The Convergence of the Approximated Derivative Function by Chebyshev Polynomials

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Abstract: Let \( f(x) \) be a differentiable function on the interval \([-1, 1]\). Finding an approximation of the derivative of the function through values of the function at points \( \{x_j\}_{j=0}^N \) is a very interesting problem. It is also important for solving differential equation. In this paper, we study the error bound, in particular for first and second derivatives by Chebyshev polynomials. Moreover, a generalisation for error bound is found.

Keywords: Chebyshev polynomials, Chebyshev interpolation, Convergence rate, Error function.

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I. Introduction

In many problems one of interested in finding the approximating the derivative of the function \( f \) depending on the value of the function \( f \) at \( x_j \). One of the method is to consider \((p_N(f))\) as an approximation to \( f' \). Let \( p_N \) be the Lagrange interpolation polynomial \( p_N \) for \( f \) which it may not converge to \( f' \) in the sup-norm. We wish to find conditions such that \( p_N \to f' \).

The Chebyshev approximation method works best when the function is smooth, and particularly when \( f(x) \) can be continued into the complex plane as a function \( f(z) \) which is analytic in an open neighborhood of \([-1,1]\).

In this case, the error

\[
E_N(x) = \max_{0 \leq |x| \leq 1} |f'(x) - p'_N(x)|,
\]

decay at least exponentially fast as \( N \to \infty \).

The Chebyshev polynomial of the first kind of degree \( N \) is defined as:

\[
T_N(x) = \cos(N \cos^{-1} x) = \cos(N \theta),
\]

where \( x = \cos \theta, -1 \leq x \leq 1, 0 \leq \theta \leq \pi \), and \( n \) is a non negative integer \([1]\).

The Chebyshev polynomials \( T_n(x) \) satisfy \(|T_n(x)| \leq 1\).

This follows from the bound \(-1 \leq \cos x \leq 1\), which leads to

\[
|T_{N+1}(x) - T_{N-1}(x)| \leq 2. \quad (1.2)
\]

The Chebyshev polynomial \( T_n(x) \) of degree \( n \geq 1 \) has \( n \) zeros on the interval \([-1, 1]\). The zeros \( x_j \) are given by:

\[
x_j = \cos \left( \frac{(2j-1)\pi}{2N} \right), \quad j=1,\ldots,N
\]

Moreover, the extrema, or points \( \bar{x}_j \) such that \( T_N(\bar{x}_j) = (-1)^j \) are given by:

\[
\bar{x}_j = \cos \left( \frac{(2j-1)\pi}{2N} \right), \quad j=1,\ldots,N
\]

The Chebyshev polynomials of the first kind have a generating function of the form

\[
\sum_{n=0}^\infty T_n(x) t^n = \frac{1 - tx}{1 - 2tx + t^2}; \quad |x| < 1, |t| < 1 \quad \ldots \quad (1 - 3)
\]

The Chebyshev polynomials of the second kind \( U_n(x) \) is defined as

\[
U_n(x) = \sin((N+1)x) = \sin \left( (N + 1) \frac{\pi}{2} \right)
\]

where \(-1 \leq x \leq 1\), \(0 \leq \theta \leq \pi\), \(x = \cos \theta\), and have a generating function of the form \([1]\)

\[
\sum_{n=0}^\infty U_n(x) t^n = \frac{1}{1 - 2tx + t^2}; \quad |x| < 1, |t| < 1 \quad \ldots \quad (1 - 4)
\]

The Chebyshev polynomials have interesting properties that make them a very attractive tool to minimize the maximum error in uniform approximation.

The derivatives of the Chebyshev polynomials satisfy the following:

\[
\left| \frac{d}{dx} T_n(x) \right| \leq N^2. \quad (1 - 5)
\]

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This comes from the definition of $T_N(x)$ and $\frac{d}{dx} T_N(x) = \frac{N \sin N \cos^{-1} x}{\sqrt{1-x^2}} = \frac{N \sin N \theta}{\sin \theta}$. We have $|\sin n\theta| \leq n |\sin \theta|$ and thus $\left| \frac{d}{dx} T_N(x) \right| \leq N^2$. For second derivative, we have

$$T_N''(x) = T_N''(\cos \theta) = \frac{N \sin N \theta \cos \theta - N^2 \cos N \sin \theta}{\sin^2 \theta}.$$  

Again by L’Hôpital’s rule, we get

$$\frac{N^3 - N}{3} \lim_{\theta \to 0 \text{ or } \pi} \frac{\sin n\theta}{\sin \theta \cos \theta} = \frac{N(N^3 - N)}{3} \lim_{\theta \to 0 \text{ or } \pi} \frac{\cos N \theta}{\cos^2 \theta - \sin^2 \theta}.$$ 

Therefore

$$|T_N''(x)| \leq \frac{N^3(N-1)(N+1)}{3}.$$ 

(1-6)

The values of $T_N(x)$ and their derivatives at some points are of interest:

$$|T_{N+1}'(x) - T_N'(x)| \leq 4N, \quad |T_{N+1}''(x) - T_N''(x)| \leq \frac{4}{3} N (2N^2 + 1).$$  

(1-7)

In general

$$T_N^{(2)}(x) \leq T_N^{(r)}(1) = \frac{N^2(N^2-1)\ldots(N^2-(r-1)^2)}{(2r-1)!}.$$  

(1-8)

II. Convergence Rate

The convergence of Chebyshev series is determined by a property of the function $f(x)$. If the function $f$ is smooth, then its Chebyshev expansion coefficients decrease rapidly. Two notions of smoothness were considered: an $r^{th}$ derivative with bounded variation, or analyticity in a neighborhood of $[-1, 1]$.

**Theorem 2.1** [2, p.66] The truncation error when approximating a function $f(x)$ in terms of Chebyshev polynomials satisfies

$$|f(x) - f_n(x)| \leq \sum_{k=n+1}^{\infty} |a_k|$$

If all $a_k$ are rapidly decreasing, then the error is dominated by the leading term $a_{n+1}T_{k+1}$.

The coefficients $a_k$ for $k > n + 1$ are negligibly small, where the rest of the terms will be neglected if $a_{n+1} \neq 0$.

**Theorem 2.2** [2, p.51] If $f, f', \ldots, f^{(r-1)}$ are absolutely continuous for $r \geq 0$ on $[-1, 1]$, where the $r^{th}$ derivative $f^{(r)}$ has bounded variation $V = \| f^{(r)} \|$, then the coefficients of the Chebyshev series satisfy the following inequality

$$|a_k| \leq \frac{2V}{\sqrt{k(k-1)\ldots(k-r)}} , \quad k \geq r + 1.$$  

(2.1)

for each $k \geq r + 1$.

**Theorem 2.3** [2, p.51] Let a function $f$ be analytic on $[-1, 1]$ and analytically continuably to the ellipse $E := \{ z \in C : z = \rho (e^{i\theta} + e^{-i\theta})/2, \, \theta \in [0, 2\pi] \}$ in which $|f(z)| \leq M$ for some $M$. For all $k \geq 0$ the Chebyshev coefficients $a_k$ of $f$ exponentially decay as $k \to \infty$ and satisfying

$$|a_k| \leq 2M \rho^{-k}, \quad \rho > 1.$$  

(2.2)

**Theorem 2.4** [2, p.53] If $f$ is absolutely continuous for $r \geq 0$ on $[-1, 1]$, where the $r^{th}$ derivative $f^{(r)}$ has bounded variation $V = \| f^{(r)} \|$, then the Chebyshev truncation satisfies

$$\| f - f_N \| \leq \frac{2V}{\sqrt{N-r}}.$$  

(2.3)

**Theorem 2.5** [2, p.58] Let a function $f$ be analytic on $[-1, 1]$ and analytically continuably to the open ellipse $E_\rho$ in which $|f| \leq M$ for some $M$. Then the Chebyshev truncation error satisfies

$$\| f - f_N \| \leq \frac{2M \rho^{-N}}{\rho - 1}.$$  

(2.4)
III. Chebyshev Interpolation

Given a function $f$ that is interpolated at $n + 1$ points in term of Chebyshev polynomials and that satisfies the interpolation condition $p_n(x_i) = f(x_i)$, we have the following theorem:

**Theorem 3.1** [2] Let $f(x)$ be a Lipschitz continuous function on $[-1, 1]$, where

$$f(x) = \sum_{k=0}^{n} a_k T_k(x), \quad a_k = \frac{2}{n} \int_{-1}^{1} f(T_k(x)) \frac{dx}{\sqrt{1-x^2}}$$

Then the function $f(x)$ can be presented by interpolation in Chebyshev points as

$$p_N = \sum_{k=0}^{n} b_k T_k(x), \quad b_k = \frac{2}{N} \sum_{i=0}^{N} f(x_i) T_k(x_i), \quad x_i = \cos \left(\frac{2i}{N}\right)$$

and

$$p_N = \sum_{k=0}^{n} c_k T_k(x), \quad c_k = \frac{2}{N} \sum_{i=0}^{N} f(x_i) T_k(x_i), \quad x_i = \cos \left(\frac{2i}{2N}\right)$$

Here $a_k$ are the exact coefficients, and $b_k$ and $c_k$ are coefficients of $p_n$.

**Theorem 3.2** [3] Assume that $\{x_j\}_{j=0}^{N}$ are distinct points in $[a,b]$ and that $f(x)$ is a function in $C^{N+1}[a,b]$ and $|f^{N+1}| \leq M$. Let $p_N$ be a sequence of polynomial interpolating $f$. Then for each $x \in [a,b]$, there is $\xi \in (a,b)$ such that

$$|f(x) - p_N(x)| \leq \prod_{k=0}^{N} |x - x_k| \left| f^{(N+1)}(\xi) \right|_{(N+1)}$$

**Theorem 3.3** Let $f(x)$ be a continuous function, $p_n(x)$ its polynomials interpolation at $n+1$ points and $(p_n(f))$ an approximation to $f$. Then

$$\|f - p_n\|_\infty \leq \left| \frac{d}{dx} \prod_{k=0}^{N} (x - x_k) \right| \left| f^{(n+1)} \right|_{(N+1)}$$

IV. Main Results

The choice of Chebyshev points minimizes the terms $\prod_{k=0}^{N} (x - x_k)$ on [-1,1]. This choice ensures uniform convergence for a Lipschitz continuous function $f$. This condition is more important than the condition of continuity of the function $f$.

**Theorem 4.1** Let $f(x)$ be a continuous function on $[a, b]$ and let $p_n(x)$ be interpolant polynomials of $f$ at Chebyshev zeros. Then the error is given by

$$\|f - p_n\|_\infty \leq \left| \frac{2(b-a)^{n+1}}{4^{n+1}(n+1)!} \right| \|f^{n+1}(\xi)\|_\infty$$

Similarly, the error at Chebyshev extrema is given by:

$$\|f - p_n\|_\infty \leq \left| \frac{1}{2^n(n+1)!} \right| \|f^{n+1}(\xi)\|_\infty$$

Now, we will investigate the interpolation convergence bound at zeros and extrema of Chebyshev polynomials:

**Theorem 4.2** If $f$ is absolutely continuous and $\|f^{(r)}\| = V < \infty$. Then for every $N \geq r + 1$,

$$\|f' - p_n'\|_\infty \leq 4V \left[ \frac{N^2(r-1)+2r(N+1)}{(r-1)(r-2)(N-r)!} \right], \quad r \geq 2$$

and

$$\|f^r - p_n^r\|_\infty \leq 2V 3 \left[ \frac{1}{(r-4)(N-r)!} + \frac{4r}{(r-3)(N-r)!} + \frac{6r^2-1}{(r-2)(N-r)!} + \frac{4r^2-2r}{(r-1)(N-r)!} - \frac{r^2-3}{r(N-r)!} \right], \quad r \geq 4$$

**Proof.**

We have

$$\|f' - p_n'\|_\infty \leq \sum_{k=0}^{N-1} |a_k - b_k| ||T_k'||_\infty + \sum_{k=N}^{n} |a_k| ||T_k'||_\infty + \sum_{k=N+1}^{n+1} |a_k| ||T_k'||_\infty$$

$$\leq 2 + \sum_{k=N+1}^{n+1} |a_k| k^2 \leq \sum_{k=N+1}^{n+1} \frac{4V}{(k-r)^4} k^2$$

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Where, \( a_k, b_k \) and \( c_k \) are defined in (3, 1), (3, 2) and (3, 3).

From the above we have that \( \|T'_k\|_\infty = k^2 \)

\[
\sum_{k=N+1}^{\infty} \frac{k^2}{(k-r)^{r+1}} \leq \int_{N}^{\infty} \frac{x^2}{(x-r)^{r+1}} dx
\]

\[
= \int_{N-r}^{\infty} \frac{(u+r)^2 du}{u^{r+1}} = \frac{N^2(r-1)-2r(N+1)}{(r-1)(r-2)(r-N)^2}.
\]

Therefore, for the second derivative \( \|f''\|_{\infty} \)

\[
\|f'' - p''\|_{\infty} \leq \sum_{k=0}^{N} |a_k - b_k| \|T''_k\|_{\infty} + \|a_N - b_N\| \|T''_N\|_{\infty} + \sum_{k=N+1}^{\infty} |a_k| \|T''_k\|_{\infty}.
\]

We have from (4) that \( \|T''_k\|_{\infty} = \frac{k^2(k^2-1)}{3} \) and so

\[
\|f'' - p''\|_{\infty} \leq \sum_{k=0}^{N} |a_k - b_k| \frac{k^2(k^2-1)}{3} + \|a_N - b_N\| \frac{N^2(N-1)(N+1)}{3} + \sum_{k=N+1}^{\infty} |a_k| \frac{k^2(k^2-1)}{3}.
\]

Similarly to the above we have

\[
\sum_{k=N+1}^{\infty} \frac{k^2(k^2-1)}{3} \leq \int_{N}^{\infty} \frac{x^2(x^2-1)dx}{(x-r)^{r+1}} = \int_{N-r}^{\infty} \frac{(u+r)^2((u+r)^2-1)du}{u^{r+1}}
\]

\[
\leq \frac{1}{(r-4)(r-N)^{r-4}} + \frac{4r^2-1}{(r-3)(r-N)^{r-3}} + \frac{6r^2-2}{(r-2)(r-N)^{r-2}} + \frac{4r^2-2}{(r-1)(r-N)^{r-1}} - \frac{r^4-r^2}{r(r-N)^{r}}.
\]

Therefore

\[
\|f'' - p''\|_{\infty} \leq \frac{2N^2\rho + (1 - 2N - 2N^2)\rho^2 + (1 + 2N + 2N^2)\rho^3}{\rho^{r+1}(1-\rho)^2} \quad r \geq 2 \quad (4, 5)
\]

and

\[
\|f'' - p''\|_{\infty} \leq \frac{N^4(\rho - 1)^4 + 4N^3(\rho - 1)^3 + 12\rho^2(1 + \rho) + \rho + N^2(\rho - 1)^2(5\rho^2 + 8\rho - 1 + 2N(\rho^3 + 9\rho^2 - 9\rho - 1)}{\rho^{r+1}(1-\rho)^2} \quad (4, 6)
\]

Proof.

As above, we arrive at

\[
\|f' - p'\|_{\infty} \leq 2 \sum_{k=N+1}^{\infty} |a_k| \|T'_k\|_{\infty} \leq \sum_{k=N+1}^{\infty} \frac{4Mk^2}{\rho^k}
\]

By the table value of the last sum \( \sum_{k=N+1}^{\infty} \frac{k^2}{\rho^k} \), which can also verified in computer algebra system ‘Mathematica’, we get the above result.

For the second derivative

\[
\|f'' - p''\|_{\infty} \leq 2 \sum_{k=N+1}^{\infty} |a_k| \|T''_k\|_{\infty} \leq \sum_{k=N+1}^{\infty} \frac{4Mk^2(k^2-1)}{\rho^k}
\]

Again by the table value of the last sum \( \sum_{k=N+1}^{\infty} \frac{k^2(k^2-1)}{\rho^k} \), which can also verified in computer algebra system ‘Mathematica’, we get the above result.

We now consider the case when the function \( f(x) \) extends to function \( f(z) \) of the complex plane which is analytic in a simple closed contour \( C \) the interval \([a, b] \). The complex equivalent to (4, 1 ) and (4, 2 ) is given by a contour integral [1, p150]:

**Theorem 4.4** [5, p.83] Assume that \( f \) is that extends to an analytic function in a domain \( \Omega \) that contains the interval [-1, 1]. Let \( C \subset \Omega \) be a simple closed contour in the complex plane and let \( x_j \subset C \), where \( f \) is an analytic function on and inside \( C \). Then

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\[ f(x) - p_N(x) = \frac{1}{2\pi i} \int_C \frac{\Phi_N(z) f(z)}{\Phi_N(z)(x-z)} \, dz, \quad x \in [-1, 1], \]  

(4, 7)

where

\[ p_N(x) = \frac{1}{2\pi i} \int_C \frac{f(z)(\Phi_N(z)-\Phi_N(x))}{\Phi_N(z)(x-z)} \, dz, \quad \Phi_N(x) = \prod_{k=0}^N (x-x_k) \]  

(4, 8)

**Remark.** In the case of interpolation at Chebyshev zeros, we have

\[ \Phi_N(x) = \prod_{k=0}^N (x-x_k) = T_N(x), \]  

whereas in the case of interpolation at Chebyshev extrema,

\[ \Phi_N(x) = \prod_{k=0}^N (x-x_k) = T_{N+1}(x) - T_{N-1}(x). \]

**Theorem 4.5** If \( f \) is a bounded analytic function such that \( |f(z)| \leq M \) in the region bounded by an ellipse \( E_\rho \) with foci \( \pm 1 \) and major semi-axis \( a = \frac{\rho+\rho^{-1}}{2} \) and minor semi-axis \( b = \frac{\rho-\rho^{-1}}{2} \) summing to \( \rho > 1 \). Then

\[ \| f' - p'_N \|_\infty \leq \left[ \frac{N^2}{(2\rho+\rho^{-1})-1} \right] \frac{M \sqrt{\rho^2+\rho^{-2}}}{(\rho^N-\rho^{-N})} \]  

(4, 9)

And, for second derivative

\[ \| f'' - p''_N \|_\infty \leq \left[ \frac{N^2(N^2-1)}{(2\rho+\rho^{-1})-1} \right] \frac{M \sqrt{\rho^2+\rho^{-2}}}{(\rho^N-\rho^{-N})} \]  

(4, 10)

Where \( p_N \) is the polynomial interpolant of degree \( \leq N \) at Chebyshev zeros.

**Proof.**

By differentiating (4, 7) we obtain

\[ f'(x) - p'_N(x) = \frac{1}{2\pi i} \int_{E_\rho} \left[ \Phi_N(z)(f(z) - f(x)) \right] \Phi_N(z)(x-z)^2 \frac{dz}{(x-z)^2} \]  

(4, 7)

From (1, 2), (1, 5), we have \( |\Phi_N(x)| \leq 1, |\Phi_N(x)| \leq N^2 \) and \( |z - x| \geq a - 1 = \frac{1}{2}(\rho + \rho^{-1}) - 1, \) so

\[ \| f' - p'_N \|_\infty \leq \left[ \frac{N^2}{(2\rho+\rho^{-1})-1} \right] \frac{M \sqrt{\rho^2+\rho^{-2}}}{(\rho^N-\rho^{-N})} \]  

(4, 11)

For the second part, we differentiate (4, 7) twice to get

\[ f'' - p''_N = \frac{1}{2\pi i} \int_{E_\rho} \left[ \frac{\Phi''_N(x)}{(x-z)^2} + \frac{2\Phi'_N(x)}{(x-z)^3} - \frac{2\Phi_N(x)}{(x-z)^3} \right] \frac{f(z)}{\Phi_N(z)} \, dz \]  

From the above, we have \( |\Phi''_N(x)| \leq \frac{N^2(N^2-1)}{3}, \) thus

\[ \| f'' - p''_N \|_\infty \leq \left[ \frac{N^2(N^2-1)}{(2\rho+\rho^{-1})-1} \right] \frac{M \sqrt{\rho^2+\rho^{-2}}}{(\rho^N-\rho^{-N})} \]  

(4, 12)

**Theorem 4.6** If \( f \) is a bounded analytic function such that \( |f(z)| \leq M \) in the region bounded by an ellipse \( E_\rho \) with foci \( \pm 1 \) and major semi-axis \( a = \frac{\rho+\rho^{-1}}{2} \) and minor semi-axis \( b = \frac{\rho-\rho^{-1}}{2} \) summing to \( \rho > 1 \). Then

\[ \| f' - p'_N \|_\infty \leq \left[ \frac{N^2}{(2\rho+\rho^{-1})-1} \right] \frac{M \sqrt{\rho^2+\rho^{-2}}}{(\rho^N-\rho^{-N})} \]  

(4, 11)

And, for second derivative

\[ \| f'' - p''_N \|_\infty \leq \left[ \frac{N^2(N^2-1)}{(2\rho+\rho^{-1})-1} \right] \frac{M \sqrt{\rho^2+\rho^{-2}}}{(\rho^N-\rho^{-N})} \]  

(4, 12)
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Where \( p_N \) is the polynomial interpolant of degree \( \leq N \) at Chebyshev extrema.

**Proof.**

By differentiating (\ref{equation}) we obtain

\[
 f'(x) - p'_N(x) = \frac{1}{2\pi i} \int_{E_p} \left[ \frac{\Phi_N(x)}{(z-x)} + \frac{\Phi_N(x)}{(z-x)^2} \right] f(z) \, dz
\]

From \( |\Phi_N(x)| \leq 2 \), \( |\Phi'_N(x)| \leq 4N \), then

\[
 \|f' - p'_N\|_\infty \leq \frac{N^2}{2(2r+2)^{-1}} + \frac{1}{(2r+2)^{-1}} \int_E |x| f(z) \, dz
\]

For the second part

\[
 f'' - p''_N = \frac{1}{2\pi i} \int_{E_p} \left[ \frac{\Phi''_N(x)}{(z-x)^2} + \frac{2\Phi'_N(x)}{(z-x)^3} + \frac{\Phi_N(x)}{(z-x)^4} \right] f(z) \, dz
\]

From above, we have \( |\Phi''_N(x)| \leq \frac{4N(2N^2+1)}{3} \), we have

\[
 \|f'' - p''_N\|_\infty \leq \frac{N(2N^2+1)}{3} + \frac{8N^2}{2(2r+2)^{-1}} + \frac{2}{(2r+2)^{-1}} \int_E f(z) \, dz
\]

**Lemma**

For Chebyshev polynomial, the estimation of \( r \)-th derivative satisfy the bound

\[
 \left\| \frac{d^r}{dx^r} (T_{N+1}(x) - T_{N-1}) \right\|_\infty \leq \frac{(N+r-2)!}{(2r-1)!(N-r+1)!} [4rN^2 + r^2]. \tag{4, 13}
\]

**Proof.**

We have \( [1] \)

\[
 \left\| f^{(r)}(x) \right\|_\infty \leq \prod_{k=0}^{r-1} N^2 - k^2 \frac{2k+1}{2k+1} \tag{4, 14}
\]

From the Stirling formula, the term \( (2r-1)! \) can be written as \( \frac{(2r)!}{2^{2r}} \) and

\[
 N^2(N^2 - 1^2)(N^2 - 2^2) \ldots (N^2 - (r-1)^2) = \frac{N(N+r)!}{N+r(N-r)!} \tag{4, 15}
\]

We use induction on \( r \). If \( r = 1 \), then we have \( N^2 \). If this hold for \( N \geq 2 \), and \( r = 1, \ldots, N-2 \), then it also hold for \( r+1 \):

\[
 \frac{N(N+r+1)!}{N+r(N-r)!} = \frac{N+r}{N+r+1} \frac{N+(r+1)(N-r)}{N+r(N-r)!!} \frac{N(N+r)!}{N+r(N-r)!!}
\]

Then by using (4, 14) and (4, 15) to estimate \( \frac{d^r}{dx^r} (T_{N+1}(x) - T_{N-1}) \), we have

\[
 \frac{d^r}{dx^r} (T_{N+1}(x) - T_{N-1}) = \frac{1}{(2r-1)!!} \frac{(N+1)(N+r+1)!}{(N+r)(N-r+1)!} - \frac{(N-1)(N+r-1)!}{(N+r-1)(N-r-1)!}
\]

\[
 = \frac{(N+r-2)!}{(2r-1)!(N-r+1)!} [4rN^2 + r^2].
\]

We may generalize the previous result as follows:

**Theorem 4.7** If \( f \) is a bounded analytic function such that \( |f(z)| \leq M \) in the region bounded by an ellipse \( E_p \) with foci \( \pm l \) and major semi-axis \( a = \frac{\rho+1}{2} \) and minor semi-axis \( b = \frac{\rho-1}{2} \) summing to \( \rho > 1 \). Then

\[
 \left\| f^{(r)}(x) - p_N^{(r)} \right\|_\infty \leq \sum_{k=0}^{r} \mathcal{N}_k \frac{r!}{(2r-1)!} \frac{(N+r-2)!}{(N-r+1)!} [4rN^2 + r^2] \times \frac{M}{(\rho^{N-r} - \rho^{N+r})} \times \sum_{k=0}^{r} \mathcal{N}_k \left( \frac{2p}{(\rho-1)^2} \right)^{-k+1} \tag{4, 16}
\]

Where \( p_N \) is the polynomial interpolant of degree \( \leq N \) at Chebyshev extrema points.

**Proof.**

By considering the error formula (\ref{equation}), we have

\[
 f^{(r)}(x) - p_N^{(r)} = \frac{1}{2\pi i} \int_{E_p} f(x) \frac{\Phi_N(x)}{(z-x)^r} \, dz.
\]

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By Leibniz's rule we have
\[ f^{(r)}(x) = \sum_{k=0}^{r} \binom{r}{k} u^{(k)} \cdot v^{(r-k)}, \]
where \( f(x) = u(x) \cdot v(x). \)

Thus
\[ f^{(r)}(x) - p_N^{(r)}(x) = \frac{1}{2\pi i} \int_{E_\rho} \frac{f(z)}{\Phi_N(z)} \sum_{k=0}^{r} \binom{r}{k} (r-k)! \left( \Phi_N(z) \right)^{k} (z-x)^{r-k-1} dz. \]

\[ = \sum_{k=0}^{r} \frac{1}{k!} \int_{E_\rho} \frac{\Phi_N(z)}{\Phi_N(z-x)^{r-k+1}} \frac{(\phi_{k,N}(z))^{(k)} f(z)}{\Phi_N(z-x)^{r-k+1}} dw \]

To estimate \( \frac{1}{x-x} \), let \( z = \frac{w + w^{-1}}{2} \), where \( w = \rho e^{i\theta} \) and \( 0 \leq \theta \leq 2\pi \). Then
\[ \frac{1}{w} = \frac{1}{w} = \frac{2}{w(1 - 2xw^{-1} + w^{-2})} \]

By the definition of the generating function of the second kind \((1, 4)\) of the Chebyshev polynomials \( U_n(x) \), we have
\[ \frac{2}{\rho} \sum_{k=0}^{\infty} U_n(x) \rho^{-k} \leq \frac{2}{\rho} \sum_{k=0}^{\infty} \frac{k + 1}{\rho^k} = \frac{2\rho}{(\rho - 1)^2}. \]

From (4, 13) we have
\[ (\Phi_N(x))^{(k)} \leq \frac{(N + r - 2)!}{((2r - 1)!(N - r + 1)!} \times \frac{4rN^2 + r^2}{} \]

Therefore
\[ \| f^{(r)}(x) - p_N^{(r)}(x) \|_\infty \leq \sum_{k=0}^{r} \frac{1}{k!} \int_{E_\rho} \frac{\Phi_N(z)}{\Phi_N(z-x)^{r-k+1}} \frac{f(z)}{w(2r - 1)!} \times \frac{1}{((2r - 1)!(N - r + 1)!} \times \frac{4rN^2 + r^2}{\rho^N - \rho^{-N}} \times \frac{M}{(\rho^N - \rho^{-N})} \times \sum_{k=0}^{r} \frac{2\rho}{(\rho - 1)^2} \]

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References