Wight noise using environmental risk cost formula

$$>Y1 := 0.9191 \cdot \begin{bmatrix} Y[1] \\ Y[2] \end{bmatrix};$$

$$Y1 := \begin{bmatrix} 0.9191\ \_R \\ 0.9191\ \_R0 \end{bmatrix}$$

$$>Y2 := 1.6021 \cdot \begin{bmatrix} Y[1] \\ Y[2] \end{bmatrix};$$

$$Y2 := \begin{bmatrix} 1.6021\ \_R \\ 1.6021\ \_R0 \end{bmatrix}$$

$$>Y3 := 1.4182 \cdot \begin{bmatrix} Y[1] \\ Y[2] \end{bmatrix};$$

$$Y3 := \begin{bmatrix} 1.4182\ \_R \\ 1.4182\ \_R0 \end{bmatrix}$$

$$>Y6 := 14.129 \cdot \begin{bmatrix} Y[1] \\ Y[2] \end{bmatrix};$$

$$Y6 := \begin{bmatrix} 14.129\ \_R \\ 14.129\ \_R0 \end{bmatrix}$$

$$>A := \begin{bmatrix} 0.05376344087 & 0.0069 \\ 0.0069 & -0.2725072043 \end{bmatrix};$$

$$A := \begin{bmatrix} 0.05376344087 & 0.0069 \\ 0.0069 & -0.2725072043 \end{bmatrix}$$

$$>z[0] := \begin{bmatrix} 5 \\ 0.05 \end{bmatrix};$$

$$z_0 := \begin{bmatrix} 5 \\ 0.05 \end{bmatrix}$$

>p[1] := MatrixPower(A, 1);

$$p_1 := \begin{bmatrix} 0.05376344087 & 0.0069 \\ 0.0069 & -0.2725072043 \end{bmatrix}$$

>$p[2] := MatrixPower(A, 2);$

$$p_2 := \begin{bmatrix} 0.00293811757418199 & -0.00150933196766700 \\ -0.00150933196766700 & 0.0743077863954020 \end{bmatrix}$$

>$p[3] := MatrixPower(A, 3);$

$$p_3 := \begin{bmatrix} 0.000147548919891739 & 0.000431576846131408 \\ 0.000431576846131408 & -0.0202598215189095 \end{bmatrix}$$

>$p[6] := MatrixPower(A, 6);$

$$p_6 := \begin{bmatrix} 2.08029257877952\,10^{-7} & -0.00000867999117681921 \\ -0.00000867999117681921 & 0.000410646626552184 \end{bmatrix}$$

computes iterative solutions to the model using the environmental risk cost values in the Table 1

>$MatrixMatrixMultiply(p[1], z[0]);$

$$\begin{bmatrix} 0.269162204350000 \\ 0.0208746397850000 \end{bmatrix}$$

$z[1] := MatrixMatrixMultiply(p[1], z[0]) + Y1;$

$$z_1 := \begin{bmatrix} 0.269162204350000 + 0.9191\_R \\ 0.0208746397850000 + 0.9191\_R0 \end{bmatrix}$$

>$z[2] := MatrixMatrixMultiply(p[2], z[0]) + Y2;$

$$z_2 := \begin{bmatrix} 0.0146151212725266 + 1.6021\_R \\ -0.00383127051856490 + 1.6021\_R0 \end{bmatrix}$$

>$z[3] := MatrixMatrixMultiply(p[3], z[0]) + Y3$

$$z_3 := \begin{bmatrix} 0.000759323441765264 + 1.4182\_R \\ 0.00114489315471157 + 1.4182\_R0 \end{bmatrix}$$

>$z[6] := MatrixMatrixMultiply(p[6], z[0]) + Y6$

$$z_6 := \begin{bmatrix} 6.06146730548798\,10^{-7} + 14.129\_R \\ -0.0000228676245564868 + 14.129\_R0 \end{bmatrix}$$

>

>$N := 4;$

$N := 4$

>$z[i+1] := (A, i) \rightarrow MatrixPower((A, i), z[0]);$

$z_{i+1} := (A, i) \rightarrow LinearAlgebra{:}\text{-}MatrixPower(A, i, z_0)$

> **for** i **from** 0 **by** 1 **while** i < N − 1 **do** $H(A, i) := unapply(MatrixMatrixMultiply(z[i+1],$
$z[0]));$**end do**;

$H := (A, i) \rightarrow unapply(LinearAlgebra{:}\text{-}MatrixMatrixMultiply(z_{i+1}, z_0))$

$H := (A, i) \rightarrow unapply(LinearAlgebra{:}\text{-}MatrixMatrixMultiply(z_{i+1}, z_0))$

$H := (A, i) \rightarrow unapply(LinearAlgebra{:}\text{-}MatrixMatrixMultiply(z_{i+1}, z_0))$

>

>$H(A, 0)$

( ) → Vector(2, {(1) = `+`(`*`(5, `*`(z[1]))), (2) = `+`(`*`(0.5e-1, `*`(z[1])))})

>$H(z[1], z[0]);$

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